AN UNAVOIDABLE CONTRACTION LOOP IN MONOTONE DEEP INFERENCE

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Consider the following linear inference from [Das13]:

$$(1)\qquad \frac{[a \vee (b \wedge b')] \wedge [(c \wedge c') \vee (d \wedge d')] \wedge [(e \wedge e') \vee f]}{(a \wedge [c \vee e]) \vee (c' \wedge e') \vee (b' \wedge d') \vee ([b \vee d] \wedge f)}$$

Recall that this is not derivable in $\{\mathsf{s}, \mathsf{m}\}$. Indeed there could be no final step (in CoS-style), since any step would break soundness.[1] It remains underivable in the presence of units, since they can only help when there is some triviality around (see [Das13]), which is not the case by inspection of the semantics.

It turns out we may derive it using a single contraction loop, on either a or f. The derivation for a is below, the one for f being symmetric:

$$\mathsf{2s}\,\cfrac{=\,\cfrac{\mathsf{m}\,\cfrac{\mathsf{ac}{\uparrow}\,\cfrac{a}{a \wedge a} \vee (b \wedge b')}{[a \vee b] \wedge [a \vee b']} \wedge \mathsf{m}\,\cfrac{(c \wedge c') \vee (d \wedge d')}{[c \vee d] \wedge [c' \vee d']}}{\mathsf{2s}\,\cfrac{[a \vee b] \wedge [c \vee d]}{(a \wedge c) \vee [b \vee d]} \wedge \mathsf{2s}\,\cfrac{[a \vee b'] \wedge [c' \vee d']}{(b' \wedge d') \vee [a \vee c']}} \wedge =\,\cfrac{(e \wedge e') \vee f}{f \vee (e \wedge e')}}{=\,\cfrac{\mathsf{s}\,\cfrac{[(a \wedge c) \vee [b \vee d]] \wedge f}{(a \wedge c) \vee ([b \vee d] \wedge f)} \vee \mathsf{s}\,\cfrac{[(b' \wedge d') \vee [a \vee c']] \wedge e \wedge e'}{(b' \wedge d') \vee \mathsf{2s}\,\cfrac{[a \vee c'] \wedge e \wedge e'}{(a \wedge e) \vee (c' \wedge e')}}}{\mathsf{m}\,\cfrac{(a \wedge c) \vee (a \wedge e)}{\mathsf{ac}{\downarrow}\,\cfrac{a \vee a}{a} \wedge [c \vee e]} \vee (c' \wedge e') \vee (b' \wedge d') \vee ([b \vee d] \wedge f)}}$$

Here we have used different colours to distinguish the two a-paths forming the loop. This contraction loop cannot be eliminated, since (1) is underivable in $\{\mathsf{s}, \mathsf{m}\}$. Indeed, even if we added (1) to SKS, we could find another similar situation with a more complicated inference, and so on for any (non-deterministic) polynomial-time decidable system extending SKS by linear rules unless $\mathbf{coNP} = \mathbf{NP}$, due to [DS16].

Notice from the above that (1) is not actually minimal, due to the available application of medial to the middle conjunct of the premiss. The resulting linear inference is indeed minimal.

References

[Das13] Anupam Das. Rewriting with linear inferences in propositional logic. In Femke van Raamsdonk, editor, *24th International Conference on Rewriting Techniques and Applications (RTA)*, volume 21 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 158–173. Schloss Dagstuhl–Leibniz-Zentrum für Informatik, 2013.

[DS16] Anupam Das and Lutz Straßburger. On linear rewriting systems for boolean logic and some applications to proof theory. *Logical Methods in Computer Science*, 12(4):9:1–27, 2016.

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[1] For this, it is helpful to appeal to the pigeonhole intuition from [Das13].