A Subatomic Proof System

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In this work we show a proof system, called SA, that generates propositional proofs by employing a single, linear, simple and regular inference rule scheme. The main idea is to consider atoms as self-dual, noncommutative binary logical relations and build formulae by freely composing units by atoms, disjunction and conjunction. If we restrict proofs to formulae where no atom occurs in the scope of another atom, we fully and faithfully recover a deduction system for propositional logic in the usual sense, where traditional proof analysis and transformation techniques such as cut elimination can be studied. In this extended abstract we do not present complete technical details, but the reader can rather easily reconstruct them from the indications provided.

One can quickly grasp the main idea by considering the occurrences of an atom a as interpretations of more primitive expressions involving a noncommutative binary relation, still denoted by a. We use the Polish notation for atoms, so that two formulae A and B in the relation a, in this order, are denoted by aAB. In SA we have an enumerable supply of atoms, denoted by lowercase Latin letters, and formulae are built over the two units for disjunction and conjunction, respectively 0 and 1. For example, the following two expressions are SA formulae:

$$a01 \vee a10 \qquad \text{and} \qquad a(b01)(c1(d10)) \wedge 0 \wedge (a00 \vee b11) \quad .$$

We call *tame* the formulae where atoms do not appear in the scope of other atoms, such as the formula at the left, and *wild* the others, such as the formula at the right.

At this point we need an interpretation $\mapsto$, which we take as a map from tame SA formulae to ordinary formulae such that

$$a01 \mapsto a \qquad \text{and} \qquad a10 \mapsto \bar{a} \quad ,$$

where $\bar{a}$ denotes the negation of a. We then stipulate that

$$a00 \mapsto 0 \qquad \text{and} \qquad a11 \mapsto 1 \quad .$$

Note that self-duality, *i.e.*, $\overline{aAB} \equiv a\bar{A}\bar{B}$, and noncommutativity, *i.e.*, $aAB \not\equiv aBA$ whenever $A \not\equiv B$, are coherent with the interpretation. We extend the interpretation $\mapsto$ to all the tame SA formulae in the natural way. We then obtain, for example,

$$a01 \vee a10 \mapsto a \vee \bar{a} \qquad \text{and} \qquad a\,(0 \vee 0)\,(1 \vee 1) \mapsto a \quad ,$$

where we assumed that the classical logic equivalences $0 \vee 0 \equiv 0$ and $1 \vee 1 \equiv 1$ are incorporated into the definition of $\mapsto$.

In order to understand how the idea applies to inference rules, let us consider now the usual contraction rule for an atom:

$$\frac{a \vee a}{a} \quad .$$

We can obtain this rule as the result of applying $\mapsto$ to the formulae of some proof system where the following inference rule instances are expressed:

$$\frac{a01 \vee a01}{a(0 \vee 0)(1 \vee 1)} \mapsto \frac{a \vee a}{a} \qquad \text{and} \qquad \frac{a10 \vee a10}{a(1 \vee 1)(0 \vee 0)} \mapsto \frac{\bar{a} \vee \bar{a}}{\bar{a}} \quad .$$

As we can see, the rule instances of whatever SA system we are using could be special cases of the linear rule

$$\frac{aAC \vee aBD}{a(A \vee B)(C \vee D)} \quad .$$

We find this interesting partly because it seems that the nonlinearity of the contraction rule has been pushed from the atoms to the units and partly because the shape of the new rule is typical of logical rules (as opposed to structural ones).

Obviously what we have just seen would be trivial unless it worked for all rules. Let us see then two more examples: the identity and cut rules in atomic form:

$$\frac{a(0 \vee 1)(1 \vee 0)}{a01 \vee a10} \mapsto \frac{1}{a \vee \bar{a}} \qquad \text{and} \qquad \frac{a01 \wedge a10}{a(0 \wedge 1)(1 \wedge 0)} \mapsto \frac{a \wedge \bar{a}}{0} \quad .$$

Here we used the equivalences $0 \vee 1 \equiv 1 \vee 0 \equiv 1$ and $0 \wedge 1 \equiv 1 \wedge 0 \equiv 0$, which again we can incorporate into the definition of $\mapsto$.

A pattern for the shape of inference rules should now become apparent, but before we proceed we need to address the following two objections that apply to Gentzen proof theory:

- A proof system only containing atomic contraction instead of the generic one is incomplete [1].
- Although proof systems with an atomic cut are complete, they entail at least an exponential penalty in the size of proofs compared to proof systems with unrestricted cut [8,9].

In deep inference [4,6] those two objections are easily overcome, because, given any proof in a Gentzen system, every contraction and cut instances can be locally transformed into their atomic variants by a local procedure of polynomial-size complexity [2]. We shall not define deep inference here, but the reader only needs to know that, in deep inference, proofs can be composed via the same connectives over which formulae are composed. In other words, if

$$\Phi = \begin{array}{c} A \\ \| \\ B \end{array} \qquad \text{and} \qquad \Psi = \begin{array}{c} C \\ \| \\ D \end{array}$$

are two proofs with, respectively, premisses A and C and conclusions B and D, then

$$\Phi \wedge \Psi = \begin{array}{c} A \wedge C \\ \| \\ B \wedge D \end{array} \qquad \text{and} \qquad \Phi \vee \Psi = \begin{array}{c} A \vee C \\ \| \\ B \vee D \end{array}$$

are valid proofs with, respectively, premisses $A \wedge C$ and $A \vee C$, and conclusions $B \wedge D$ and $B \vee D$. Significantly, while $\Phi \wedge \Psi$ can be represented in Gentzen, $\Phi \vee \Psi$ cannot. This is basically the definition of deep inference and it holds for every language, not just propositional classical logic.

A further advantage of deep inference is that, contrary to Gentzen theory [10], self-dual noncommutative connectives such as the ones that we use for atoms here can easily be accommodated into proof systems enjoying cut elimination. For all these reasons we adopt deep inference and in the rest we assume this implicitly.

Let us now consider the following partial order C of logical relations

$$\begin{array}{ccccccc} & & & \wedge & & & \\ & & / & & \backslash & & \\ a_1 & \diagup & a_2 & & a_3 & \cdots & , \\ & \diagdown & & \backslash & & & \\ & & & \vee & & & \end{array}$$

where $\vee <_{\mathsf{C}} a_i <_{\mathsf{C}} \wedge$. Intuitively, $>_{\mathsf{C}}$ corresponds to implication, as in, *e.g.*, $A \wedge B \Rightarrow A \vee B$ and $(0 \wedge 1) \Rightarrow a01 \Rightarrow (0 \vee 1)$. On C we define the involution $\bar{\cdot}$ such that $\bar{\vee} = \wedge$ and $\bar{a}_i = a_i$, and we also define for each of its elements α the set $\mathsf{i}(\alpha) = \{\alpha, \bar{\alpha}\}$. We then define the (infinite) set of quadruples

$$\mathsf{Q}_{\mathsf{C}} = \{\, \langle \alpha\beta\gamma\delta \rangle \mid \alpha \leq_{\mathsf{C}} \delta, \delta \in \mathsf{i}(\alpha), \gamma \leq_{\mathsf{C}} \beta, \beta \in \mathsf{i}(\gamma) \,\} \setminus \{\langle \vee\wedge\wedge\vee \rangle\} \quad .$$ [1]

We define *system* SA as the deep inference system whose only inference rule is

$$\frac{\beta\ \alpha AC\ \delta BD}{\alpha\ \beta AB\ \gamma CD}\ \langle \alpha\beta\gamma\delta \rangle \in \mathsf{Q}_{\mathsf{C}} \quad ;$$

for uniformity, in the scheme $\vee$ and $\wedge$ are represented in Polish notation.

Soundness for this system can easily be proved by checking that each inference rule instance involving tame formulae corresponds to a valid implication between premiss and conclusion. To prove *completeness* we need to make sure that every valid tautology A can be proved by SA in the form of a tame formula B such that $B \mapsto A$. This can be done rather easily by showing that each inference rule of a

[1] Q_{C} is obtained by substituting a and b with every possible atom in

$$\{\langle \vee,\vee,\vee,\vee \rangle, \langle \vee,\vee,\vee,\wedge \rangle, \langle \vee,a,a,\vee \rangle, \langle \vee,a,a,\wedge \rangle, \langle \vee,\wedge,\vee,\vee \rangle, \langle \vee,\wedge,\vee,\wedge \rangle, \langle \vee,\wedge,\wedge,\wedge \rangle,$$
$$\langle a,\vee,\vee,a \rangle, \langle a,a,a,a \rangle, \langle a,b,b,a \rangle, \langle a,\wedge,\vee,a \rangle, \langle a,\wedge,\wedge,a \rangle, \langle \wedge,\vee,\vee,\wedge \rangle, \langle \wedge,a,a,\wedge \rangle,$$
$$\langle \wedge,\wedge,\vee,\wedge \rangle, \langle \wedge,\wedge,\wedge,\wedge \rangle\} \quad .$$

complete system for propositional logic, such as KS [2], can be represented by one or more rules of SA.

Since we have seen how to deal with identity and contraction above, suffice to see how we can represent weakening (with $\langle a, \wedge, \vee, a\rangle$), switch (with $\langle \vee, \wedge, \vee, \vee\rangle$ or $\langle \vee, \wedge, \wedge, \wedge\rangle$) and medial (with $\langle \wedge, \vee, \vee, \wedge\rangle$):

$$\frac{a01 \wedge a00}{a(0 \wedge 0)(1 \vee 0)}\ , \quad \frac{a10 \wedge a00}{a(1 \wedge 0)(0 \vee 0)}\ , \quad \frac{(A \vee C) \wedge (B \vee D)}{(A \wedge B) \vee (C \vee D)}\ , \quad \frac{(A \wedge C) \vee (B \wedge D)}{(A \vee B) \wedge (C \vee D)}\ .$$

We can then state the following proposition.

Proposition 1. SA *is sound and complete for propositional logic.*

It is rather intriguing that a sound and complete system for propositional logic can be obtained with a highly regular system such as SA, and we do not have a 'moral' explanation for this (yet). However, there is more. Obviously, SA is a conservative extension of propositional logic, because it proves more, namely it proves wild formulae. We could ask ourselves what happens to proofs if we are only interested in tame formulae. It is very easy to prove the following:

Proposition 2. *If the conclusion of a proof in* SA *is a tame formula, then no wild formula appears in the proof.*

In other words, what we see as propositional logic proofs are just special observations obtained from a more general and more regular collection of proofs. In the near future we will study SA and will try to see whether its regularity could be exploited for a better understanding of proof normalisation. Certainly, given the close correspondence between SA and KS, the whole proof theory of KS can be inherited by SA, and in particular the modern notions of normalisation based on atomic flows [5,7]. Translating KS into SA might shed light on why the purely structural information of atomic flows is sufficient to achieve normalisation in KS.

We are also interested in the possibility of extending the idea behind SA to other logics. For example, let us consider linear logic [3]. If we had to interpret the SA contraction rule in linear logic we would not be able to obtain contraction because in linear logic $1 \parr 1 \not\equiv 1$. Therefore the interpretation of the atomic contraction rule instance would be something like

$$\frac{a \parr a}{a0(1 \parr 1)}\ ,$$

which does not correspond to any linear logic proof and which we could consider 'wild'. This hints at the possibility of controlling the 'resource consciousness' or 'substructurality' of a logic not only by a careful choice of inference rules, but also by an appropriate choice of the equivalences (or lack thereof) governing the units, and implementing them into the interpretation map.

If all this turns out to be viable, we would then have the chance to study normalisation for a wide range of logics by working at the 'subatomic' level of SA and then specialising the proof systems and their proof theory by simply tuning the interpretation map.

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