Interaction and Depth against Nondeterminism in Proof Search

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The Calculus of Structures

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A proof in the sequent system GS1p

$$\dfrac{\dfrac{}{\vdash a\,,\,\bar{a}}\,\mathrm{Ax} \qquad \dfrac{\dfrac{}{\vdash b\,,\,\bar{b}}\,\mathrm{Ax}}{\vdash b \vee \bar{b}}\,\mathrm{R}\vee}{\dfrac{\vdash a\,,\,\bar{a} \wedge (b \vee \bar{b})}{\vdash a \vee (\bar{a} \wedge (b \vee \bar{b}))}\,\mathrm{R}\vee}\,\mathrm{R}\wedge$$

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$\leadsto$

A proof in system KSg

$$\mathsf{ai}\!\downarrow \dfrac{\mathsf{ai}\!\downarrow \dfrac{\mathsf{tt}\!\downarrow \dfrac{}{\vdash \mathsf{tt}}}{\vdash a \vee \bar{a}}}{\vdash a \vee (\bar{a} \wedge (b \vee \bar{b}))}$$

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Deep inference brings shorter proofs.

[Polynomial Size Deep-Inference Proofs Instead of Exponential Size Shallow-Inference Proofs, Guglielmi, 2004]

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is derivable in the calculus of structures system for classical logic.

$$\mathsf{s}\,\frac{S((R \vee U) \wedge T)}{S((R \wedge T) \vee U)} \qquad \mathsf{ai}{\downarrow}\,\frac{S\{\mathtt{tt}\}}{S(a \vee \bar{a})}$$

However...

Consider the instance of the sequent calculus inference rule:

$$\frac{\vdash a, \bar{a} \quad \vdash b, \bar{b}}{\vdash a, b, \bar{a} \wedge \bar{b}} \, \mathsf{R}\wedge$$

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In the calculus of structures this rule is simulated by the switch rule:

$$\mathsf{s}\,\frac{\mathsf{s}\,\dfrac{(a \vee \bar{a}) \wedge (b \vee \bar{b})}{a \vee (\bar{a} \wedge (b \vee \bar{b}))}}{a \vee b \vee (\bar{a} \wedge \bar{b})}$$

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Switch rule can be applied to $a \vee b \vee (\bar{a} \wedge \bar{b})$ in 27 different ways, and

to $a_1 \vee b_1 \vee (\bar{a}_1 \wedge \bar{b}_1 \wedge (a_2 \vee b_2 \vee (\bar{a}_2 \wedge \bar{b}_2)))$ in 69 different ways.

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Deep inference causes redundant nondeterminism.

System BV

- System BV: [Guglielmi,99] smallest technically nontrivial system

  MLL $+$ mix $+$ mix0 $+$ a non-commutative self-dual operator

  resembling prefix operator of process algebra: $a.b.P$

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- BV structures:

  $$S ::= \circ \mid a \mid [\underbrace{S, \ldots, S}_{>0}] \mid (\underbrace{S, \ldots, S}_{>0}) \mid \langle \underbrace{S; \ldots ; S}_{>0} \rangle \mid \bar{S}$$

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  $[(\bar{a}, \bar{b}), a, b]$ corresponds to $((a^{\perp} \otimes b^{\perp}) \parr a \parr b)$

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- Structures are considered equivalent modulo an equational theory.

Syntactic Equivalence of BV Structures

Associativity

$$[R, [T, U]] = [[R, T], U]$$
$$(R, (T, U)) = ((R, T), U)$$
$$\langle R; \langle T; U\rangle\rangle = \langle\langle R; T\rangle; U\rangle$$

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$$\langle \circ ; R \rangle = R$$

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$$(R, T) = (T, R)$$

Unit

$$[\circ, R] = R$$
$$(\circ, R) = R$$
$$\langle R ; \circ \rangle = R$$
$$\langle \circ ; R \rangle = R$$

Negation

$$\overline{[R, T]} = (\bar{R}, \bar{T})$$
$$\overline{(R, T)} = [\bar{R}, \bar{T}]$$
$$\overline{\langle R; T \rangle} = \langle \bar{R}; \bar{T} \rangle$$

$$\bar{\circ} = \circ$$
$$\bar{\bar{R}} = R$$

System BV of the Calculus of Structures

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]} \qquad \mathsf{q}\!\downarrow \frac{S\langle [R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}$$

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$$\text{MLL} \left\{ \mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar{a}]} \qquad \mathsf{s}\, \frac{S([R,U],T)}{S[(R,T),U]} \right.$$

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All the systems in the calculus of structures follows this scheme.

Classical Logic in the Calculus of Structures

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$$\mathsf{KSg}\left\{\begin{array}{cc} \mathsf{ai}{\downarrow}\,\dfrac{S\{\mathrm{tt}\}}{S[a,\bar{a}]} & \mathsf{s}\,\dfrac{S([R,U],T)}{S[(R,T),U]} \\[2ex] \mathsf{w}{\downarrow}\,\dfrac{S\{\mathrm{ff}\}}{S\{R\}} & \mathsf{c}{\downarrow}\,\dfrac{S[R,R]}{S\{R\}} \end{array}\right.$$

[Brünnler,CSL'03]

Reducing Nondeterminism

In BV, the rule s can be applied to $[(a, b), \bar{a}, \bar{b}]$ in 12 different ways:

$$\mathsf{s}\,\frac{([\bar{a}, a, b], \bar{b})}{[(\bar{a}, \bar{b}), a, b]} \qquad \mathsf{s}\,\frac{[([\bar{a}, b], \bar{b}), a]}{[(\bar{a}, \bar{b}), a, b]} \qquad \mathsf{s}\,\frac{[(\bar{a}, \bar{b}, a), b]}{[(\bar{a}, \bar{b}), a, b]}$$

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Observation: Switch rule breaks the "interaction" between atoms.

$$\mathsf{s}\,\frac{S([R, W], T)}{S[(R, T), W]}$$

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Observation: Switch rule breaks the "interaction" between atoms.

$$\mathsf{lis}\frac{S([R, W], T)}{S[(R, T), W]} \qquad \text{if} \quad \mathsf{at}\,\bar{R} \cap \mathsf{at}\,W \neq \emptyset$$

Definition: System BVsl is the system $\{\mathsf{ai}{\downarrow}, \mathsf{lis}, \mathsf{q}{\downarrow}\}$.

Lazy Interaction Switch

Consider:

$$[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]$$

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- The rule s can be applied to this structure in 42 different ways.
  (In system KSg, in 111 different ways.)

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$$\text{lis}\,\frac{S([R, W], T)}{S[(R, T), W]} \qquad \text{if} \quad \text{at}\,\bar{R} \cap \text{at}\,W \neq \emptyset$$

- The rule lis can be applied in 14 different ways.

Lazy Interaction Switch

Consider:

$$\frac{[b,([a,\bar{a}],\bar{b},[c,d,(\bar{c},\bar{d},[e,f,(\bar{e},\bar{f})])])]}{[a,b,(\bar{a},\bar{b},[c,d,(\bar{c},\bar{d},[e,f,(\bar{e},\bar{f})])])]}$$

$$\mathsf{lis}\,\frac{S([R,W],T)}{S[(R,T),W]} \qquad \text{if} \quad \mathsf{at}\,\bar{R} \cap \mathsf{at}\,W \neq \emptyset$$

- The rule lis can be applied in 14 different ways.

$$\{a\} \cap \{a\} \neq \emptyset$$

Lazy Interaction Switch

Consider:

$$\frac{[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [f, ([e, \bar{e}], \bar{f})])])]}{[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]}$$

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- The rule lis can be applied in 14 different ways.

$$\{e\} \cap \{e\} \neq \emptyset$$

Lazy Interaction Switch

Consider:

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$$\mathsf{lis}\,\frac{S([R, W], T)}{S[(R, T), W]} \qquad \text{if} \quad \text{at}\,\bar{R} \cap \text{at}\,W \neq \emptyset$$

- The rule s can be applied to this structure in 42 different ways. (In system KSg, in 111 different ways.)
- The rule lis can be applied in 14 different ways.

$$\{a\} \cap \{\bar{b}, c, \bar{c}, d, \bar{d}, e, \bar{e}, f, \bar{f}\} = \emptyset$$

Reducing Nondeterminism

$$\mathsf{lis}\,\dfrac{\mathsf{ai}\!\downarrow\dfrac{\mathsf{ai}\!\downarrow\dfrac{\circ}{[b,\bar{b}]}}{[([a,\bar{a}],\bar{b}),b]}}{[(\bar{a},\bar{b}),a,b]}$$

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In system $\{\mathsf{s},\mathsf{ai}{\downarrow}\}$ in the proof search space of $[(\bar{a},\bar{b}),a,b]$, there are 358 derivations including these 6 proofs, and no other proofs.

Reducing Nondeterminism

Definition: System BVsl is the system $\{\mathsf{ai}\downarrow, \mathsf{lis}, \mathsf{q}\downarrow\}$.

Theorem: Systems $\{\mathsf{ai}\downarrow, \mathsf{s}, \mathsf{q}\downarrow\}$ (BV) and BVsl are equivalent. [LPAR'06]

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Theorem: The cut rule is admissible for system BVsl. [Tech.Rep.06]

Theorem: System BV is NP-Complete. [WOLLIC'06]

Classical Logic in the Calculus of Structures

$$\text{ai}{\downarrow}\,\frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \text{s}\,\frac{S([R,U],T)}{S[(R,T),U]} \qquad \text{q}{\downarrow}\,\frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}$$

$\downarrow$ $\downarrow$

$$\text{KSg}\left\{\begin{array}{ll} \text{ai}{\downarrow}\,\dfrac{S\{\mathsf{tt}\}}{S[a,\bar{a}]} & \text{s}\,\dfrac{S([R,U],T)}{S[(R,T),U]} \\[2ex] \text{w}{\downarrow}\,\dfrac{S\{\mathsf{ff}\}}{S\{R\}} & \text{c}{\downarrow}\,\dfrac{S[R,R]}{S\{R\}} \end{array}\right.$$

[Brünnler,CSL'03]

Reducing Nondeterminism in Classical Logic System KSg

Theorem: A structure R has a proof in KSg iff

$$\begin{array}{l} \| \{\mathsf{s}, \mathsf{ai}{\downarrow}\} \\ R'' \\ \| \{\mathsf{w}{\downarrow}\} \\ R' \\ \| \{\mathsf{c}{\downarrow}\} \\ R \end{array} \qquad \begin{array}{l} [\mathsf{tt}, \mathsf{tt}] = \mathsf{tt} \\ (\mathsf{ff}, \mathsf{ff}) = \mathsf{ff} \end{array}$$

Reducing Nondeterminism in Classical Logic System KSg

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$$\begin{array}{l} \parallel \{\mathsf{s},\mathsf{ai}\downarrow\} \\ R'' \\ \parallel \{\mathsf{w}\downarrow\} \\ R' \\ \parallel \{\mathsf{c}\downarrow\} \\ R \end{array}$$

$$[\mathrm{tt}, \mathrm{tt}] = \mathrm{tt}$$
$$(\mathrm{ff}, \mathrm{ff}) = \mathrm{ff}$$

Definition: System KSgi is the system resulting from replacing the switch rule in system KSg with the lazy interaction switch rule.

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Theorem: Systems KSg and KSgi are equivalent. [LPAR'06]

Implementation in Maude

- Systems in the calculus of structures can be expressed as term rewriting systems modulo equational theories.
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- Language Maude allows implementing term rewriting systems modulo associativity, commutativity and unit(s).
- Maude has a built-in breadth-first search function.
- Systems of the calculus of structures can be easily implemented by resorting to the simple high level language of Maude. [ESSLLI'04,ISCIS'04]

Example: Maude Module for System BV

```
mod BV is
 sorts Atom Unit Structure .
 subsort Atom < Structure .
 subsort Unit < Structure .

  op o  : -> Unit .
  op -_      : Atom -> Atom [ prec 50 ] .
  op [_,_]   : Structure Structure -> Structure [assoc comm id: o] .
  op {_,_}   : Structure Structure -> Structure [assoc comm id: o] .
  op <_;_>   : Structure Structure -> Structure [assoc id: o] .
  ops a b c d e : -> Atom .

  var R T U V : Structure .      var A : Atom .

 rl [ai-down] : [ A , - A ]              => o .
 rl [s]       : [ { R , T } , U ]        => { [ R , U ] , T } .
 rl [q-down]  : [ < R ; T > , < U ; V > ] => < [R,U] ; [T,V] > .
endm
```

Automated Proof Search

1. $[a, b, (\bar{a}, \bar{b}, [a, b, (\bar{a}, \bar{b})])]$
2. $[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]$

Query	System	# states explored	finds a proof in # ms (cpu)
1.	$\{$s, ai$\downarrow\}$	1041	100
	$\{$lis, ai$\downarrow\}$	264	0
2.	$\{$s, ai$\downarrow\}$	–	–
	$\{$lis, ai$\downarrow\}$	6595	1370

Lazy Interaction Switch Revisited

Consider:

$$\frac{[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [f, ([e, \bar{e}], \bar{f})])])]}{[b, a, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]}$$

$$\mathsf{lis}\,\frac{S([R, W], T)}{S[(R, T), W]} \qquad \text{if} \quad \text{at}\,\bar{R} \cap \text{at}\,W \neq \emptyset$$

- The rule s can be applied to this structure in 42 different ways. (In system KSg, in 111 different ways.)
- The rule lis can be applied in 14 different ways.

$$\{a\} \cap \{\bar{b}, c, \bar{c}, d, \bar{d}, e, \bar{e}, f, \bar{f}\} = \emptyset$$

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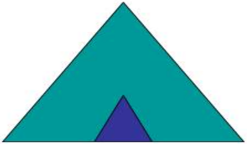
The condition of the rule must be performed for 42 such substructures.

This is expensive in proof search.

Deep Inference vs. Deepest Inference

Idea: When we restrict the application of the inference rules to the deepest redexes, we are restricted to the smaller substructures.



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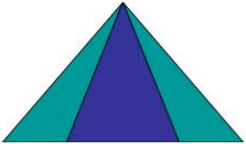

- Sequent calculus (shallow inference) is complete.

$$[a, b, (\bar{a}, \bar{b}, [c, d, (\bar{c}, \bar{d}, [e, f, (\bar{e}, \bar{f})])])]$$

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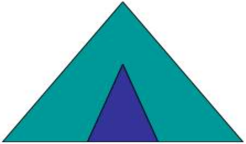

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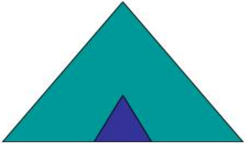

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Is there a plausible notion of "deepest inference" that is complete?

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Deepest Switch

Definition: A instance of the switch rule

$$\mathsf{s}\,\frac{S([R, W], T)}{S[(R, T), W]}$$

is an instance of deep switch (ds) if $R \neq (R_1, R_2)$ and $W \neq [W_1, W_2]$.

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Proposition: Every proof in system $\{\mathsf{ai}\!\downarrow, \mathsf{s}\}$ can be transformed to a proof in $\{\mathsf{ai}\!\downarrow, \mathsf{ds}\}$ in linear time.

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- Providing a confluent deductive system for MLL for structures consisting of pairwise distinct atoms. [Guerrini, 1999]

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- The technique for reducing nondeterminism provides a performance improvement in implementations.