Geometric Normalisation with Atomic Flows

Alessio Guglielmi

University of Bath and INRIA Nancy-Grand Est

Joint work with
Paola Bruscoli, Tom Gundersen, Michel Parigot and Lutz Straßburger

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This talk is available at `http://cs.bath.ac.uk/ag/t/GNAF.pdf`

Outline

(Proof) System SKS [Brünnler & Tiu(2001)]

- Atomic rules:

$$\mathsf{ai}{\downarrow}\,\frac{\mathsf{t}}{a \vee \bar{a}} \qquad \mathsf{aw}{\downarrow}\,\frac{\mathsf{f}}{a} \qquad \mathsf{ac}{\downarrow}\,\frac{a \vee a}{a}$$

identity *weakening* *contraction*

$$\mathsf{ai}{\uparrow}\,\frac{a \wedge \bar{a}}{\mathsf{f}} \qquad \mathsf{aw}{\uparrow}\,\frac{a}{\mathsf{t}} \qquad \mathsf{ac}{\uparrow}\,\frac{a}{a \wedge a}$$

cut *coweakening* *cocontraction*

- Linear rules:

$$\mathsf{s}\,\frac{A \wedge [B \vee C]}{(A \wedge B) \vee C} \qquad \mathsf{m}\,\frac{(A \wedge B) \vee (C \wedge D)}{[A \vee C] \wedge [B \vee D]}$$

switch *medial*

- Plus an '$=$' linear rule (associativity, commutativity, units).
- Rules are applied anywhere inside formulae.
- Negation on atoms only.
- Cut is atomic.
- SKS is complete and implicationally complete for propositional logic.

Examples in Open Deduction [Guglielmi et al.(2010)Guglielmi, Gundersen, & Parigot]

- $$\mathsf{m}\,\dfrac{\dfrac{a}{a \wedge a} \vee \dfrac{b}{b \wedge b}}{[a \vee b] \wedge [a \vee b]} \wedge \dfrac{a}{a \wedge a}$$

- $$\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{\dfrac{\mathsf{t}}{a \vee \bar{a}}}{[a \vee \mathsf{t}] \wedge [\mathsf{t} \vee \bar{a}]}}{\left[\mathsf{s}\,\dfrac{[a \vee \mathsf{t}] \wedge \bar{a}}{\dfrac{a \wedge \bar{a}}{\mathsf{f}} \vee \mathsf{t}} \vee \mathsf{t}\right]}$$

Proofs are composed by the same operators as formulae.

Top-down symmetry: so inference steps can be made atomic (the medial rule, m, is impossible in the sequent calculus).

Locality

- Deep inference allows locality,
- *i.e.*, inference steps can be checked in constant time (so, inference steps are small).

Example, atomic cocontraction:

$$\mathsf{m}\frac{\left(\frac{a}{a \wedge a} \vee \frac{b}{b \wedge b}\right)}{[a \vee b] \wedge [a \vee b]} \quad \wedge \quad \frac{a}{a \wedge a}$$

Note: the sequent calculus

- does not allow locality in contraction (counterexample in [Brünnler(2004)]), and
- does not allow local reduction of cut into atomic form.

Goal of This Talk

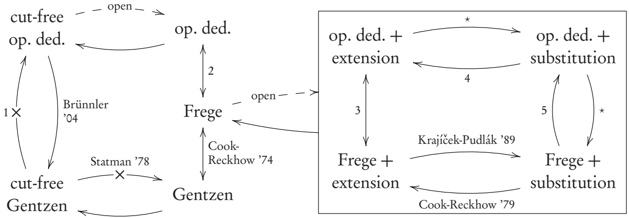
Slogans:

- Deep inference = locality (+ symmetry).
- Locality = linearity + atomicity.
- Geometry = syntax independence (elimination of bureaucracy).
- Locality $\to$ geometry $\to$ semantics of proofs (Lamarche *dixit*).

To show that:

- We can normalise in a largely syntax-independent way.
- Normalisation is a very robust phenomenon.
- We can start thinking about characterising cut-free formalisms in a robust way (in the sense of Cook and Reckhow).

Big Picture on Proof Complexity

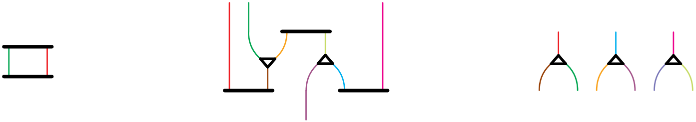


$\longrightarrow =$ 'polynomially simulates'.

Open deduction has as small proofs as the best formalisms
and
it has a normalisation theory
and
its cut-free proof systems are more powerful than Gentzen ones
and
cut elimination is quasipolynomial (instead of exponential).
(See [Jeřábek(2009), Bruscoli & Guglielmi(2009),
Bruscoli et al.(2010)Bruscoli, Guglielmi, Gundersen, & Parigot]).

(Atomic) Flows

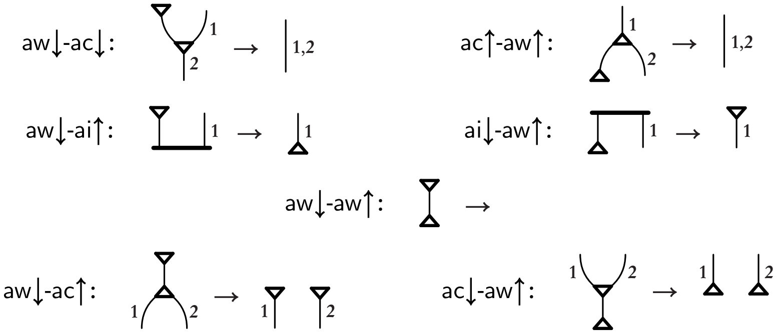
$$\mathsf{s}\,\dfrac{\mathsf{m}\,\dfrac{\dfrac{\mathsf{t}}{a\vee\bar a}}{[a\vee \mathsf{t}]\wedge[\mathsf{t}\vee\bar a]}}{\left[\mathsf{s}\,\dfrac{[a\vee\mathsf{t}]\wedge\bar a}{\dfrac{a\wedge\bar a}{\mathsf{f}}\vee\mathsf{t}}\quad\vee\quad\mathsf{t}\right]}
= \dfrac{\left(\mathsf{s}\,\dfrac{a\wedge\left[\bar a\vee\dfrac{\mathsf{t}}{\bar a\vee a}\right]}{\dfrac{a\wedge\dfrac{\bar a\vee\bar a}{\bar a}}{\mathsf{f}}\quad\vee\quad\dfrac{a}{a\wedge a}}\quad\wedge\quad\bar a\right)}{a\wedge\dfrac{a\wedge\bar a}{\mathsf{f}}}
\qquad
\mathsf{m}\,\dfrac{\dfrac{a}{a\wedge a}\vee\dfrac{b}{b\wedge b}}{[a\vee b]\wedge[a\vee b]}\quad\wedge\quad\dfrac{a}{a\wedge a}$$



- Below derivations, their (atomic) flows are shown.
- Only structural information is retained in flows.
- Logical information is lost.
- Flow size is polynomially related to derivation size.

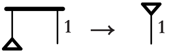
Flow Reductions: (Co)Weakening (1)

Consider these flow reductions:



Each of them corresponds to a correct derivation reduction.

Flow Reductions: (Co)Weakening (2)

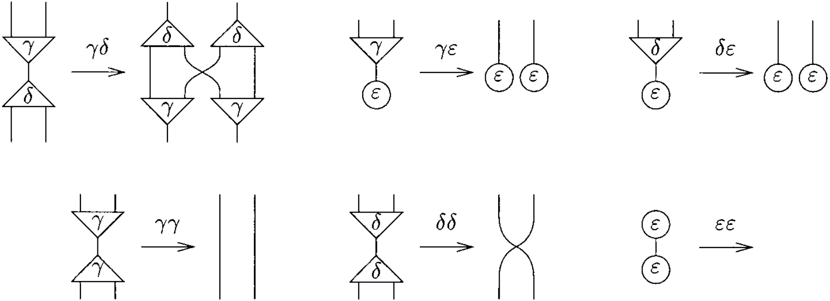
For example, ai↓-aw↑:  specifies that

$$\begin{array}{c} \Pi'' \Big\Vert \\ \xi\left\{\dfrac{\mathsf{t}}{a^{\epsilon} \vee \bar{a}}\right\} \\ \Phi \Big\Vert \\ \zeta\left\{\dfrac{a^{\epsilon}}{\mathsf{t}}\right\} \\ \Psi \Big\Vert \\ \alpha \end{array} \qquad \text{becomes} \qquad \begin{array}{c} \Pi'' \Big\Vert \\ \xi\left[\mathsf{t} \vee \dfrac{\mathsf{f}}{\bar{a}}\right] \\ \Phi\{a^{\epsilon}/\mathsf{t}\} \Big\Vert \\ \zeta\{\mathsf{t}\} \\ \Psi \Big\Vert \\ \alpha \end{array}$$

We can operate on flow reductions instead than on derivations: it is much easier and we get natural, syntax-independent induction measures.

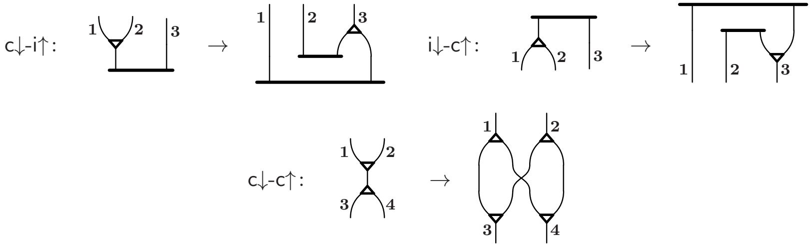
Relation With Interaction Combinators?

Lots of coincidences, but also differences: no apparent logical meaning for two 'contractions':



Flow Reductions: (Co)Contraction

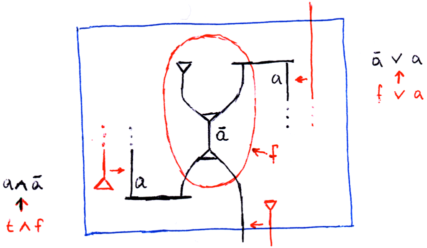
Consider these flow reductions:



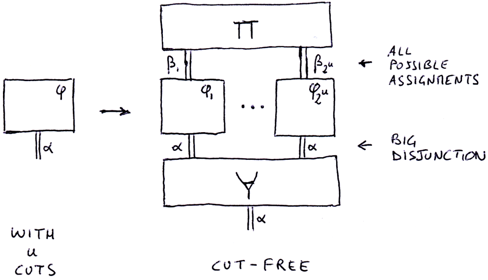
- They conserve the number and length of paths.
- Note that they can blow up a derivation exponentially.
- It's a good thing: cocontraction is a new compression mechanism (sharing?).
- Open problem: does cocontraction provide exponential compression? Conjecture: yes.

Cut Elimination by 'Experiments'

Experiment:



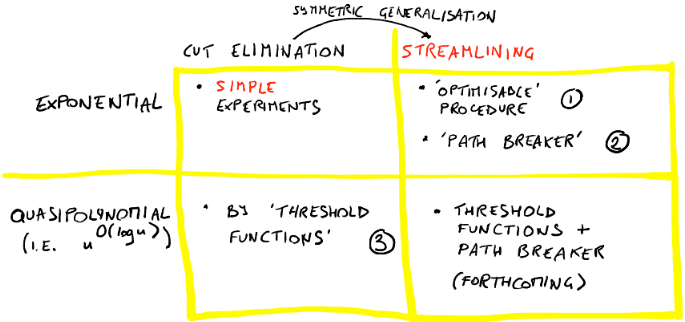
We do:



Simple, exponential cut elimination; proof generates 2^n experiments.

Normalisation

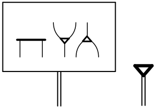
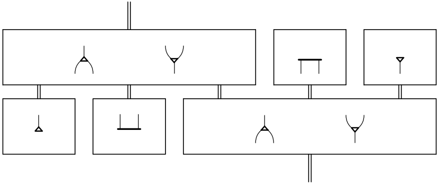
Overview



- None of these methods existed before atomic flows, none of them requires permutations or other syntactic devices.
- Quasipolynomial procedures are surprising.

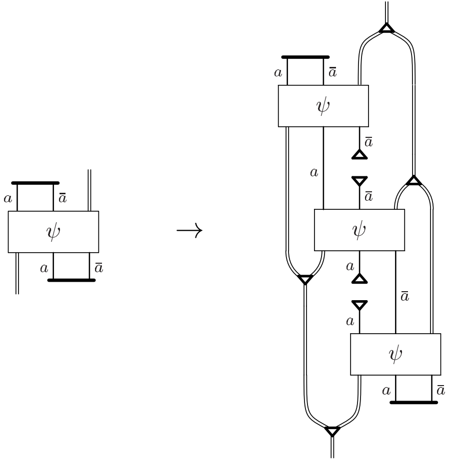
(1) [Guglielmi & Gundersen(2008)]; (2) LICS 2010 submission; (3) [Bruscoli et al.(2010)Bruscoli, Guglielmi, Gundersen, & Parigot].

Generalising the Cut-Free Form

- Normalised proof: 
- Normalised derivation: 
- The symmetric form is called streamlined.
- Cut elimination is a corollary of streamlining.
- We just need to break the paths between identities and cuts, and (co)weakenings do the rest.

How Do We Break Paths?

With the path breaker:



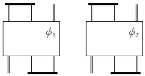
Even if there is a path between identity and cut on the left, there is none on the right.

We Can Do This on Derivations, of Course

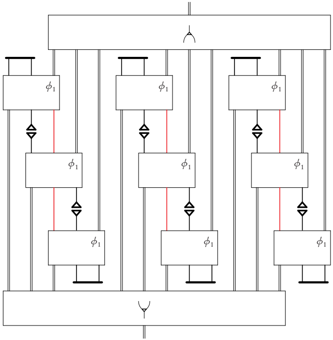
$$
\begin{array}{c}
A \\
\hline
[a \vee \bar{a}] \wedge A \\
\Psi \Big\| \\
B \vee (a \wedge \bar{a}) \\
\hline
B
\end{array}
\quad \longrightarrow \quad
\begin{array}{c}
A \\
\Big\| \{\mathsf{c}\uparrow, \mathsf{ai}\downarrow, =\} \\
(([a \vee \bar{a}] \wedge A) \wedge A) \wedge A \\
(\Psi \wedge A) \wedge A \Big\| \\
([B \vee (a \wedge \bar{a})] \wedge A) \wedge A \\
\Phi_a \wedge A \Big\| \\
[B \vee ([a \vee \bar{a}] \wedge A)] \wedge A \\
[B \vee \Psi] \wedge A \Big\| \\
B \vee ([B \vee (a \wedge \bar{a})] \wedge A) \\
B \vee \Phi_{\bar{a}} \Big\| \\
B \vee [B \vee ([a \vee \bar{a}] \wedge A)] \\
B \vee [B \vee \Psi] \Big\| \\
B \vee [B \vee [B \vee (a \wedge \bar{a})]] \\
\Big\| \{\mathsf{c}\downarrow, \mathsf{ai}\uparrow, =\} \\
B
\end{array}
$$

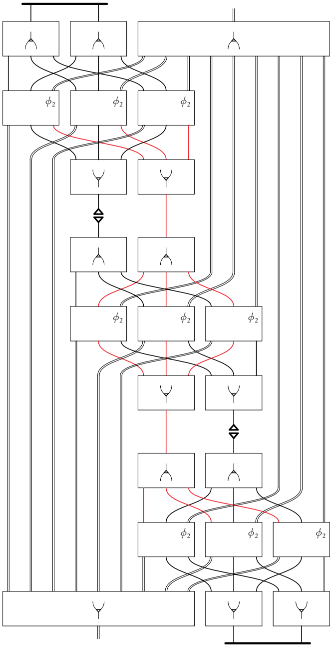
- We can compose this as many times as there are paths between identities and cut.
- We obtain a family of normalisers that only depends on n.
- The construction is exponential.
- Note: finding something like this is *unthinkable* without flows.

Example for $n = 2$

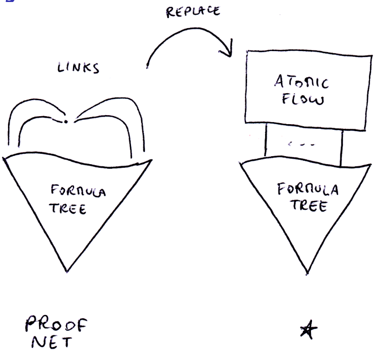


$\rightarrow$





Conjecture



- We think that (*) might make for a proof system (see also recent work by Straßburger).
- This means that there should exist a polynomial algorithm to check the correctness of (*).
- If this is true, we have an excellent bureaucracy-free formalism.
- Note: if such a thing existed for proof nets, then coNP $=$ NP.

Conclusion

- Cut elimination does not depend on logical rules.
- It only depends on structural information, *i.e.*, geometry.
- Normalisation is extremely robust.
- Deep inference's locality is key.
- Complexity-wise, deep inference is as powerful as the best formalisms,
- and more powerful if analyticity is requested.

This talk is available at `http://cs.bath.ac.uk/ag/t/GNAF.pdf`

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