MAKING SEVERAL COMPUTATION/NORMALISATION MECHANISMS COEXIST

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INVITED TALK AT FISP KICK-OFF 2016 (INNSBRUCK)
16/11/16

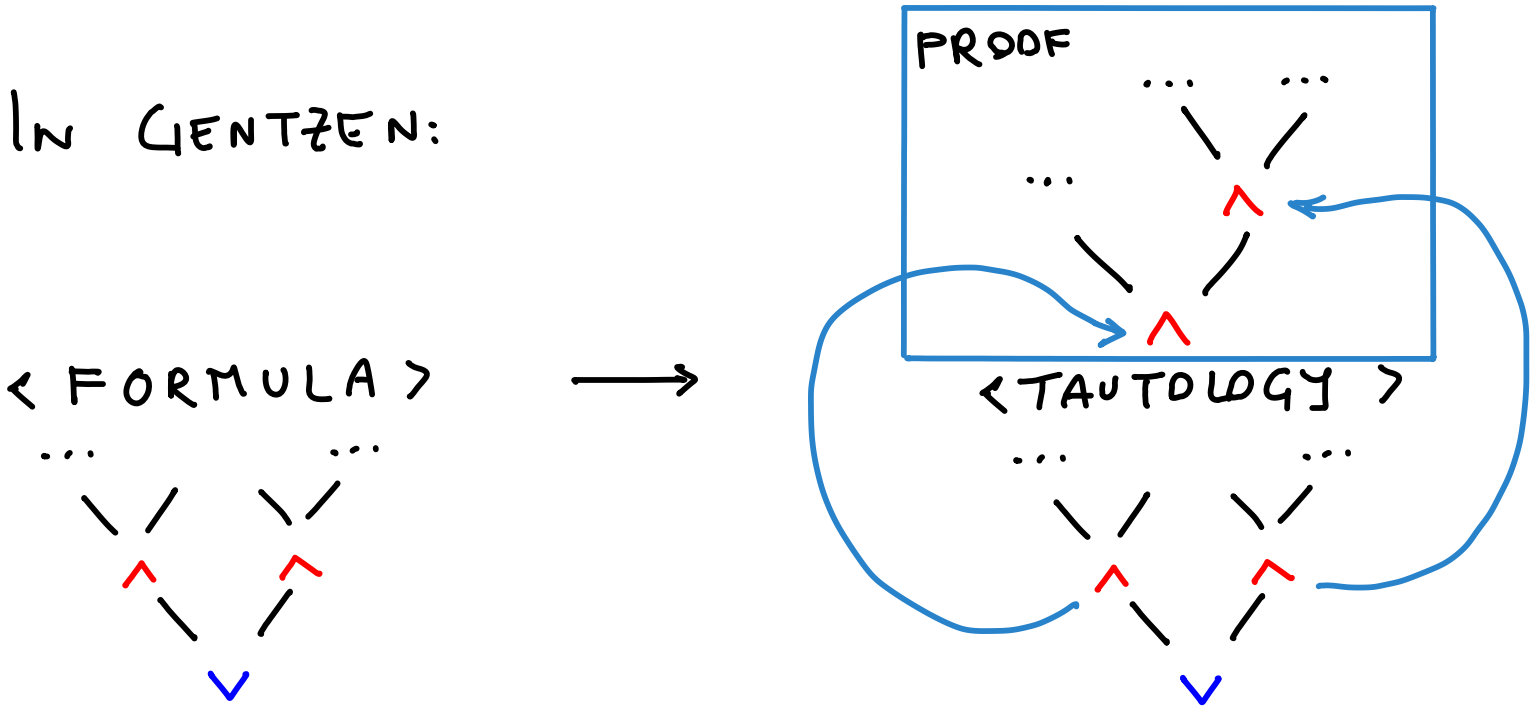
TALK AVAILABLE AT
http://cs.bath.ac.uk/ag/t/MSCNMC.pdf
AND FROM MY HOME PAGE

GENTZEN PROOF THEORY

COMPRESSION? MANIPULATION? LOCALITY? MODULARITY? ...

... NAMES! SCOPE! QUANTIFIERS! SUBSTITUTIONS! TREES!

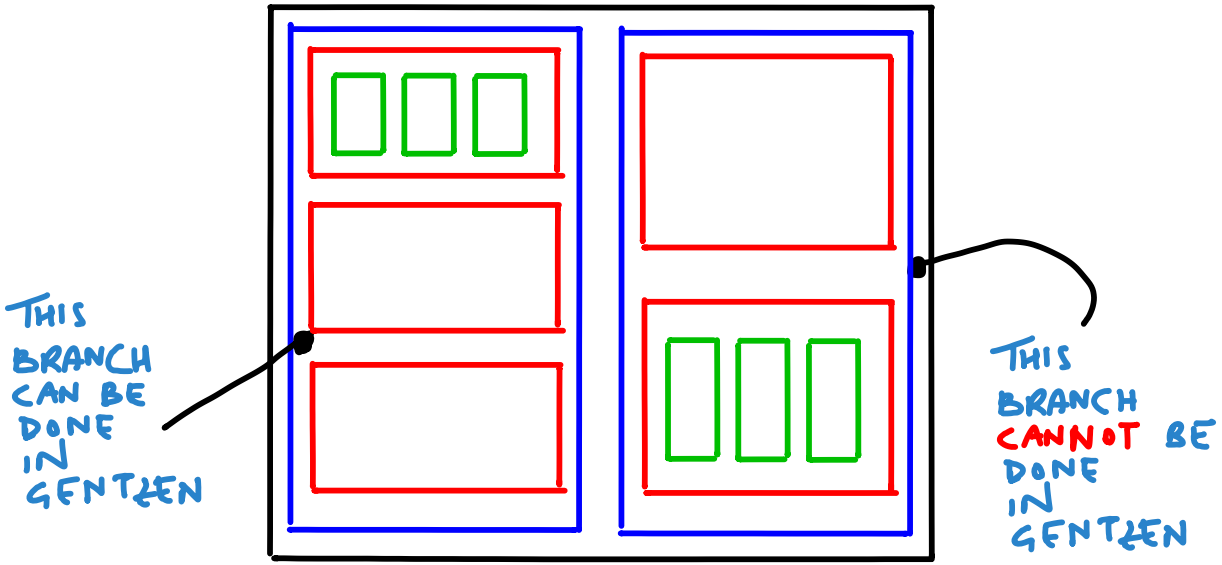
IN GENTZEN:



THE $\underbrace{\text{FORMULA TREE}}_{\text{GOOD}}$ GIVES SHAPE TO THE $\underbrace{\text{PROOF TREE}}_{\text{NOT GOOD ENOUGH}}$.

DEEP INFERENCE: BETTER TREES

A BETTER PROOF TREE LOOKS LIKE THIS:



VERTICAL COMPOSITION: INFERENCE

HORIZONTAL COMPOSITION: ANY (!) CONNECTIVE

IT'S MORE GENERAL THAN GENTZEN TREES AND YIELDS:

- EXPONENTIAL SPEED-UPS NONELEMENTARY (?)
- ANALYTICITY FOR MODAL AND EXOTIC LOGICS
- CANONICAL FORMS
- COMPUTATIONAL INTERPRETATIONS FOR CONCURRENCY

... AND MUCH MORE

COMPRESSION AND NATURALITY

$$
\begin{array}{rl}
\text{IDENTITY} & \dfrac{}{\vdash P(z)\ ,\ \overline{P(z)}} \\[2ex]
\text{WEAKENING} & \dfrac{\vdash P(z)\ ,\ \overline{P(z)}}{\vdash P(z)\ ,\ \overline{P(z)}\ ,\ P(y)} \\[2ex]
\vee & \dfrac{\vdash \overline{P(x)}\ ,\ P(z)\ ,\ \overline{P(z)}\ ,\ P(y)}{\vdash \overline{P(x)}\ ,\ P(z)\ ,\ \overline{P(z)} \vee P(y)} \\[2ex]
\vee & \dfrac{\vdash \overline{P(x)}\ ,\ P(z)\ ,\ \overline{P(z)} \vee P(y)}{\vdash \overline{P(x)} \vee P(z)\ ,\ \overline{P(z)} \vee P(y)} \\[2ex]
\forall & \dfrac{\vdash \overline{P(x)} \vee P(z)\ ,\ \overline{P(z)} \vee P(y)}{\vdash \overline{P(x)} \vee P(z)\ ,\ \forall y\,(\overline{P(z)} \vee P(y))} \\[2ex]
\exists & \dfrac{\vdash \overline{P(x)} \vee P(z)\ ,\ \forall y\,(\overline{P(z)} \vee P(y))}{\vdash \overline{P(x)} \vee P(z)\ ,\ \exists x\,\forall y\,(\overline{P(x)} \vee P(y))} \\[2ex]
\forall & \dfrac{\vdash \overline{P(x)} \vee P(z)\ ,\ \exists x\,\forall y\,(\overline{P(x)} \vee P(y))}{\vdash \forall y\,(\overline{P(x)} \vee P(y))\ ,\ \exists x\,\forall y\,(\overline{P(x)} \vee P(y))} \\[2ex]
\text{CONTRACTION} & \dfrac{\vdash \forall y\,(\overline{P(x)} \vee P(y))\ ,\ \exists x\,\forall y\,(\overline{P(x)} \vee P(y))}{\vdash \exists x\,\forall y\,(\overline{P(x)} \vee P(y))}
\end{array}
$$

SEQUENT CALCULUS

DEEP INFERENCE

SOUND!

APPARENTLY THE SPEED-UP IS NONELEMENTARY

(SEE AGUILERA – BAAZ, WE ARE CHECKING THE DETAILS)

$$
\begin{array}{rl}
\text{IDENTITY} & \dfrac{t}{\exists x\,\overline{P(x)} \vee \forall y\,P(y))} \\[2ex]
\text{RULES OF PASSAGE} & \dfrac{\exists x\,(\overline{P(x)} \vee \forall y\,P(y))}{\exists x\,\forall y\,(\overline{P(x)} \vee P(y))}
\end{array}
$$

TRIVIAL CUT ELIMINATION

ASSIGNMENT $A_\sigma = a \wedge \bar{b} \wedge \bar{c} \wedge d \wedge \bar{e} \wedge f \wedge \ldots$

$$\boxed{\text{CUT}\ \frac{B \wedge \bar{B}}{f}} \xrightarrow{\text{APPLY } \sigma} \boxed{\frac{B\sigma \wedge \bar{B}\sigma}{f} = \frac{f \wedge t}{f}}$$

THE CUT DISAPPEARS

PROOF φ WITH CUTS

C

CONCLUSION OF φ

APPLY ALL $\sigma_1, \ldots, \sigma_{2^n}$

CANONICAL PROOF WITH NO CUTS

A_{σ_1} $\quad$ $A_{\sigma_{2^n}}$

$\varphi\sigma_1$ NO CUTS $\quad \vee \cdots \vee \quad$ $\varphi\sigma_{2^n}$ NO CUTS

C $\quad$ C

SEVERAL CONTRACTIONS

C

CANONICAL FORM! NON-COMPUTATIONAL NORMALISATION

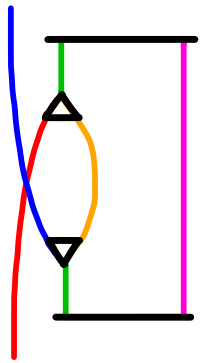
ATOMIC FLOWS

$$
\dfrac{\dfrac{\dfrac{\dfrac{a \quad \dfrac{t}{a \vee \bar{a}}}{a \wedge a} \vee (a \wedge \bar{a})}{a \vee a} \wedge (a \vee \bar{a})}{a} \quad \dfrac{a \wedge \bar{a}}{f}}{a \vee}
$$

PROOF

$\longrightarrow$

TRACING OCCURRENCES



ATOMIC FLOW

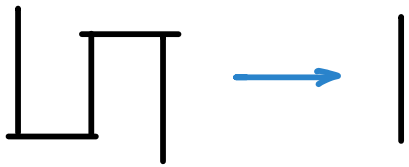
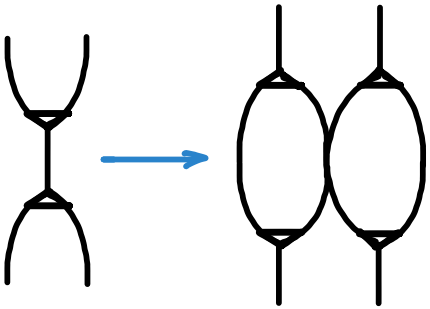
THERE'S ENOUGH INFORMATION IN AN ATOMIC FLOW TO ELIMINATE CUTS IN PROPOSITIONAL CLASSICAL LOGIC

COMPUTATION VS NORMALISATION IN DEEP INFERENCE

NORMALISATION NOT NECESSARILY COMPUTATION

HOWEVER

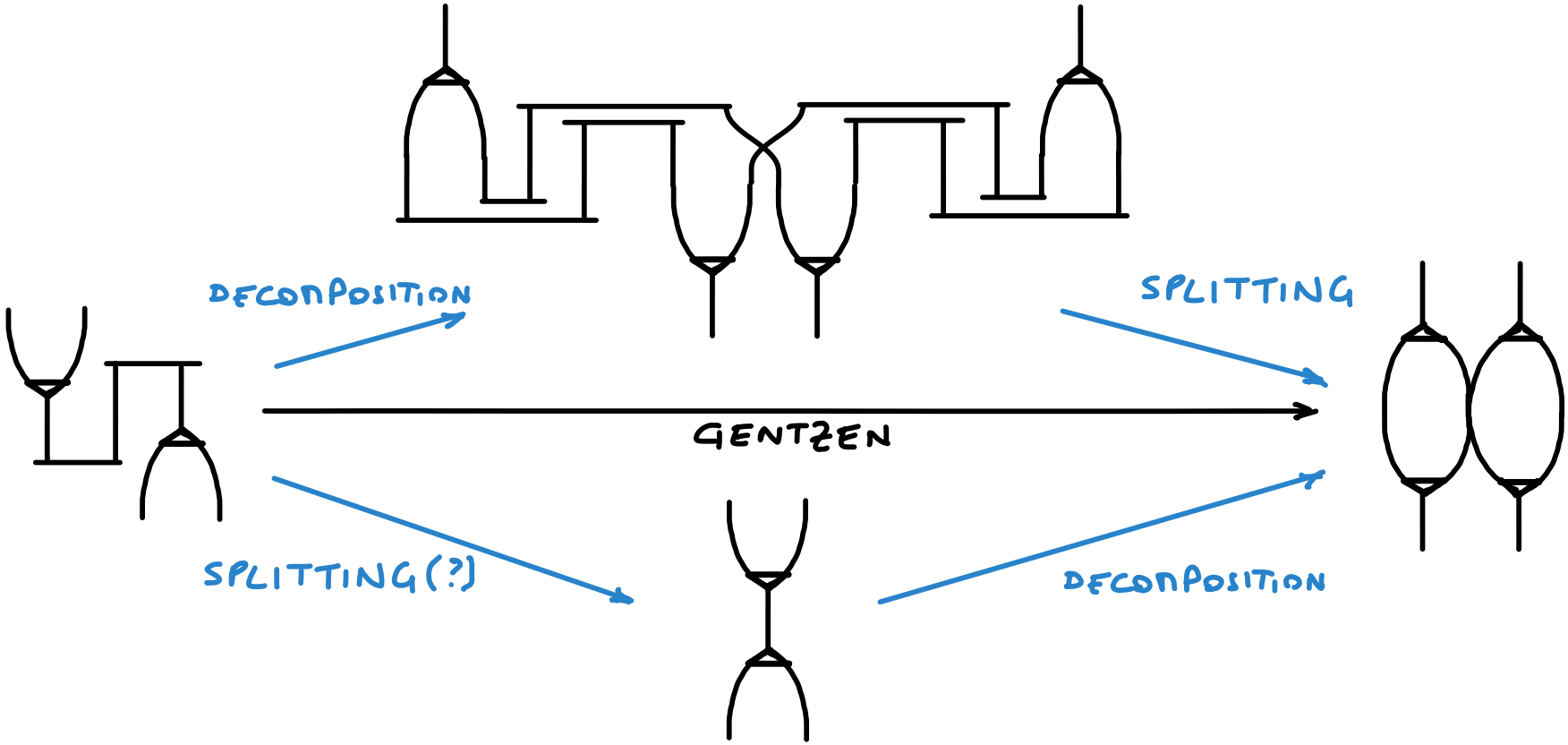
SEVERAL INDEPENDENT NORMALISATION/COMPUTATION MECHANISM :

- CUT ELIMINATION : (SPLITTING)  (YANKING) NO COMPLEXITY (LINEAR LOGIC)
- CONTRACTION : (DECOMPOSITION)  (UNFOLDING) EXPONENTIAL
- SUBSTITUTION EXPONENTIAL
- HERBRAND EXPANSION NON-ELEMENTARY ?

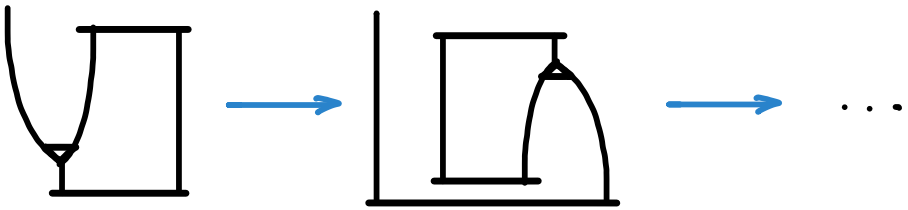
IN GENTZEN THESE MECHANISMS ARE CONFLATED: $\dfrac{\vdash \Gamma, mA \quad \vdash nA^{\perp}, \Delta}{\vdash \Gamma, \Delta}$ ↑ CUT + CONTRACTION

INDEPENDENCE OF CUT ELIMINATION AND CONTRACTION

IDEA:



PROBLEM:

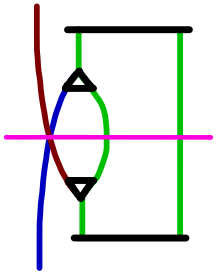


CYCLES!

Cycle elimination I

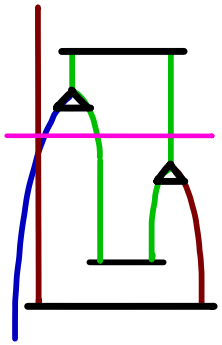
We will remove the green cycle by breaking its cut.

$$
\begin{array}{l}
\quad a \wedge \dfrac{t}{a \vee \bar{a}} \\
s \;\dfrac{\qquad\qquad}{\dfrac{a}{a \wedge a} \vee (a \wedge \bar{a})} \\
m \;\dfrac{\qquad\qquad}{\dfrac{a \vee a}{a} \wedge (a \vee \bar{a})} \\
s \;\dfrac{\qquad\qquad}{a \vee \dfrac{a \wedge \bar{a}}{f}}
\end{array}
$$

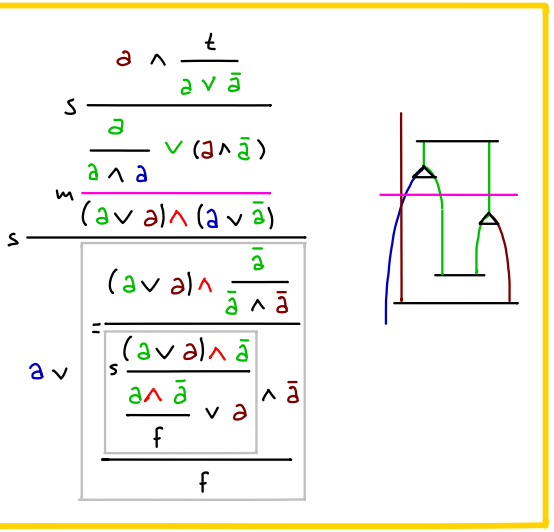


We move the contraction past the cut

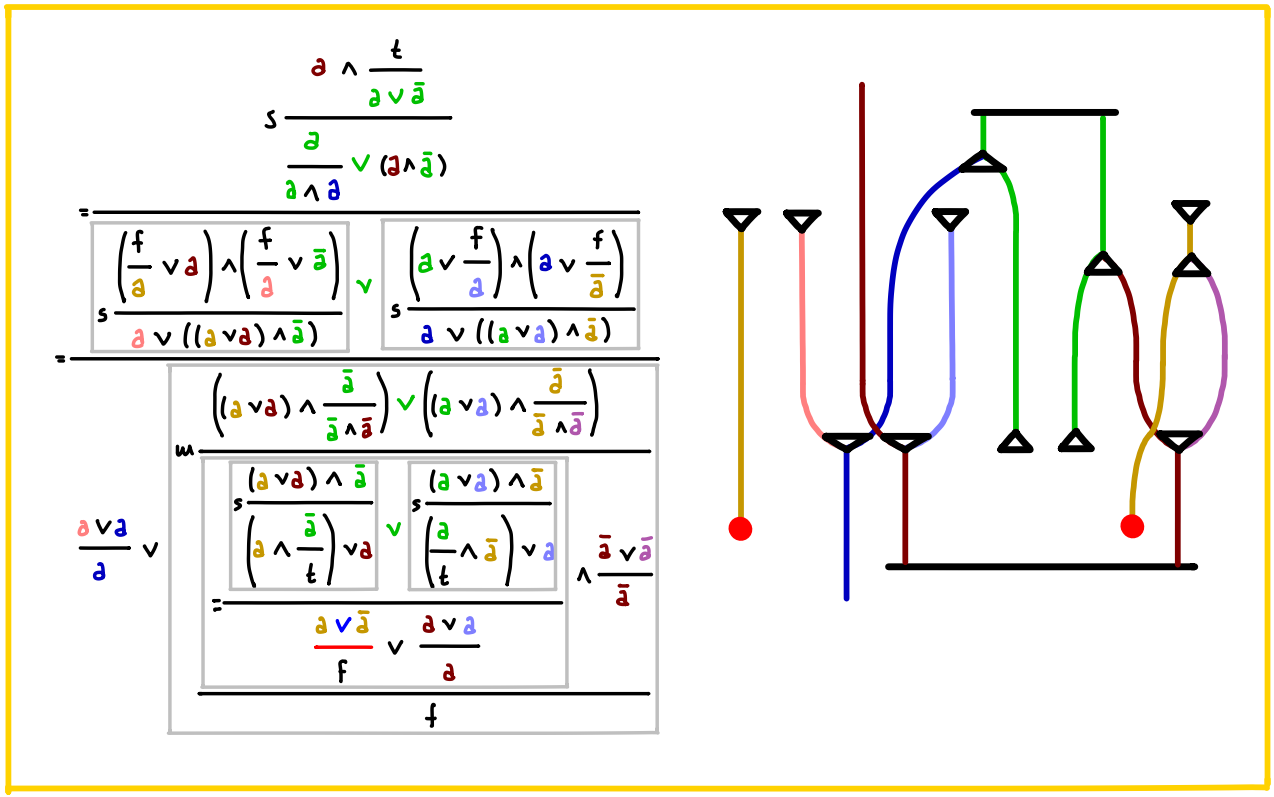
$$
\begin{array}{l}
\quad a \wedge \dfrac{t}{a \vee \bar{a}} \\
s \;\dfrac{\qquad\qquad}{\dfrac{a}{a \wedge a} \vee (a \wedge \bar{a})} \\
m \;\dfrac{\qquad\qquad}{(a \vee a) \wedge (a \vee \bar{a})} \\
s \;\dfrac{\qquad\qquad}{a \vee \dfrac{\dfrac{(a \vee a) \wedge \dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}}{s\;\dfrac{(a \vee a) \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{f} \vee a}} \wedge \bar{a}}{f}}
\end{array}
$$



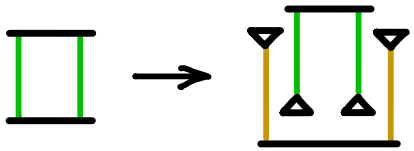
Cycle elimination 2



We get a derivation whose red step is invalid.

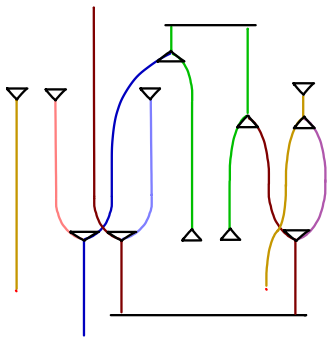


Idea:



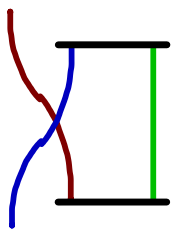
Cycle elimination 3

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{a \wedge \dfrac{t}{a \vee \bar{a}}}{a}\,\mathsf{s}}{a \wedge a} \vee (a \wedge \bar{a})}{\left(\dfrac{f}{a} \vee a\right) \wedge \left(\dfrac{f}{a} \vee \bar{a}\right)}\,\mathsf{s}}{a \vee ((a \vee a) \wedge \bar{a})} \vee \dfrac{\left(a \vee \dfrac{f}{a}\right) \wedge \left(a \vee \dfrac{f}{\bar{a}}\right)}{a \vee ((a \vee a) \wedge \bar{a})}\,\mathsf{s}}{\left((a \vee a) \wedge \dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}\right) \vee \left((a \vee a) \wedge \dfrac{\bar{a}}{\bar{a} \wedge \bar{a}}\right)}\,\mathsf{m}}{\dfrac{a \vee a}{a} \vee \dfrac{\dfrac{\dfrac{(a \vee a) \wedge \bar{a}}{\left(a \wedge \dfrac{\bar{a}}{t}\right) \vee a}\,\mathsf{s} \vee \dfrac{(a \vee a) \wedge \bar{a}}{\left(\dfrac{a}{t} \wedge \bar{a}\right) \vee a}\,\mathsf{s}}{\dfrac{a \vee \bar{a}}{f} \vee \dfrac{a \vee a}{a}} \wedge \dfrac{\bar{a} \vee \bar{a}}{\bar{a}}}{f}}
$$



We let the (co)weakenings go. This fixes the invalid step and simplifies the derivation.

$$
\dfrac{a \wedge \dfrac{t}{a \vee \bar{a}}}{a \vee \dfrac{a \wedge \bar{a}}{f}}\,\mathsf{s}
$$



Ok: complexity? We don't know yet

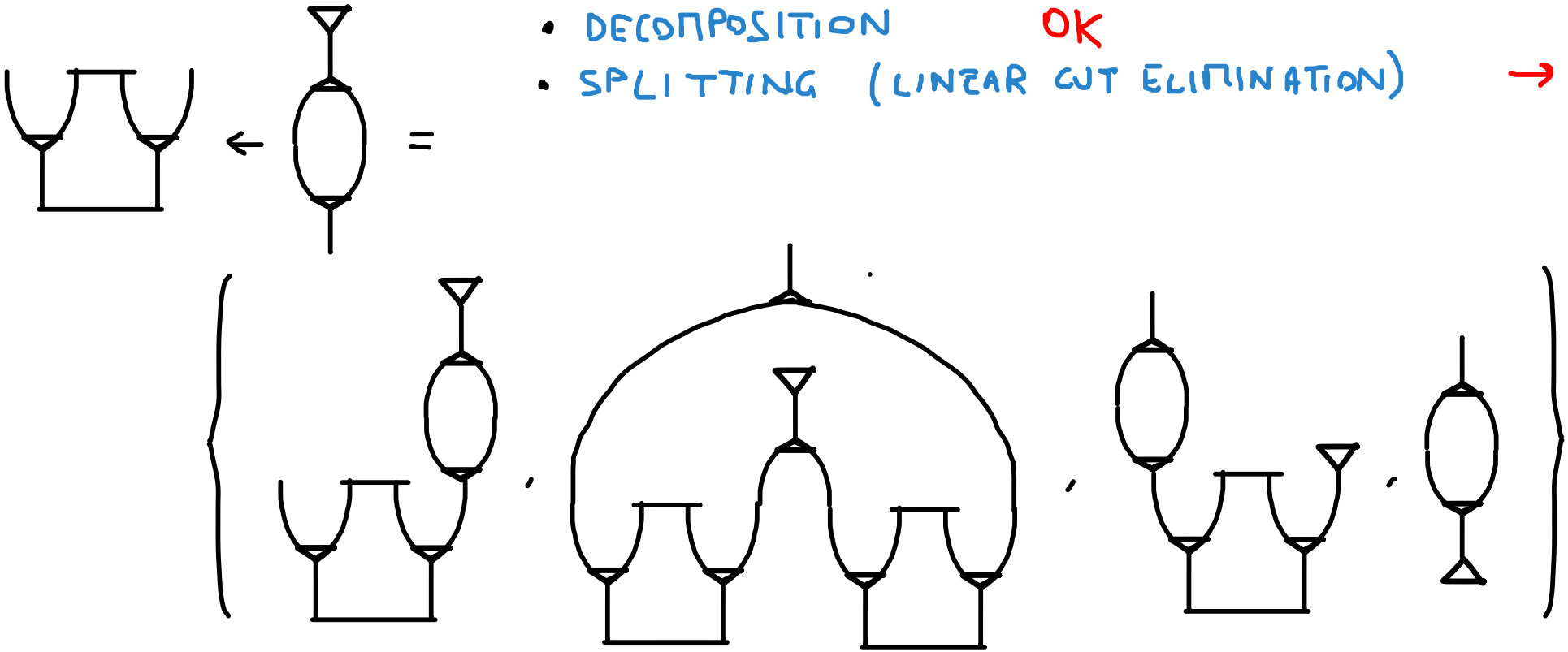
SUBSTITUTION

$$\frac{a \wedge \bar{a}}{f} \Big/ \left\{ a \leftarrow \frac{b \vee b}{b} \right\} = \left\{ \frac{\dfrac{b \vee b}{b} \wedge \bar{b}}{f} \approx \frac{(b \vee b) \wedge \dfrac{\bar{b}}{\bar{b} \wedge \bar{b}}}{f} \right\}$$

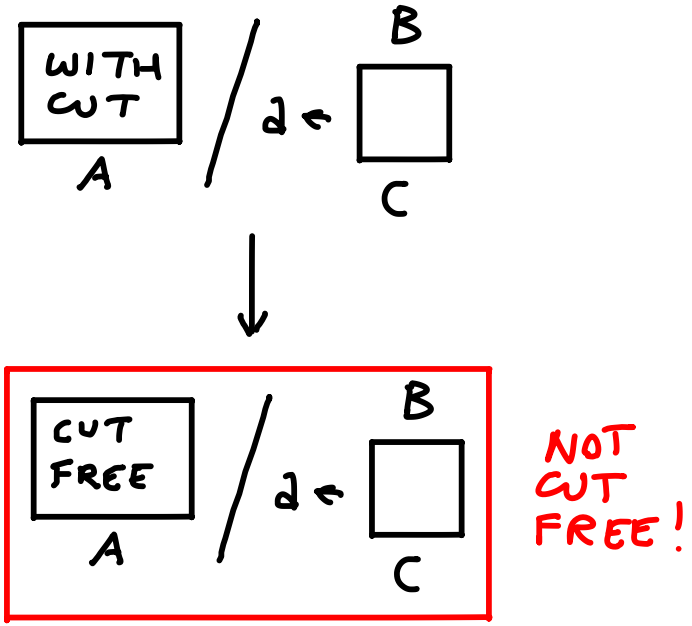
EXPONENTIAL COMPRESSION MECHANISM

NEEDS TO BE INDEPENDENT OF

- DECOMPOSITION OK
- SPLITTING (LINEAR CUT ELIMINATION) →

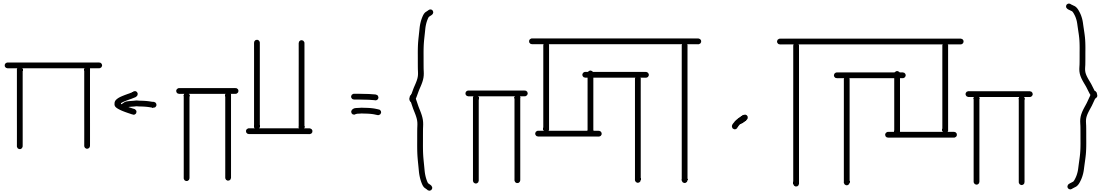


INDEPENDENCE OF CUT ELIMINATION AND SUBSTITUTION I



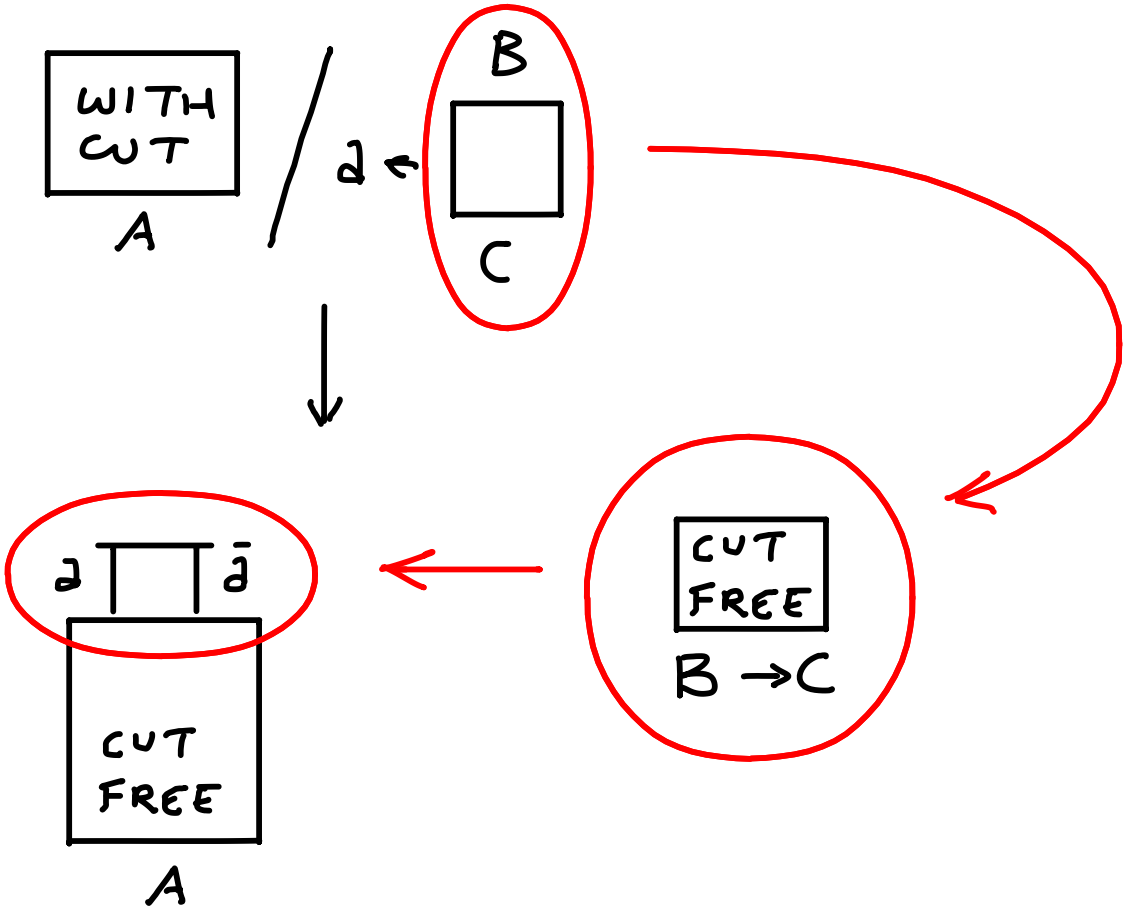
NOT CUT FREE!

BECAUSE



THIS DOES NOT WORK.

INDEPENDENCE OF CUT ELIMINATION AND SUBSTITUTION 2



THIS WORKS.

SUMMARY

- GENTZEN'S PROOF THEORY SUFFERS FROM A MASSIVE ARTEFACT: A WEAK TREE STRUCTURE.

- CUT ELIMINATION IN GENTZEN IS THE CONFLATION OF TWO MECHANISMS: SPLITTING AND DECOMPOSITION.

- THERE ARE SEVERAL, INDEPENDENT NORMALISATION MECHANISMS.

References

Nonelementary speed-up [1]; atomic flows [4, 5]; deep inference [2]; splitting-decomposition [3, 6, 7]. The rest is on manuscripts, available on request.

[1] Juan P. Aguilera & Matthias Baaz (2016): *Unsound Inferences Make Proofs Shorter*.
Available at https://arxiv.org/pdf/1608.07703.pdf.

[2] Alessio Guglielmi: *Deep Inference*.
Web site at http://alessio.guglielmi.name/res/cos.

[3] Alessio Guglielmi (2007): *A System of Interaction and Structure*.
ACM Transactions on Computational Logic 8(1), pp. 1:1–64, doi:10.1145/1182613.1182614.
Available at http://cs.bath.ac.uk/ag/p/SystIntStr.pdf.

[4] Alessio Guglielmi & Tom Gundersen (2008): *Normalisation Control in Deep Inference Via Atomic Flows*.
Logical Methods in Computer Science 4(1), pp. 9:1–36, doi:10.2168/LMCS-4(1:9)2008.
Available at http://arxiv.org/pdf/0709.1205.pdf.

[5] Alessio Guglielmi, Tom Gundersen & Lutz Straßburger (2010): *Breaking Paths in Atomic Flows for Classical Logic*.
In Jean-Pierre Jouannaud, editor: *25th Annual IEEE Symposium on Logic in Computer Science (LICS)*, IEEE, pp. 284–293, doi:10.1109/LICS.2010.12.
Available at http://www.lix.polytechnique.fr/~lutz/papers/AFII.pdf.

References (cont.)

[6] Alessio Guglielmi & Lutz Straßburger (2011): *A System of Interaction and Structure V: The Exponentials and Splitting.*
Mathematical Structures in Computer Science 21(3), pp. 563–584,
doi:10.1017/S096012951100003X.
Available at
http://www.lix.polytechnique.fr/~lutz/papers/NEL-splitting.pdf.

[7] Lutz Straßburger & Alessio Guglielmi (2011): *A System of Interaction and Structure IV: The Exponentials and Decomposition.*
ACM Transactions on Computational Logic 12(4), pp. 23:1–39, doi:10.1145/1970398.1970399.
Available at http://arxiv.org/pdf/0903.5259.pdf.