*This note is superseded by **A First Order System with Finite Choice of Premises** by Kai Brünnler and Alessio Guglielmi, in Hendricks et al., editor, **First-Order Logic Revisited**, Logos Verlag, Berlin, 2004, pp. 59-74, URL: [http://www.iam.unibe.ch/~kai/Papers/FinitaryFOL.pdf](http://www.iam.unibe.ch/~kai/Papers/FinitaryFOL.pdf).*



# FINITARY CUT

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We all know that the traditional cut rule is considered infinitary; take

```
      |- S1, F    |- S2, ¬F
cut-------------------- :
           |- S1, S2
```

the formula F can be any formula, the choice is infinite. But if we reduce the cut rule to atomic form, as we always can do in the calculus of structures [WS], is atomic cut still infinitary? Not really, and this is the subject of this note.

In any system in the calculus of structures the above rule becomes

```
  S(F,¬F)
i↑------- ,
   S{⊥}
```

where (_,_) denotes conjunction (whatever kind), ⊥ is the unit of disjunction (defined as dual to conjunction), S{ } is a structure with a hole, S(F,¬F) is a shortcut for S{(F,¬F)} and F is a generic structure corresponding to a generic formula.

We know that, as a consequence of top-down symmetry, we always can define rules that are able to decompose the generic cut rule into its atomic version

```
   S(a,¬a)
ai↑------- .
    S{⊥}
```

In other words, we always can replace a generic cut by an atomic one with no loss of derivability (we can prove the same under the same premises; in particular, we prove the same theorems from empty premises). This works trivially for classical logic [BT], linear logic with all its fragments [LS], NEL and all fragments [GS], etc.

Is atomic cut infinitary? Of course, in principle there is an infinite choice for the atom a, but if we're only interested in provability, then we only want to choose the atom a among those appearing in S{ } (this can be shown by a trivial argument, but common sense is enough). Then, a system where ai↑ is replaced by the following rule fai↑ conserves provability:

```
        S(a,¬a)
    fai↑-------   where a or ¬a appears in S{ }.
         S{⊥}
```

The proviso is not elegant. We could stipulate the language of atoms we start with is *finite*, for example it can correspond to the language of atoms necessary to prove a given formula. In any case, rule fai↑ enjoys the subformula property, of course.

Where does infinity hide? In the only place where infinite choice should be found: in the choice of inference rules to apply. A normal cut would produce a generic formula, which only needs to contain atoms already present in the context. We can consider this formula as built from its atoms, and there is an infinite choice in building it. In the calculus of structures, this corresponds to building a corresponding derivation, where each step is finite.

## References


[BT] Kai Brünnler and Alwen Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347-361. Springer-Verlag, 2001. URL: http://www.iam.unibe.ch/~kai/Papers/lcl-lpar.pdf.

[GS] Alessio Guglielmi and Lutz Straßburger. A non-commutative extension of MELL. In M. Baaz and A. Voronkov, editors, *LPAR 2002*, volume 2514 of *Lecture Notes in Artificial Intelligence*, pages 231-246. Springer-Verlag, 2002. URL: http://www.lix.polytechnique.fr/~lutz/papers/NEL.pdf.

[LS] Lutz Straßburger. A local system for linear logic. In M. Baaz and A. Voronkov, editors, *LPAR 2002*, volume 2514 of *Lecture Notes in Artificial Intelligence*, pages 388-402. Springer-Verlag, 2002. URL: http://www.lix.polytechnique.fr/~lutz/papers/lls-lpar.pdf.


## Web Site


[WS] http://alessio.guglielmi.name/res/cos.