Introduction to Deep Inference

Alessio Guglielmi

University of Bath

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Deep inference web site: `http://alessio.guglielmi.name/res/cos/`

Outline

Deep inference: Free composition of proofs

Normalisation Phase 1: Reduction of cut to atomic form

Normalisation Phase 2: Splitting

Substructural example: The noncommutative linear logic BV

Perspectives

What is deep inference?

It's the free composition of proofs via the same connectives as formulae.

If

$$\Phi = \begin{array}{c} A \\ \| \\ B \end{array} \qquad \text{and} \qquad \Psi = \begin{array}{c} C \\ \| \\ D \end{array}$$

are two proofs with, respectively, premisses A and C and conclusions B and D, then

$$(\Phi \wedge \Psi) = \begin{array}{c} (A \wedge C) \\ \| \\ (B \wedge D) \end{array} \qquad \text{and} \qquad [\Phi \vee \Psi] = \begin{array}{c} [A \vee C] \\ \| \\ [B \vee D] \end{array}$$

are valid proofs with, respectively, premisses $(A \wedge C)$ and $[A \vee C]$, and conclusions $(B \wedge D)$ and $[B \vee D]$.

Why deep inference?

- To recover a De Morgan premiss-conclusion symmetry that is lost in Gentzen [2].
- To obtain new notions of normalisation in addition to cut elimination [11, 10].
- To shorten analytic proofs by exponential factors compared to Gentzen [6, 8].
- To obtain quasipolynomial-time normalisation for propositional logic [7].
- To express logics that cannot be expressed in Gentzen [22, 3].
- To make the proof theory of a vast range of logics regular and modular [3].
- To get proof systems whose inference rules are local, which is usually impossible in Gentzen [19].
- To inspire a new generation of proof nets and semantics of proofs [20].

Why deep inference? (cont.)

- To investigate the nature of cut elimination [10, 12].
- To type optimal versions of the λ-calculus that are not typeable in Gentzen [13, 14].
- To model process algebras [5, 16, 17, 18].
- To model quantum causal evolution [1] …
- … and much more.

Several formalisms can be designed in deep inference: Calculus of Structures (CoS), Nested Sequents, Open Deduction, Formalism B, …

CoS and open deduction are equivalent under any reasonable point of view, so we adopt open deduction. (CoS is convenient for certain technical aspects.)

Nested sequents is not full deep inference.

Formalism B is still in development.

Open deduction system SKS

- Atomic/structural rules:

$$\mathsf{i}\!\downarrow \frac{\mathsf{t}}{a \vee \bar{a}} \quad \mathsf{w}\!\downarrow \frac{\mathsf{f}}{a} \quad \mathsf{c}\!\downarrow \frac{a \vee a}{a}$$

identity *weakening* *contraction*

$$\mathsf{i}\!\uparrow \frac{a \wedge \bar{a}}{\mathsf{f}} \quad \mathsf{w}\!\uparrow \frac{a}{\mathsf{t}} \quad \mathsf{c}\!\uparrow \frac{a}{a \wedge a}$$

cut *coweakening* *cocontraction*

- Linear/logical rules:

$$\mathsf{s} \frac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \quad \mathsf{m} \frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}$$

switch *medial*

- Plus an ‘=’ linear rule (associativity, commutativity, units).
- Negation on atoms only.

The cut is atomic.

SKS is complete for propositional logic. See [4].

Example

$$
\begin{array}{c}
[a \vee b] \wedge a \\
\| \\
([a \vee b] \wedge a) \wedge ([a \vee b] \wedge a)
\end{array}
=
\boxed{\mathsf{m}\,\dfrac{\boxed{\mathsf{c}\!\uparrow \dfrac{a}{a \wedge a}} \vee \boxed{\mathsf{c}\!\uparrow \dfrac{b}{b \wedge b}}}{[a \vee b] \wedge [a \vee b]}}
\wedge
\boxed{\mathsf{c}\!\uparrow \dfrac{a}{a \wedge a}}
$$

Structural rules on generic formulae can be replaced by structural rules on atoms.

Example with quantifiers

$$
\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{\exists x \forall y \left[\boxed{\mathsf{w}{\downarrow}\,\dfrac{\mathsf{f}}{\overline{p(x)}}} \vee p(y) \right] \vee \exists x \forall y \left[\overline{p(x)} \vee \boxed{\mathsf{w}{\downarrow}\,\dfrac{\mathsf{f}}{p(y)}} \right]}
$$

$$
\mathsf{c}{\downarrow}\,\dfrac{\exists x \forall y \left[\boxed{\mathsf{w}{\downarrow}\,\dfrac{\mathsf{f}}{\overline{p(x)}}} \vee p(y) \right] \vee \exists x \forall y \left[\overline{p(x)} \vee \boxed{\mathsf{w}{\downarrow}\,\dfrac{\mathsf{f}}{p(y)}} \right]}{\exists x \forall y \left[\overline{p(x)} \vee p(y) \right]}
$$

This is more natural than in Gentzen because there is no waste in the proof.

Analyticity costs much less (I)

Statman tautologies:

$$
\begin{aligned}
S_1 &= (a \wedge b) \vee \bar{a} \vee \bar{b} \quad , \\
S_2 &= (c \wedge d) \vee \left(\left[\bar{c} \vee \bar{d} \right] \wedge a \wedge \left[\bar{c} \vee \bar{d} \right] \wedge b \right) \vee \bar{a} \vee \bar{b} \quad , \\
S_3 &= (e \wedge f) \vee \left(\left[\bar{e} \vee \bar{f} \right] \wedge c \wedge \left[\bar{e} \vee \bar{f} \right] \wedge d \right) \vee \\
&\qquad \left(\left[\bar{e} \vee \bar{f} \right] \wedge \left[\bar{c} \vee \bar{d} \right] \wedge a \wedge \left[\bar{e} \vee \bar{f} \right] \wedge \left[\bar{c} \vee \bar{d} \right] \wedge b \right) \vee \bar{a} \vee \bar{b}
\end{aligned}
$$

and so on…

In the cut-free sequent calculus proofs grow exponentially.

Analyticity costs much less (2)

Open deduction proof of S_1:

$$
\begin{array}{c}
\mathsf{t} \\
\Big\Vert\, \mathsf{i}{\downarrow},\mathsf{s} \\
\boxed{(a \wedge b) \vee \bar{a} \vee \bar{b}}
\end{array}
\quad = \quad
\mathsf{s}\,\dfrac{\boxed{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{a \vee \bar{a}}} \wedge \boxed{\mathsf{i}{\downarrow}\,\dfrac{\mathsf{t}}{b \vee \bar{b}}}}{\boxed{\mathsf{s}\,\dfrac{[a \vee \bar{a}] \wedge b}{(a \wedge b) \vee \bar{a}}} \vee \bar{b}}
$$

Analyticity costs much less (3)

Open deduction proof of S_2:

$$
\begin{array}{c}
\mathsf{t} \\
\Big\Vert\, \mathsf{i}{\downarrow},\mathsf{s} \\
\boxed{
\boxed{
\begin{array}{c}
\boxed{
\boxed{\begin{array}{c} \mathsf{t} \\ \Vert\, \mathsf{i}{\downarrow},\mathsf{s} \\ (c \wedge d) \vee \bar{c} \vee \bar{d} \end{array}}
\wedge a \wedge
\boxed{\begin{array}{c} \mathsf{t} \\ \Vert\, \mathsf{i}{\downarrow},\mathsf{s} \\ (c \wedge d) \vee \bar{c} \vee \bar{d} \end{array}}
\wedge b
} \\
\Big\Vert\, \mathsf{s} \\
\boxed{
\boxed{\begin{array}{c} (c \wedge d) \vee (c \wedge d) \\ \Vert\, \mathsf{c}{\downarrow},\mathsf{m} \\ \boxed{c \wedge d} \end{array}}
\vee \Big(\big[\bar{c} \vee \bar{d}\big] \wedge a \wedge \big[\bar{c} \vee \bar{d}\big] \wedge b \Big)
}
\end{array}
}
\vee \bar{a} \vee \bar{b}
}
\end{array}
$$

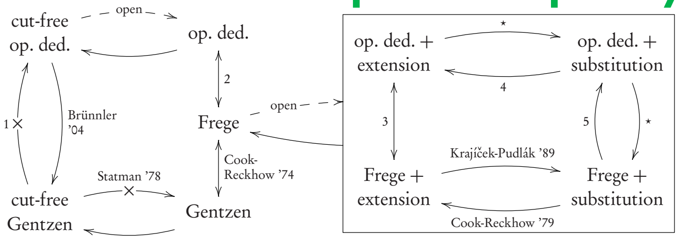
Analyticity costs much less (4)

Open deduction proof of S_3:

$$
\begin{array}{c}
\mathsf{t} \\
\Big\| {\scriptstyle \mathsf{i}\downarrow,\mathsf{s}} \\
\boxed{
\boxed{
\begin{array}{c}
\boxed{
\boxed{\begin{array}{c}\mathsf{t}\\ \Big\| {\scriptstyle \mathsf{i}\downarrow,\mathsf{s}} \\ (c\wedge d)\vee\bar c\vee\bar d\end{array}}
\wedge a \wedge
\boxed{\begin{array}{c}\mathsf{t}\\ \Big\| {\scriptstyle \mathsf{i}\downarrow,\mathsf{s}} \\ (c\wedge d)\vee\bar c\vee\bar d\end{array}}
\wedge b
} \\
\Big\| {\scriptstyle \mathsf{s}} \\
\boxed{
\boxed{
\begin{array}{c}
(c\wedge d)\vee(c\wedge d) \\
\Big\| {\scriptstyle \mathsf{c}\downarrow,\mathsf{m}} \\
\boxed{
\boxed{\begin{array}{c}\mathsf{t}\\ \Big\| {\scriptstyle \mathsf{i}\downarrow,\mathsf{s}} \\ (e\wedge f)\vee\bar e\vee\bar f\end{array}}
\wedge c \wedge
\boxed{\begin{array}{c}\mathsf{t}\\ \Big\| {\scriptstyle \mathsf{i}\downarrow,\mathsf{s}} \\ (e\wedge f)\vee\bar e\vee\bar f\end{array}}
\wedge d
}
\end{array}
}
\vee
\left(
\boxed{\begin{array}{c}\mathsf{t}\\ \Big\| {\scriptstyle \mathsf{i}\downarrow,\mathsf{s}} \\ (e\wedge f)\vee\bar e\vee\bar f\end{array}}
\wedge [\bar c\vee\bar d] \wedge a \wedge
\boxed{\begin{array}{c}\mathsf{t}\\ \Big\| {\scriptstyle \mathsf{i}\downarrow,\mathsf{s}} \\ (e\wedge f)\vee\bar e\vee\bar f\end{array}}
\wedge [\bar c\vee\bar d] \wedge b
\right)
} \\
\Big\| {\scriptstyle \mathsf{s}} \\
\boxed{
\boxed{
\begin{array}{c}
(e\wedge f)\vee(e\wedge f)\vee(e\wedge f)\vee(e\wedge f) \\
\Big\| {\scriptstyle \mathsf{c}\downarrow,\mathsf{m}} \\
e\wedge f
\end{array}
}
\vee \left([\bar e\vee\bar f]\wedge c\wedge[\bar e\vee\bar f]\wedge d\right) \vee \left([\bar e\vee\bar f]\wedge[\bar c\vee\bar d]\wedge a\wedge[\bar e\vee\bar f]\wedge[\bar c\vee\bar d]\wedge b\right)
}
\end{array}
}
\vee \bar a \vee \bar b
}
\end{array}
$$

In open deduction analytic Statman proofs grow polynomially.

Open deduction and proof complexity (size)



$\longrightarrow$ = 'polynomially simulates'.

Open deduction:

- in the cut-free case, thanks to deep inference, has an exponential speed-up over the cut-free sequent calculus (e.g., over Statman tautologies)—1, see [6];
- has as small proofs as the best formalisms—2, 3, 4, 5, see [6];
- thanks to dagness, has quasipolynomial cut elimination (instead of exponential) [7, 15].
- *CUT FREE DEEP INFERENCE OUTPERFORMS THE CUT FREE SEQUENT CALCULUS.*

Open deduction and proof search complexity

Unconstrained bottom-up formula-driven proof search has horrendous complexity due to deep inference, because every connective can make the search tree branch.

However:

1. Das proved that in the presence of distributivity, a depth 2 proof system polynomially simulates any unbounded depth proof system [8]. This means that a very moderate increase of nondeterminism buys exponentially smaller proofs.
2. Focusing techniques should be facilitated by the more liberal proof composition.
3. In particular it should be possible to confine the search inside small sub-spaces of canonical proofs.
4. *THE SEQUENT CALCULUS WAS DESIGNED TO MAKE PROOF SEARCH FINITE, NOT NECESSARILY TO MAKE IT EFFICIENT.*

Normalisation Phase 1: Reduction of cut to atomic form

Apply repeatedly—and locally:

$$\mathsf{i}{\uparrow}\,\frac{[A \vee B] \wedge \bar{A} \wedge \bar{B}}{\mathsf{f}} \quad = \quad \mathsf{s}\,\frac{\boxed{\mathsf{s}\,\dfrac{[A \vee B] \wedge \bar{B}}{A \vee (B \wedge \bar{B})}} \wedge \bar{A}}{\boxed{\mathsf{i}{\uparrow}\,\dfrac{A \wedge \bar{A}}{\mathsf{f}}} \vee \boxed{\mathsf{i}{\uparrow}\,\dfrac{B \wedge \bar{B}}{\mathsf{f}}}}$$

Proof complexity does not increase!

Normalisation Phase 2: Splitting

Theorem (Splitting) *For every proof* $\begin{array}{c} \mathbf{t} \\ \| \\ K\{A \wedge B\} \end{array}$

there are proofs

$$\begin{array}{c} K_A \vee K_B \vee \{\ \} \\ \| \\ K\{\ \} \end{array} \qquad \begin{array}{c} \mathbf{t} \\ \| \\ K_A \vee A \end{array} \qquad \begin{array}{c} \mathbf{t} \\ \| \\ K_B \vee B \end{array}$$

An alike theorem holds for every logic expressed in deep inference so far (including logics that for Gentzen theory are hopeless).

Splitting for an atomic cut

Therefore for every cut-free proof
$$\begin{array}{c}\mathsf{t}\\ \Vert\\ K\{a \wedge \bar{a}\}\end{array}$$

there are cut-free proofs

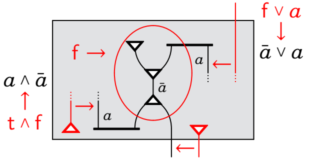
$$\begin{array}{c}K'\{\bar{a}\} \vee K''\{a\} \vee \{\ \}\\ \Vert\\ K\{\ \}\end{array} \qquad \begin{array}{c}\mathsf{t}\\ \Vert\\ K'\{\bar{a}\} \vee a\end{array} \qquad \begin{array}{c}\mathsf{t}\\ \Vert\\ K''\{a\} \vee \bar{a}\end{array}$$

and so we can build
$$\begin{array}{c} \mathrm{i}{\downarrow}\dfrac{\mathsf{t}}{\boxed{\boxed{\begin{array}{c}a\\ \Vert\\ K'\{\bar{a}\}\end{array}} \vee \boxed{\begin{array}{c}\bar{a}\\ \Vert\\ K''\{a\}\end{array}}}}\\ \Vert\\ K\{\mathsf{f}\}\end{array}$$

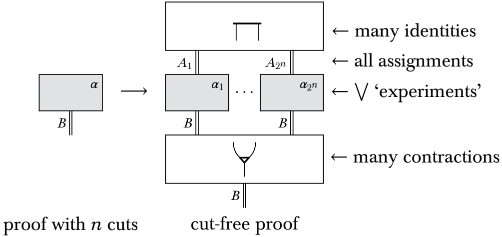
and a cut at the bottom would be admissible.

Cut elimination by 'experiments' (for logics with contraction)

Experiment over a proof:



We do:



- Simple, exponential cut elimination;
- 2^n experiments, where n is the number of atoms;
- fairly syntax independent method.

The secret of success is in the proof composition mechanism.

WHY IS THIS IMPOSSIBLE IN THE SEQUENT CALCULUS?

System BV

BV = MLL + self-dual noncommutative operator [9, 22]:

- Equations:

$$\overline{A \otimes B} = \bar{A} \parr \bar{B} \qquad \overline{A \parr B} = \bar{A} \otimes \bar{B} \qquad \overline{A \triangleleft B} = \bar{A} \triangleleft \bar{B}$$

$$A \otimes (B \otimes C) = (A \otimes B) \otimes C$$
$$A \triangleleft \langle B \triangleleft C \rangle = \langle A \triangleleft B \rangle \triangleleft C$$
$$A \parr [B \parr C] = [A \parr B] \parr C$$

$$A \otimes B = B \otimes A \qquad A \parr B = B \parr A$$

$$A \otimes \circ = A \triangleleft \circ = \circ \triangleleft A = A \parr \circ = A$$

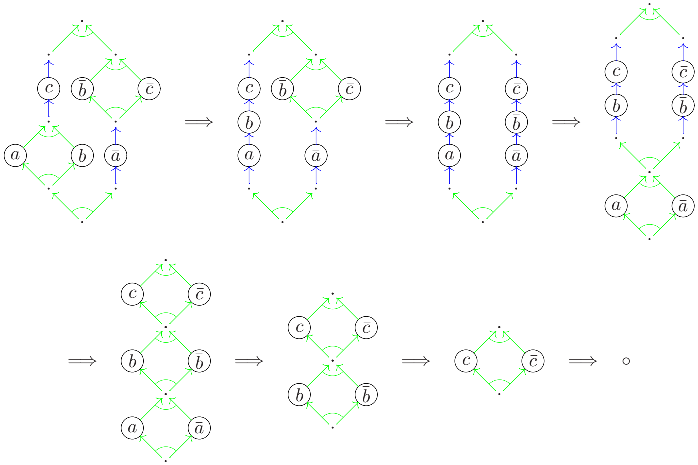
- Rules:

$$\mathsf{i}\!\uparrow \frac{a \otimes \bar{a}}{\circ} \qquad \mathsf{q}\!\uparrow \frac{\langle A \triangleleft B \rangle \otimes \langle C \triangleleft D \rangle}{(A \otimes C) \triangleleft (B \otimes D)}$$

$$\mathsf{i}\!\downarrow \frac{\circ}{a \parr \bar{a}} \qquad \mathsf{s} \frac{A \otimes [B \parr C]}{(A \otimes B) \parr C} \qquad \mathsf{q}\!\downarrow \frac{[A \parr C] \triangleleft [B \parr D]}{\langle A \triangleleft B \rangle \parr \langle C \triangleleft D \rangle}$$

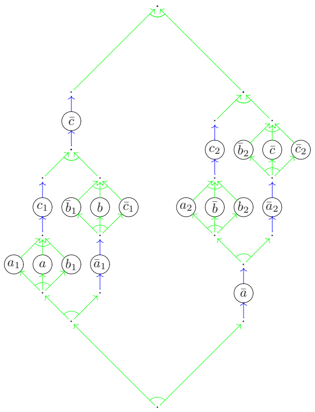
Tiu's counterexample:
BV is not expressible in Gentzen

Graphical representation of a proof in BV:



Tiu’s counterexample: BV is not expressible in Gentzen (cont.)

We can build a growing fractal of growing depth; the next step is:



…and each of its cut-free proofs has to start deeper inside.

THEREFORE BV CANNOT BE CAPTURED BY SHALLOW INFERENCE!

Splitting for BV

Theorem (Splitting) *For every proof* $\begin{array}{c}\circ\\ \parallel\\ K\{A \otimes B\}\end{array}$ *there are proofs*

$$\begin{array}{c}K_A \parr K_B \parr \{\ \}\\ \parallel\\ K\{\ \}\end{array} \qquad \begin{array}{c}\circ\\ \parallel\\ K_A \parr A\end{array} \qquad \begin{array}{c}\circ\\ \parallel\\ K_B \parr B\end{array}$$

and for every proof $\begin{array}{c}\circ\\ \parallel\\ K\{A \triangleleft B\}\end{array}$ *there are proofs*

$$\begin{array}{c}\langle K_A \triangleleft K_B\rangle \parr \{\ \}\\ \parallel\\ K\{\ \}\end{array} \qquad \begin{array}{c}\circ\\ \parallel\\ K_A \parr A\end{array} \qquad \begin{array}{c}\circ\\ \parallel\\ K_B \parr B\end{array}$$

Splitting recovers Gentzen's notion of analyticity without imposing it on the meta-level of the formalism.

(Personal) perspectives

- Formalism B: proof theory and proof complexity together in a formalism which is by design as powerful as Frege + substitution.
- Reasonable solution to the proof identity problem [21].
- Logical interpretation of expressive process algebras.
- Typing and compiling optimal functional computations [13].

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