

ON ANALYTICITY IN DEEP INFERENCE

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Joint work with Paola Bruscoli
Work in progress with Benjamin Ralph } Both Bath

Talk available from my home page and from <http://cs.bath.ac.uk/ag/t/OAIDI.pdf>
All about deep inference at <http://alessio.guglielmi.name/res/cos>

THANK YOU DALE

Everything I do in research has been determined by two things Dale did in the winter of 1994, in Philadelphia, while I was a PhD student.

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This paper gave me a thesis and allowed me to survive through difficult years. Why was I struggling? Because of:

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2 Still Dale, in the winter of 1994, in Philadelphia.

He did not discourage me to pursue the following idea:

- design a logic about the physics of virtual particles,
- such that a black hole is modeled by the cut rule.

It took me five more years (well beyond my PhD) to do BV ($V = \text{virtual}$) and then deep inference.

Ten years later Prakash Panangaden saw that indeed BV models quantum causal evolution.

We still don't know about black holes but I trust Dale.

PROVING EXISTENTIALS GENTZEN-STYLE

$$\begin{array}{c}
 \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\lambda_R \frac{}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}} \\
 \\
 \frac{2w_{L,R}}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \\
 \frac{}{\text{---}} \\
 \\
 \frac{\frac{\frac{R\sqrt{2}^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}}{\neg_R \frac{}{\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}} \quad \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{w_{L,R} \frac{}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}}} \quad \frac{R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R(J_2^{\sqrt{2}})^{\sqrt{2}}}{w_{L,R} \frac{}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2}^{\sqrt{2}}, R(J_2^{\sqrt{2}})^{\sqrt{2}}}}} \\
 \frac{2\exists_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2} \wedge R(J_2^{\sqrt{2}})^{\sqrt{2}}} \\
 \\
 \frac{2\exists_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \\
 \frac{\lambda_R}{\frac{}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}} \\
 \\
 \frac{2\exists_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \\
 \frac{c_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}
 \end{array}$$

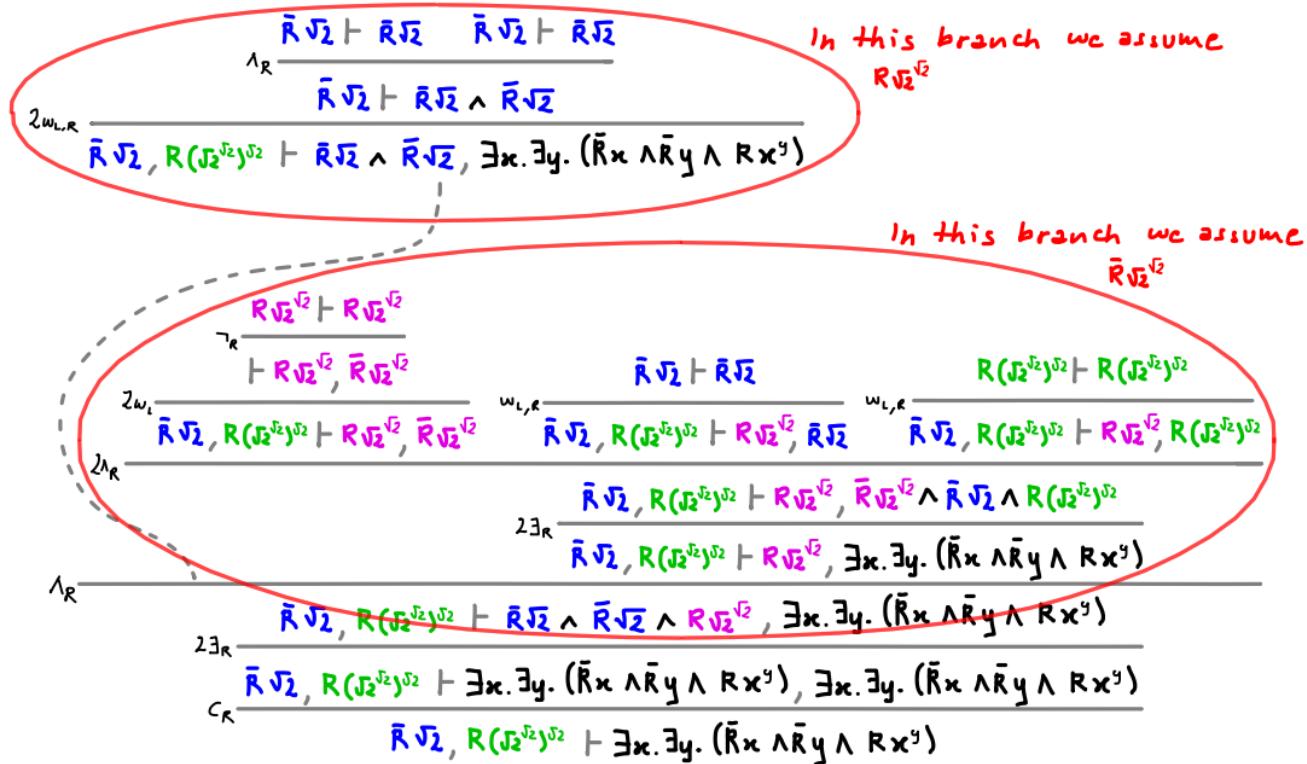
There are two irrationals x and y such that x^y is rational.

PROVING EXISTENTIALS GENTZEN-STYLE

	$\frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}$	$\bar{R}\sqrt{2}$ means $\sqrt{2}$ is irrational; $R(J_2^{\sqrt{2}})^{\bar{J}_2}$ means $(J_2^{\sqrt{2}})^{\bar{J}_2}$ is rational. We assume both.
λ_R	$\frac{2w_{L,R}}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$	
	$\frac{R\sqrt{2}^{\bar{J}_2} \vdash R\sqrt{2}^{\bar{J}_2}}{\vdash R\sqrt{2}^{\bar{J}_2}, \bar{R}\sqrt{2}^{\bar{J}_2}}$	$R\sqrt{2}^{\bar{J}_2}$ means $\sqrt{2}^{\bar{J}_2}$ is rational. We don't know, so we try both ways.
λ_R	$\frac{2w_i}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash R\sqrt{2}^{\bar{J}_2}, \bar{R}\sqrt{2}^{\bar{J}_2}}$	
λ_R	$\frac{}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}$	$\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}$
λ_R	$\frac{}{R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash R(J_2^{\sqrt{2}})^{\bar{J}_2}}$	$R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash R(J_2^{\sqrt{2}})^{\bar{J}_2}$
	$\frac{2\exists_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash R\sqrt{2}^{\bar{J}_2}, \bar{R}\sqrt{2}^{\bar{J}_2} \wedge \bar{R}\sqrt{2} \wedge R(J_2^{\sqrt{2}})^{\bar{J}_2}}$	
λ_R	$\frac{2\exists_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash R\sqrt{2}^{\bar{J}_2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$	
	$\frac{2\exists_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\bar{J}_2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$	
λ_R	$\frac{2\exists_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$	
c_R	$\frac{}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\bar{J}_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$	

There are two irrationals x and y such that x^y is rational.

PROVING EXISTENTIALS GENTZEN-STYLE



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$$\begin{array}{c}
 \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\lambda_R \frac{}{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}} \\
 \\
 \frac{2w_{L,R}}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \\
 \\
 \frac{}{\frac{\frac{R\sqrt{2}^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}}{\neg_R \frac{}{\vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}}} \quad \frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{w_{L,R} \frac{}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}}} }{R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2} \vdash R\sqrt{2}^{\sqrt{2}}, R(J_2^{\sqrt{2}})^{\sqrt{2}}} \\
 \\
 \frac{x = \sqrt{2}^{\sqrt{2}}, y = \sqrt{2}}{R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R(J_2^{\sqrt{2}})^{\sqrt{2}}} \\
 \\
 \frac{2w_1}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}^{\sqrt{2}}} \quad \frac{w_{L,R}}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \bar{R}\sqrt{2}} \quad \frac{}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, R(J_2^{\sqrt{2}})^{\sqrt{2}}} \\
 \\
 \frac{2\lambda_R}{\frac{\frac{x = y = \sqrt{2}}{2\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}}{\bar{R}\sqrt{2}^{\sqrt{2}} \wedge \bar{R}\sqrt{2} \wedge R(J_2^{\sqrt{2}})^{\sqrt{2}}}}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash R\sqrt{2}^{\sqrt{2}}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}} \\
 \\
 \frac{\lambda_R}{\frac{2\exists_R \frac{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R\sqrt{2}^{\sqrt{2}}}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{\frac{C_R}{\bar{R}\sqrt{2}, R(J_2^{\sqrt{2}})^{\sqrt{2}} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}} \\
 \end{array}$$

There are two irrationals x and y such that x^y is rational.

WHAT DO WE WANT?

- 1 Canonicity of proofs, no bureaucracy, good proof semantics.
Gentzen systems are helpers, but this is for another talk.
- 2 Low complexity:
 - small proofs,
 - efficient proof search.

This is what the talk is about.

PROVING EXISTENTIALS GENTZEN-STYLE

$$\frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\lambda_R \quad \bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}}$$

$$\frac{2_{WL,R} \quad \bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

Complexity comes from

\forall_R : not much to do here,

\exists_R : let's see what we can do...

$$\frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\neg_R \quad \vdash \bar{R}\sqrt{2}, \bar{R}\sqrt{2}}$$

$$\frac{2_{WL} \quad \bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2}, \bar{R}\sqrt{2}}{\bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2}, \bar{R}\sqrt{2}}$$

$$\frac{\bar{R}\sqrt{2} \vdash \bar{R}\sqrt{2}}{\omega_{L,R} \quad \bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2}}$$

$$\frac{\bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2}, \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R(J_2^{J_2})^{J_2}}{2\exists_R \quad \bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$\frac{\bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge R(J_2^{J_2})^{J_2}, \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{2\exists_R \quad \bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \bar{R}\sqrt{2} \wedge \bar{R}\sqrt{2} \wedge \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

$$\frac{\bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y), \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{C_R \quad \bar{R}\sqrt{2}, R(J_2^{J_2})^{J_2} \vdash \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}$$

There are two irrationals x and y such that x^y is rational.

PROVING EXISTENTIALS IN DEEP INFERENCE

$$\frac{i \frac{\frac{t}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}}{= \left\{ \left(\frac{i \frac{\frac{t}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y}}{Rx \wedge \bar{R}y \wedge Rx^y}}{Rx \vee Ry \vee \exists z. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \right) \vee \left(\frac{i \frac{\frac{t}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)}}{Ry \wedge \bar{R}y \wedge Rx^y}}{Ry \vee \bar{R}x^y \vee \exists z. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \right) \right\} \frac{\sqrt{2}^{\sqrt{2}}}{x} \vee \frac{\sqrt{2}^{\sqrt{2}}}{y}}{= \frac{i \frac{\frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{R\sqrt{2}}}{R\sqrt{2}}}$$

The red elements are work in progress.

There are two irrationals x and y such that x^y is rational.

PROVING EXISTENTIALS IN DEEP INFERENCE

Much cleaner than this:

$$i \frac{\epsilon}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}$$

$$\begin{array}{c}
 \frac{KQ_1 \vdash RQ \quad KQ_2 \vdash RQ}{KQ_1 \wedge KQ_2 \vdash RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge KQ_2, 3x.3y.((RQ \wedge RQ) \wedge RQ)}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{RQ \vdash RQ \quad RQ \vdash RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \frac{KQ_1, RQ \text{open} \vdash RQ \wedge RQ}{KQ_1, RQ \text{open} \vdash RQ \wedge RQ} \\
 \end{array}$$

$$= \frac{\left\{ \begin{array}{c} i \frac{\epsilon}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y} \\ 1s \frac{Rx \vee Ry \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y}{Rx \vee Ry \vee \exists x.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right\}_{\sqrt{2}}^{\sqrt{2}} \vee \left\{ \begin{array}{c} \bar{R}x \wedge i \frac{\epsilon}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)} \\ 1s \frac{Ry \vee \bar{R}x^y \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y}{Ry \vee \bar{R}x^y \vee \exists y.(\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right\}_{\sqrt{2}}^{\sqrt{2}}} {\frac{2c \frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2})^{\sqrt{2}} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}$$

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PROVING EXISTENTIALS IN DEEP INFERENCE

In this branch we assume
 $R\sqrt{2}^{\sqrt{2}}$, with $x=y=\sqrt{2}$

In this branch we assume
 $\bar{R}\sqrt{2}^{\sqrt{2}}$, with $x=\sqrt{2}^{\sqrt{2}}$, $y=\sqrt{2}$

$$\begin{array}{c}
 \text{---} \\
 \text{---} \\
 = \quad \frac{\text{---}}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}} \\
 \text{---} \\
 = \quad \left\{ \frac{\text{---}}{\frac{i \quad \frac{\text{---}}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y}}{1s \quad \frac{\text{---}}{Rx \wedge \bar{R}y \wedge Rx^y}}} \right\}_{\sqrt{2}^{\sqrt{2}} \atop x} \vee \left\{ \frac{\text{---}}{\frac{i \quad \frac{\text{---}}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)}}{1s \quad \frac{\text{---}}{Ry \wedge \bar{R}y \wedge Rx^y}}} \right\}_{\sqrt{2}^{\sqrt{2}} \atop y} \\
 = \quad \frac{\text{---}}{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}} \vee \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}} \vee \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}
 \end{array}$$

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$$\begin{array}{c}
 i \frac{}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}} \\
 = \frac{}{\left\{ \begin{array}{c} i \frac{}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y} \\ 1s \frac{}{Rx \vee Ry \vee \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right\} \vee \left\{ \begin{array}{c} \bar{R}x \wedge i \frac{}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)} \\ 1s \frac{}{Ry \vee \bar{R}x^y \vee \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right\}}{= \frac{\begin{array}{c} R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2} \\ 2c \frac{}{R\sqrt{2}} \end{array} \vee \begin{array}{c} \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \\ 2c \frac{}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array}}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}
 \end{array}$$

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PROVING EXISTENTIALS IN DEEP INFERENCE

We use substitutions: $\{\varphi\}_x^z = \varphi(x \leftarrow z)$ where φ is a proof and z is a term
 indicated actual

$$\frac{i \frac{t}{R\sqrt{2}^{j_2} \vee \bar{R}\sqrt{2}^{j_2}}}{= \left\{ \left(\frac{i \frac{t}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y)} \wedge Rx^y}{1s \frac{Rx \vee Ry \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y}{Rx \vee Ry \vee \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}} \right) \vee \left(\frac{i \frac{t}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)} \wedge Ry^x}{1s \frac{Ry \vee \bar{R}x^y \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y}{Ry \vee \bar{R}x^y \vee \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}} \right) \right\} = \frac{i \frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{j_2})^{j_1} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{R\sqrt{2}}$$

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 indicated actual and z is a term

$$\frac{i \frac{t}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}}{= \left\{ \left(\begin{array}{c} i \frac{t}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y} \\ 1s \frac{}{Rx \vee Ry \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y} \end{array} \right)_{\substack{\sqrt{2} \\ x}} \vee \left(\begin{array}{c} i \frac{t}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)} \\ 1s \frac{}{Ry \vee \bar{R}x^y \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y} \end{array} \right)_{\substack{\sqrt{2} \\ y}} \right\}_{\sqrt{2}}}$$

$$= \frac{\frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{R\sqrt{2}}$$

There are two irrationals x and y such that x^y is rational.

PROVING EXISTENTIALS IN DEEP INFERENCE

We use substitutions: $\{\varphi\}_x^z = \varphi(x \leftarrow z)$ where φ is a proof and z is a term
 indicated actual

$$\frac{= \frac{i \frac{\frac{t}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}}{R_x \vee R_y \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y}}_{1s} \wedge \frac{R_x \wedge i \frac{\frac{t}{R_y \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)}}{R_y \vee \bar{R}x^y \vee (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}_{1s} \wedge \frac{R_y \vee \bar{R}x^y \vee i \frac{\frac{t}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}_{23}}_{23}}{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}$$

There are two irrationals x and y such that x^y is rational.

PROVING EXISTENTIALS IN DEEP INFERENCE

No proof complexity and no proof-search complexity in the \exists rule!

$$\begin{array}{c}
 i \frac{\epsilon}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}} \\
 = \frac{}{\left\{ \left(\begin{array}{c} i \frac{\epsilon}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y} \\ 1s \frac{Rx \vee Ry \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right) \wedge_x \vee \left(\begin{array}{c} \bar{R}x \wedge i \frac{\epsilon}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)} \\ 1s \frac{Ry \vee \bar{R}x^y \vee \bar{R}x \wedge \bar{R}y \wedge Rx^y}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right) \wedge_y \right\}}{= \frac{\lambda c \frac{R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{R\sqrt{2}}$$

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$$\frac{= \frac{i \frac{\frac{\frac{\epsilon}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}}}{R\sqrt{2}^{\sqrt{2}} \wedge Rx^y}}{\left\{ \begin{array}{l} i \frac{\frac{\epsilon}{Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y)} \wedge Rx^y}{\bar{R}x \wedge \bar{R}y \wedge Rx^y} \\ 1s \frac{Rx \vee Ry \vee \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{Rx \vee Ry \vee \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right\}_{\bar{x}} \vee \left\{ \begin{array}{l} i \frac{\frac{\epsilon}{Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y)}}{\bar{R}x \wedge \bar{R}y \wedge Rx^y} \\ 1s \frac{Ry \vee \bar{R}x^y \vee \exists x. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{Ry \vee \bar{R}x^y \vee \exists x. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right\}_{\bar{y}}} \right\}_{\bar{x}} \vee \left\{ \begin{array}{l} i \frac{\frac{\epsilon}{R\sqrt{2}^{\sqrt{2}} \vee R\sqrt{2}^{\sqrt{2}} \vee R\sqrt{2}^{\sqrt{2}}}}{\bar{R}(R\sqrt{2}^{\sqrt{2}})^{\sqrt{2}}} \vee c \frac{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \end{array} \right\}_{\bar{R}\sqrt{2}^{\sqrt{2}}}}{R\sqrt{2}^{\sqrt{2}}}$$

Moral reasons:

- 1 There is more information in the proof: assignments.
- 2 That information is shared, therefore it costs less.

PROVING EXISTENTIALS IN DEEP INFERENCE

This is also good for proof canonicity, bureaucracy and semantics.

$$\begin{array}{c}
 \frac{i \quad t}{R\sqrt{2}^{\sqrt{2}} \vee \bar{R}\sqrt{2}^{\sqrt{2}}} \\
 = \frac{}{\left\{ \left(\frac{\begin{array}{c} i \quad t \\ Rx \vee Ry \vee (\bar{R}x \wedge \bar{R}y) \wedge Rx^y \end{array}}{Rx \wedge Ry \vee (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \right)_{\substack{\sqrt{2} \\ x}} \vee \left(\frac{\begin{array}{c} \bar{R}x \wedge i \quad t \\ Ry \vee \bar{R}x^y \vee (\bar{R}y \wedge Rx^y) \end{array}}{Ry \vee \bar{R}x^y \vee (\bar{R}x \wedge \bar{R}y \wedge Rx^y)} \right)_{\substack{\sqrt{2} \\ x}} \right\}_{\substack{\sqrt{2} \\ y}} } \\
 = \frac{\frac{\begin{array}{c} R\sqrt{2} \vee R\sqrt{2} \vee R\sqrt{2} \\ R\sqrt{2} \end{array}}{R\sqrt{2}} \vee \bar{R}(\sqrt{2}^{\sqrt{2}})^{\sqrt{2}} \vee c \frac{\begin{array}{c} \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \vee \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \\ \exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y) \end{array}}{\exists x. \exists y. (\bar{R}x \wedge \bar{R}y \wedge Rx^y)}}{R\sqrt{2}}
 \end{array}$$

See how dealing with terms is relegated to the top and bottom.
 The coloured atoms behave as propositional variables.

THE IDEOLOGY BEHIND

Definition An inference rule

$$r \frac{A}{B}$$

is analytic iff for every B there is an n such that the number of possible A s is bounded by n .

THE IDEOLOGY BEHIND

Definition An inference rule

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i.e., proof-search nondeterminism does not depend on the context.

In Gentzen, $\exists_R \frac{\Gamma \vdash \Delta, A(x \leftarrow z)}{\Gamma \vdash \Delta, \exists x.A}$ is not analytic (no subformula property).

In deep inference $\exists \frac{A}{\exists x.A}, \frac{\{A\}_x^z}{A(x \leftarrow z)}$ and $= \frac{A(x \leftarrow z)}{\{A\}_x^z}$ are analytic.

Complete analytic system for predicate logic?
Compatible with the existing normalisation theory?
Gives Herbrand theorem?

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Give me three solid reasons for investigating proof search
in deep inference.

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Answer

- 1 You can bound at will the **depth level** (see also work by Anupam Das). Even at low depth / low nondeterminism you can profit from freedom in composing proofs.
- 2 **Splitting theorems** allow you to decide where to **focus proof search** (in the same sense as focussing). See the accompanying note for details and pointers.

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Answer

- 1 You can bound at will the depth level (see also work by Anupam Das). Even at low depth / low nondeterminism you can profit from freedom in composing proofs.
- 2 Splitting theorems allow you to decide where to focus proof search (in the same sense as focussing). See the accompanying note for details and pointers.
- 3 You can get nontrivial speed-ups!

NONELEMENTARY SPEED-UP

Corollary of Aguilera-Baaz*: Deep inference has a nonelementary speed-up over cut-free Gentzen proofs of the predicate calculus.

Moral reason:

$$\begin{array}{c}
 \frac{\omega \vdash P_z, \quad \overline{P_z}}{\omega \vdash P_z, \quad \overline{P_z}, Py} \\
 \frac{\omega \vdash \overline{P_x}, P_z, \quad \overline{P_z}, Py}{\omega \vdash \overline{P_x}, P_z, \quad \overline{P_z} \vee Py} \\
 \frac{\checkmark \vdash \overline{P_x}, P_z, \quad \overline{P_z} \vee Py}{\checkmark \vdash \overline{P_x} \vee P_z, \quad \overline{P_z} \vee Py} \\
 \frac{\checkmark \vdash \overline{P_x} \vee P_z, \quad \overline{P_z} \vee Py}{\checkmark \vdash \overline{P_x} \vee P_z, \quad \forall y.(\overline{P_z} \vee Py)} \\
 \frac{\exists \vdash \overline{P_x} \vee P_z, \quad \exists x. \forall y.(\overline{P_x} \vee Py)}{\exists \vdash \forall y.(\overline{P_x} \vee Py), \quad \exists x. \forall y.(\overline{P_x} \vee Py)} \\
 \frac{\exists \vdash \exists x. \forall y.(\overline{P_x} \vee Py), \quad \exists x. \forall y.(\overline{P_x} \vee Py)}{\exists \vdash \exists x. \forall y.(\overline{P_x} \vee Py)}
 \end{array}$$

Take proofs of the drinker formula.

Gentzen bureaucracy demands contractions (and nonanalytic \exists rules)...

... deep inference does not.

$$\begin{aligned}
 t &= \frac{}{\exists x. \overline{P_x} \vee \forall y. Py} \\
 &= \frac{}{\exists x. (\overline{P_x} \vee \forall y. Py)} \\
 &= \exists x. \forall y. (\overline{P_x} \vee Py)
 \end{aligned}$$

* J.P. Aguilera and M. Baaz. Unsound Inferences Make Proofs Shorter. 2016, t.b. on JSL.

HAPPY BIRTHDAY DALE!