#### BUTTERFLIES

Alessio Guglielmi (TU Dresden) 26.11.2001 - updated on 1.11.2002 and 29.7.2005

There is a trivial, but fatal, mistake in this note. I leave the note unchanged because it is a nice exercise to try and find the mistake. I still believe that PCP can be used to prove the undecidability of NEL. In the meantime, Lutz Straßburger proved the result, with other methods, in [LS].

It's not known whether MELL is decidable. I conjecture here that system NEL [GS], i.e., MELL plus non-commutativity, is undecidable. I propose an encoding of Post's Correspondence Problem (PCP) into NEL that has the merit of making non-commutativity (in NEL) correspond to sequential composition (in PCP), which is intuitive.

The following theorem is well-known:

**1 Definition** An instance of *Post's Correspondence Problem* (PCP) consists of two lists  $v_1, \ldots, v_h$  and  $w_1, \ldots, w_h$  of strings of positive atoms. This instance of PCP *has a solution* if there is any sequence of integers  $i_1, \ldots, i_m$ , with m > 0, such that

$$V_{i1}; V_{i2}; \dots; V_{im} = W_{i1}; W_{i2}; \dots; W_{im}$$
.

**2** Example Let  $v_1 = a$ ,  $v_2 = abaaa$ ,  $v_3 = ab$  and  $w_1 = aaa$ ,  $w_2 = ab$ ,  $w_2 = b$ . In this case

 $v_{2};v_{1};v_{1};v_{3} = w_{2};w_{1};w_{1};w_{3} = abaaaaaab$ ,

so this PCP has a solution.

**3 Theorem** *PCP* is undecidable.

Now let's encode a generic PCP into NEL. I will use lower case letters to denote atoms in the same alphabet of the PCP strings; upper case letters other than R and S are still atoms, different from the previous ones, and they serve bookkeeping purposes.

We build a set of rules, i.e., a set of structures of NEL, that will be prefixed by ? and that will always be available while going up in the search for a proof. There are three phases in the proof, the first generates strings, the second generates two equal matching strings, the third one recognizes their equality.

# Phase 1: Generation of words

The strings  $v_i$  and  $w_i$  share a common alphabet. Let us build two disjoint alphabets, one for v's and the other for w's, to be used in the encoding, so as to know whether a certain atom is used for a v or for a w. For every couple ( $v_i$ ,  $w_i$ ) create a structure  $S_i$  and a

structure S'<sub>i</sub>: Let V<sub>i</sub> be the seq concatenation of the atoms in v<sub>i</sub>, and let W<sub>i</sub> be the seq concatenation of the atoms in w<sub>i</sub>, using atoms from the respective alphabets; let M and M' be special atoms; then

```
 \begin{array}{l} S_{1} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a > \end{array}, \begin{array}{c} <a ';a ';a '> \end{array} \right] > \right) , \\ S_{1}^{'} &= \left( \begin{array}{c} \neg M ' \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a > \end{array}, \begin{array}{c} <a ';a ';a '> \end{array} \right] > \right) , \\ S_{2}^{'} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a;b;a;a;a > \end{array}, \begin{array}{c} <a ';a ';a '> \end{array} \right] > \right) , \\ S_{2}^{'} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a;b;a;a;a > \end{array}, \begin{array}{c} <a ';b '> \end{array} \right] > \right) , \\ S_{2}^{'} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a;b;a;a;a > \end{array}, \begin{array}{c} <a ';b '> \end{array} \right] > \right) , \\ S_{3}^{'} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a;b;a;a;a > \end{array}, \begin{array}{c} <a ';b '> \end{array} \right] > \right) , \\ S_{3}^{'} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a;b > a;b > \end{array}, \begin{array}{c} <a ';b > \end{array} \right] > \right) , \\ S_{3}^{'} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a;b > a;b > \end{array}, \begin{array}{c} <b '> \end{array} \right] > \right) , \\ S_{3}^{'} &= \left( \begin{array}{c} \neg M \end{array}, \begin{array}{c} < M \end{array}; \left[ \begin{array}{c} <a;b > a;b > \end{array}, \begin{array}{c} <b '> \end{array} \right] > \right) ) . \end{array}
```

### Phase 2: Generation of matching strings

Let N be one more special atom. For every atom z in the alphabet of the PCP create a structure  $T_{z} = (-N, < N; [-z, -z'] > )$ .

**5 Example** For our case above:

 $T_{a} = (\neg N , < N ; [\neg a , \neg a'] > ) ,$  $T_{b} = (\neg N , < N ; [\neg b , \neg b'] > ) .$ 

#### Phase 3: Matching

The strings generated in Phase 1 and Phase 2 are matched one against the other.

### Encoding

Given a PCP P, the encoding is then the structure

 $\mathsf{P}_{_{anc}} = [ \ldots, ?\mathsf{S}_{_i} , \ldots, ?\mathsf{S'}_{_i} , \ldots, ?\mathsf{T}_{_z} , \ldots, \mathsf{M'} , \mathsf{N} , \neg\mathsf{M} , \neg\mathsf{N} ] \ .$ 

**6** Conjecture Given a PCP P, then P has a solution iff  $P_{enc}$  is provable in NEL.

The difficulty of course is in seeing whether, given a proof, the corresponding problem has a solution. In other words, we have to check whether anything perverse might happen in a proof. Proving this conjecture should anyway be rather easy.

## References

[GS] Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In M. Baaz and A. Voronkov, editors, *LPAR 2002*, volume 2514 of *Lecture Notes in*  Artificial Intelligence, pages 231-246. Springer-Verlag, 2002. URL: <a href="http://www.ps.uni-sb.de/~lutz/papers/NEL.pdf">http://www.ps.uni-sb.de/~lutz/papers/NEL.pdf</a>.

[LS] Lutz Straßburger. System NEL is undecidable. In Ruy De Queiroz, Elaine Pimentel, and Lucília Figueiredo, editors, 10th Workshop on Logic, Language, Information and Computation (WoLLIC), volume 84 of Electronic Notes in Theoretical Computer Science, 2003. URL: <a href="http://www.ps.uni-sb.de/~lutz/papers/NELundec\_wollic03.pdf">http://www.ps.uni-sb.de/~lutz/papers/NELundec\_wollic03.pdf</a>.

### Web Site

http://alessio.guglielmi.name/res/cos/.