Subatomic Proof Systems

Andrea Aler Tubella
IRIF, CNRS and Univ. Paris Diderot

Alessio Guglielmi
University of Bath

The proof theoretic methodology of deep inference [5] yields the widest range of analytic proof systems. In particular, several logics for which there are no analytic proof systems in Gentzen, or for which there only are cumbersome ones, admit elegant and regular analytic proof systems in deep inference. The regularity of inference rule schemes in deep inference stems from their ability to access the atoms which compose formulae.

The main idea behind this talk is based on a surprising observation: if we allow inference rules to see even deeper, inside atoms, then we are able to reduce disparate rules such as contraction, cut, identity and any logical rule like conjunction-introduction, into a unique rule scheme.

The intuition behind this new approach is to consider atoms as logical relations, and to build formulae by freely composing constants by connectives and atoms. For example, $A \equiv (\mathsf{f} \mathbf{a} \mathsf{t}) \vee \mathsf{t}$ is a subatomic formula for classical logic. The main idea is to interpret $(\mathsf{f}\ \mathbf{a}\ \mathsf{t})$ as a positive occurrence of the atom a, and $(\mathsf{t}\,\mathbf{a}\,\mathsf{f})$ as a negative occurrence of the same atom, denoted by $\bar{a}$. Intuitively, we can view subatomic formulae as a superposition of truth values. For example, $(\mathsf{f}\ \mathbf{a}\ \mathsf{t})$ is the superposition of the two possible assignments for the atom a, and $(\mathsf{t}\,\mathbf{a}\,\mathsf{f})$ is the superposition of the possible assignments for $\bar{a}$: if we read the value on the left of the atom we assign f to a and t to $\bar{a}$, and viceversa if we read the one on the right. By developping this methodology, that we call *subatomic*, we are able to achieve complete regularity on the shape of inference rules for a wide range of logical systems.

Indeed, we are able to represent a great width of proof systems in such a way that every rule, including rules such as atomic introduction or contraction, is an instance of the rule scheme

$$\frac{A\ \alpha\ B\ \nu\ C\ \beta\ D}{A\ \nu\ C\ \alpha\ B\ \gamma\ D} \quad ,$$

where $\alpha, \nu, \beta, \gamma$ are relations, and A, B, C, D are formulae. The fact that so many rules can be presented in this form remains quite intriguing, although it has proven certainly useful.

This unprecedented regularity can be exploited to reason generally about proof systems. In particular, we can apply it to study the interactions between rules and their role in normalisation procedures. We are able to generalise two complementary normalisation procedures: *splitting* and *decomposition*.

Splitting is a generalisation of a common technique employed for cut-elimination in deep inference systems [3], [6], [7], [9]. The idea behind it is rooted in deep inference methods. In the sequent calculus, formulae have a root connective that allows us to determine which rules are applied immediately above the cut. In deep inference, rules can be applied anywhere deep in a formula and as such anything can happen above a cut. As a consequence, the splitting method focuses on understanding the behaviour of the context around the cut. We show that cut-elimination via splitting can be achieved for a generalisation of linear systems [2].

The reach of the splitting technique goes beyond linear systems when combined with decomposition. In many systems, derivations can be arranged into consecutive subderivations made up of only certain rules [4], [8], [9]. By having a single inference rule shape to consider, we are able to provide generalised reduction rules to manipulate proofs through local transformations to obtain their decomposition into a linear phase followed by a phase made up only of contractions.

In this way, splitting deals with the interactions between cuts and linear rules, whereas decomposition deals with the interactions between cuts and contractions. These two phenomena, tangled in traditional Gentzen-style cut-elimination procedures through the use of a mix rule conflating cuts and contractions, turn out to be quite different complexity-wise. Splitting is a procedure of polynomial-time complexity where we need to look at a whole proof in order to eliminate the cuts, whereas decomposition has an exponential cost and can be achieved through local rewritings. By untangling these interactions and separating cut-elimination into these two procedures we can therefore gain a better control on the complexity, as well as a better understanding of the reasons behind the prevalence of cut-elimination in such a width of proof systems.

References

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