Proof Theory With Deep Inference

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Proof Theory With Deep Inference

Outline of the course

1 Why deep inference?

Motivations from:

- proof theory
- computer science

2 The calculus of structures:

a proof theory with deep inference and premise / conclusion symmetry

3 Classical logic

atomicity and locality

4 Linear logic

modularity

5 System SBV

'exotism' (and process algebras)

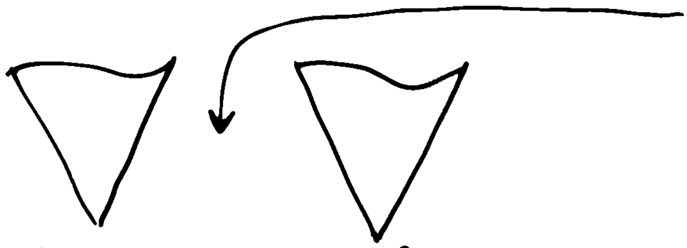
Why deep inference?

Proof theoretic motivations

Which connective does join branches?

- in classical logic:

$\wedge$!



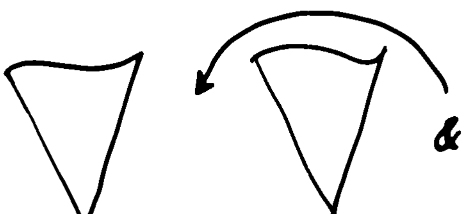
$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}$$

in fact:

$(\Gamma \vee A) \wedge (\Gamma \vee B) \Rightarrow \Gamma \vee (A \wedge B)$

OK!

- in linear logic:

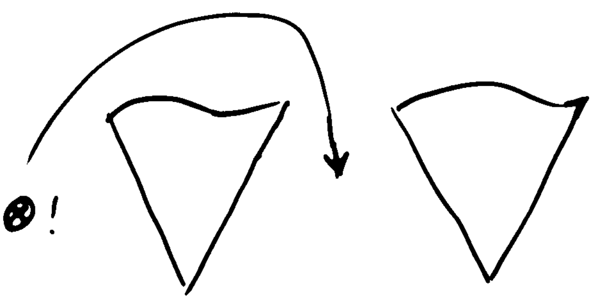


$$\frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \& B}$$

$\&$! $\neq$ $\otimes$!

$\uparrow$

!?



$$\frac{\vdash \Gamma, A \qquad \vdash \Delta, B}{\vdash \Gamma, \Delta, A \otimes B}$$

in fact:

$(\Gamma \parr A) \& (\Gamma \parr B) \multimap \Gamma \parr (A \& B)$

in fact:

$(\Gamma \parr A) \otimes (\Delta \parr B) \multimap \Gamma \parr \Delta \parr (A \otimes B)$

observe: $(\Gamma \parr A) \otimes (\Gamma \parr B) \not\multimap \Gamma \parr (A \& B)$ $\quad$ $(\Gamma \parr A) \& (\Delta \parr B) \not\multimap \Gamma \parr \Delta \parr (A \otimes B)$

! 〃 $\otimes$! 〃 $\multimap$ 〃 $\quad$! 〃 $\&$! 〃 $\multimap$ 〃

Mismatch between object level and meta-level

- Many logics suffer from the mismatch, especially modal logics
- There are two possible solutions

1 To enrich the meta-level:

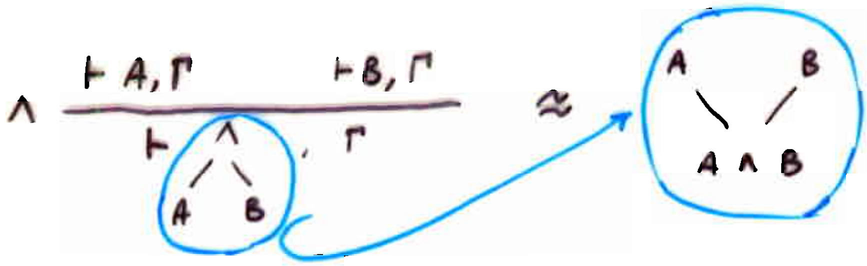
- display calculus (Belnap, Dunn, ...)
- various other calculi for modal logics (Došen, Masini, ...)
- BI logic (Pym)
- Abrusci - Ruet, Retoré for non-commutative logics ...

2 To <u>get rid</u> of the meta-level:

the calculus of structures

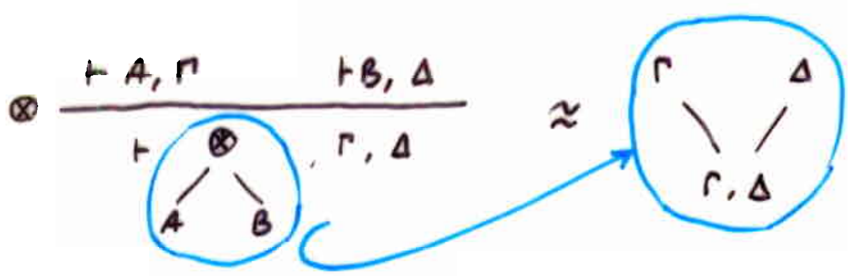
'Computer science' motivations

- Example 1 "Additive" conjunction



the formula tree shapes the proof tree

- Example 2 "Multiplicative" conjunction



the formula tree induces an unwanted tree
(in proof-search)

- Example 3 Cut elimination

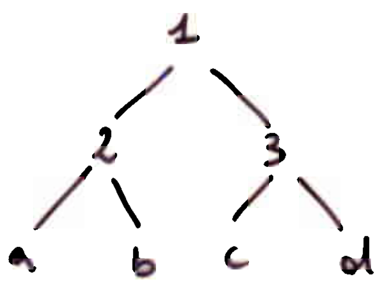
$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash A,\Gamma_1\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdash B,\Gamma_2\end{matrix}}{\vdash A\otimes B,\Gamma_1,\Gamma_2}\,\otimes \qquad \dfrac{\begin{matrix}\Pi_3\\ \vdash A^\perp,B^\perp,\Delta\end{matrix}}{\vdash A^\perp \parr B^\perp,\Delta}\,\parr}{\vdash \Gamma_1,\Gamma_2,\Delta}\,\mathrm{cut}$$

$$\downarrow$$

$$\dfrac{\dfrac{\begin{matrix}\Pi_1\\ \vdash A,\Gamma_1\end{matrix} \qquad \begin{matrix}\Pi_3\\ \vdash A^\perp,B^\perp,\Delta\end{matrix}}{\vdash B^\perp,\Gamma_1,\Delta}\,\mathrm{cut} \qquad \begin{matrix}\Pi_2\\ \vdash B,\Gamma_2\end{matrix}}{\vdash \Gamma_1,\Gamma_2,\Delta}\,\mathrm{cut}$$

the formula tree decides the order of reductions

- Example Suppose that
  - atoms are processors: a, b, c, d
  - communication flows through the tree structure



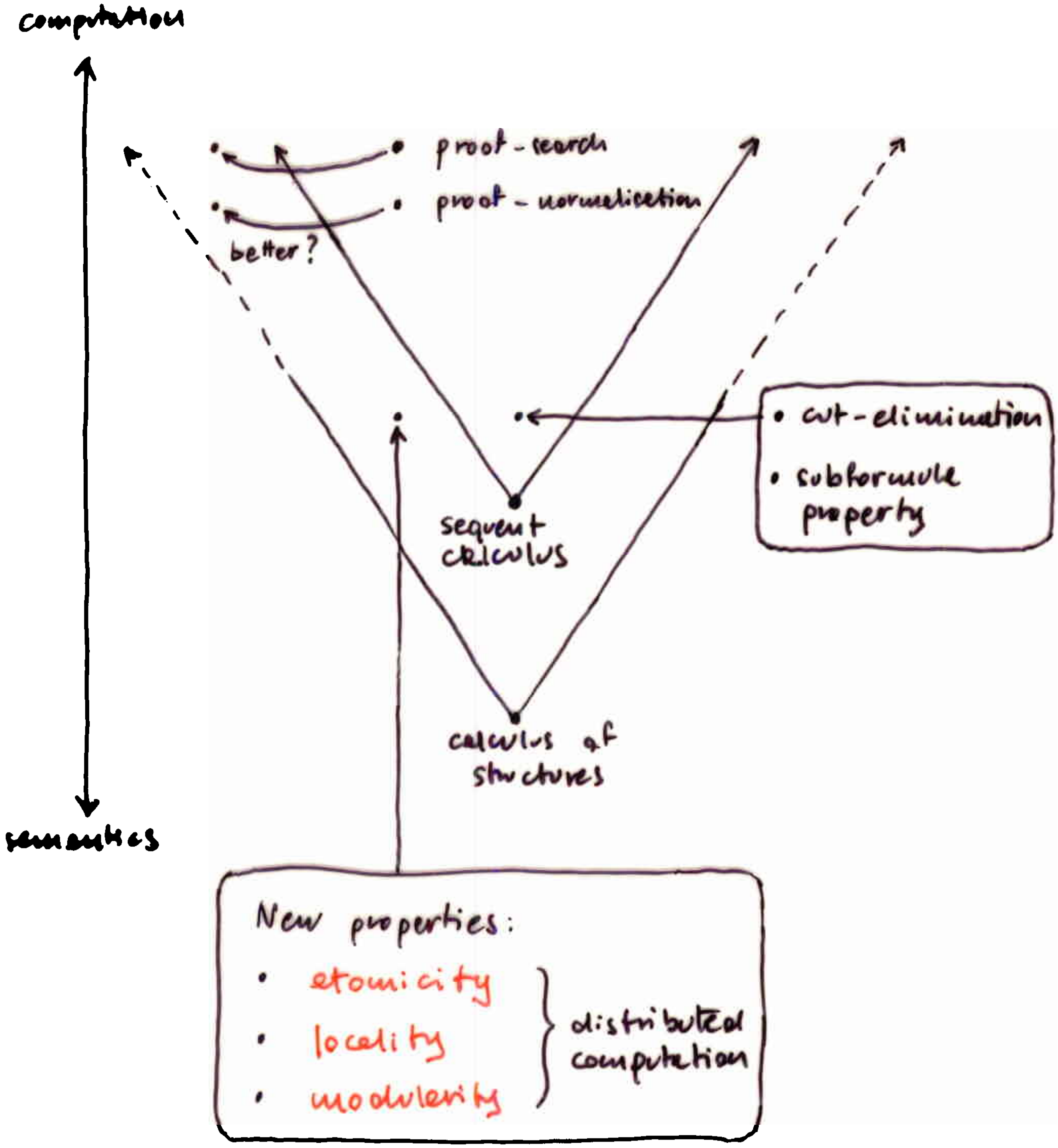
the communication workload of 1 is four times that of 2 and 3

- Main connectives create an asymmetry

- Step back: in the calculus of structures there are no main connectives

2 What is the calculus of structures?

It's a step back from the sequent calculus



Do we get a better proof theory?

2 What is the calculus of structures?

There are no main connectives

- Example 1 "Additive" conjunction

$$\frac{\vdash (A \vee C) \wedge (B \vee C)}{\vdash (A \wedge B) \vee C}$$

- Example 2 "Multiplicative" conjunction

$$\frac{\vdash (A \parr C) \otimes B}{\vdash (A \otimes B) \parr C}$$

- Inference rules can be applied deep inside formulae

- There is a new top-down symmetry

- What happens to the subformula property?

Inference rules can be applied deep inside formulae

- Example 1 "Additive" conjunction

Rule

$$\rho\,\frac{S\{(A\vee C)\wedge(B\vee C)\}}{S\{(A\wedge B)\vee C\}}$$

can be applied as in

$$\rho\,\frac{((A\vee C)\wedge(B\vee C)\wedge D)\vee E}{(((A\wedge B)\vee C)\wedge D)\vee E}$$

- Example 2 "Multiplicative" conjunction

Rule

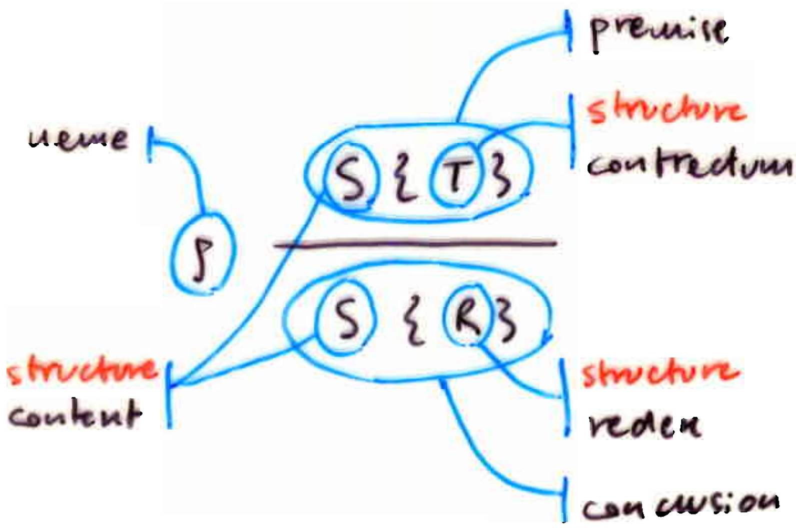
$$\rho\,\frac{S\{(A\parr C)\otimes B\}}{S\{(A\otimes B)\parr C\}}$$

can be applied as in

$$\rho\,\frac{\rho\,\frac{(A\parr C)\otimes(B\parr D)}{((A\parr C)\otimes B)\parr D}}{(A\otimes B)\parr C\parr D}$$

Inference rules can be applied deep inside formulae

- Inference rule ρ:



- The hole in $S\{\ \}$ does not appear inside a negation

- Rule ρ corresponds to $T \rightarrow R$

Structures

- **Atoms** are positive or negative: $a, b, c, \dots, \bar{a}, \bar{b}, \bar{c}, \dots$

- **Structures** $P, Q, R, S, T, U, \dots$ are

$$S ::=$$

atoms	a
disjunctions	$\mid [\underbrace{S, \dots, S}_{>0}]$
conjunctions	$\mid (\underbrace{S, \dots, S}_{>0})$
other relations	$\mid \langle \underbrace{S; \dots; S}_{>0} \rangle \mid \dots$
units	$\mid t \mid f \mid 1 \mid \bot \mid \dots$
modalized structures	$\mid ?S \mid !S \mid \dots$
quantified structures	$\mid \exists x. S \mid \forall x. S \mid \dots$
negated structures	$\mid \bar{S}$

Structures

- Equations are imposed over structures:

commutativity (not always)	$[R,T]=[T,R]$
associativity (always)	$\langle \vec{R};\langle \vec{T}\rangle;\vec{U}\rangle=\langle \vec{R};\vec{T};\vec{U}\rangle$
De Morgan (always!)	$\overline{[R,T]}=(\bar{R},\bar{T})$
contextual closure	$R=T \Rightarrow S\{R\}=S\{T\}$

- Notation Braces are dropped when unnecessary. Example:

  $S[R,T]$ instead of $S\{[R,T]\}$

There is a new top-down symmetry 1 of 3

If

$$\rho\downarrow \frac{S\{T\}}{S\{R\}}$$

is a rule, corresponding to

$$T \rightarrow R$$

then

$$\rho\uparrow \frac{S\{\bar{R}\}}{S\{\bar{T}\}}$$

is also a rule, corresponding to

$$\bar{R} \rightarrow \bar{T}$$

There is a new top-down symmetry

Example In linear logic

$$\mathsf{p}{\downarrow}\ \frac{S\{![R,T]\}}{S[!R,?T]}$$

corresponds to

$$!(R \parr T) \multimap (!R \parr ?T)$$

and

$$\mathsf{p}{\uparrow}\ \frac{S(?R,!T)}{S\{?(R,T)\}}$$

corresponds to

$$\overline{(!R \parr ?T)} \multimap \overline{!(R \parr T)}$$

There is a new top-down symmetry

- Derivations (Δ) are chains of instances of inference rules

$$\begin{array}{cc} & \vdots \\ & U \\ \pi & \overline{} \\ & T \\ \rho & \overline{} \\ & R \\ & \vdots \end{array}$$

- There is a top-down symmetry. Example

$$\begin{array}{cc} & \vdots \\ & \bar{R} \\ \bar{\rho} & \overline{} \\ & \bar{T} \\ \bar{\pi} & \overline{} \\ & \bar{U} \\ & \vdots \end{array}$$

is a valid derivation

What happens to the subformula property?

- Morally, it still holds if we design rules carefully. Example: in

$$\frac{S([R,U],T)}{S[(R,T),U]}$$

premise and conclusion are made of the same pieces

- Rules can still be finitary, either upwards, or downwards, or both

- Being finitary does not depend on having main connectives

Do we get a better proof theory?

- We have some chances because:
  - we abolished the main connective idea
  - we are free to apply rules deeply
  - then we have more freedom
  - we also have a new symmetry!
  - we should see proofs in more detail

- But:
  - we have to be careful in designing systems!
    (we shouldn't abuse freedom)
  - it's still not clear whether we can do some good distributed computation

Recipe for a good system

- Choose disjunction and conjunction and make identity and cut.

Example: linear logic

- $[R, T]$ stands for $R \parr T$

  (R, T) stands for $R \otimes T$

- Establish

$$\text{i}\!\downarrow \frac{S\{1\}}{S[R, \bar{R}]} \qquad \text{i}\!\uparrow \frac{S(R, \bar{R})}{S\{\bot\}}$$

interaction down or identity — interaction up or cut

- This is your interaction fragment

Recipe for a good system

- Take each couple of dual logical relations, for example:
  - $\{R, T\}$ stands for $R \bullet T$

    $\langle R, T \rangle$ stands for $R \,\&\, T$
  - and create the rules

$$\rho\downarrow \frac{S([R,U],[T,V])}{S[\langle R,T\rangle,\{U,V\}]} \qquad \rho\uparrow \frac{S(\{R,T\},\langle U,V\rangle)}{S\{(R,U),(T,V)\}}$$

  - or, for example

$$\rho\downarrow \frac{S\{\forall x.[R,T]\}}{S[\forall x.R,\exists x.T]} \qquad \rho\uparrow \frac{S(\exists x.R,\forall x.T)}{S\{\exists x.(R,T)\}}$$

  - This is your core structure fragment

- Add the non-core structure fragment

A one-sided system into the calculus of structures

One-sided (Gentzen–Schütte) system GS1p	A system for classical logic in the calculus of structures (the "naïf" system)
$\mathsf{id}\ \dfrac{}{\vdash A, \bar{A}}$	$\mathsf{t}\ \dfrac{}{t}$ $\mathsf{id}\ \dfrac{S}{(S,[A,\bar{A}])}$
$\vee_L\ \dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}$ $\quad \vee_R\ \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}$	$\vee\ \dfrac{(S, \Gamma)}{(S,[\Gamma, A])}$
$\wedge\ \dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}$	$\wedge\ \dfrac{(S,[\Gamma,A],[\Gamma,B])}{(S,[\Gamma,(A,B)])}$
$\mathsf{w}\ \dfrac{\vdash \Gamma}{\vdash \Gamma, A}$	
$\mathsf{c}\ \dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$	$\mathsf{c}\ \dfrac{(S,[\Gamma,A,A])}{(S,[\Gamma,A])}$
$\mathsf{cut}\ \dfrac{\vdash \Gamma, A \quad \vdash \Delta, \bar{A}}{\vdash \Gamma, \Delta}$	$\mathsf{cut}\ \dfrac{(S,[\Gamma,A],[\Delta,\bar{A}])}{(S,[\Gamma,\Delta])}$

3 Classical logic

A one-sided system into the calculus of structures

- Equations

$$[R] = (R) = R$$

$$[\vec{R}, \vec{T}] = [\vec{T}, \vec{R}]$$
$$(\vec{R}, \vec{T}) = (\vec{T}, \vec{R})$$

$$[\vec{R}, [\vec{T}], \vec{U}] = [\vec{R}, \vec{T}, \vec{U}]$$
$$(\vec{R}, (\vec{T}), \vec{U}) = (\vec{R}, \vec{T}, \vec{U})$$

$$\overline{[R,T]} = (\bar{R}, \bar{T})$$
$$\overline{(R,T)} = [\bar{R}, \bar{T}]$$

$$\bar{\bar{R}} = R$$

$$\text{if } R = T \text{ then } S\{R\} = S\{T\}$$

$$[R, f] = R = (R, t)$$

$$\bar{t} = f$$
$$\bar{f} = t$$

- Example Prove $((A \supset B) \supset A) \supset A \equiv \overline{(\overline{(\bar{A} \vee B)} \vee A)} \vee A$

$$\equiv ((\bar{A} \vee B) \wedge \bar{A}) \vee A$$

$$
\mathsf{c}\,\dfrac{\vee_R\,\dfrac{\vee_L\,\dfrac{\wedge\,\dfrac{\vee_L\,\dfrac{\mathsf{id}\,\dfrac{}{\vdash \bar{A}, A}}{\vdash \bar{A} \vee B, A} \qquad \mathsf{id}\,\dfrac{}{\vdash \bar{A}, A}}{\vdash (\bar{A} \vee B) \wedge \bar{A}, A}}{\vdash ((\bar{A} \vee B) \wedge \bar{A}) \vee A, A}}{\vdash ((\bar{A} \vee B) \wedge \bar{A}) \vee A, ((\bar{A} \vee B) \wedge \bar{A}) \vee A}}{\vdash ((\bar{A} \vee B) \wedge \bar{A}) \vee A}
$$

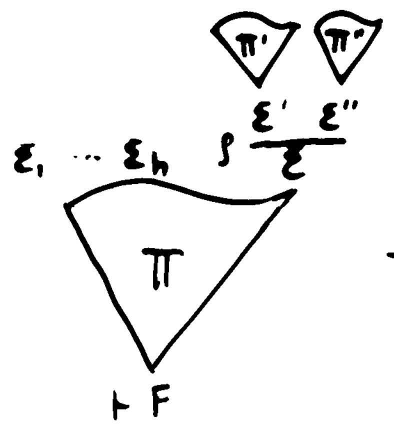
$$
= \dfrac{\mathsf{c}\,\dfrac{=;\vee\,\dfrac{\vee\,\dfrac{\wedge\,\dfrac{\mathsf{id}\,\dfrac{\vee\,\dfrac{\mathsf{id}\,\dfrac{t\,\dfrac{}{t}}{(t, [\bar{A}, A])}}{(t, [\bar{A}, B, A])}}{(t, [\bar{A}, B, A], [\bar{A}, A])}}{(t, [([\bar{A}, B], \bar{A}), A])}}{(t, [([\bar{A}, B], \bar{A}), A, A])}}{(t, [f, ([\bar{A}, B], \bar{A}), A, ([\bar{A}, B], \bar{A}), A])}}{(t, [f, ([\bar{A}, B], \bar{A}), A])}}{[([\bar{A}, B], \bar{A}), A]}
$$

3 Classical logic

The calculus of structures generalizes the one-sided sequent calculus

- It is trivial and non-interesting to port a system in the one-sided sequent calculus to the calculus of structures

- The translation works like this:



$$\longrightarrow \quad \begin{array}{c} \Pi' + \Pi'' \Big\Vert \\ \rho \, \dfrac{(\varepsilon_1, \ldots, \varepsilon_h, \varepsilon', \varepsilon'')}{(\varepsilon_1, \ldots, \varepsilon_h, \varepsilon)} \\ \Pi \Big\Vert \\ F \end{array}$$

- Symmetry is not exploited!

- Deepness is not exploited!

- Can we do better than the sequent calculus?

A deep, symmetric system

- Let's apply our recipe!

- We keep the equations we have already

- Interaction

$$\text{i}\!\downarrow \frac{S\{t\}}{S[R,\bar{R}]} \qquad \text{i}\!\uparrow \frac{S(R,\bar{R})}{S\{f\}}$$

- Core structure

$$\text{s}\!\downarrow \frac{S([R,U],[T,V])}{S[(R,T),U,V]} \qquad \text{s}\!\uparrow \frac{S([R,T],U,V)}{S[(R,U),(T,V)]}$$

- Non-core structure (here we have to be creative)

$$\text{w}\!\downarrow \frac{S\{f\}}{S\{R\}} \qquad \text{w}\!\uparrow \frac{S\{R\}}{S\{t\}}$$

$$\text{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \qquad \text{c}\!\uparrow \frac{S\{R\}}{S(R,R)}$$

A deep, symmetric system

- **Definition** A *system* ($\mathcal{S}$) is a set of inference rules

- **Definition** A rule ρ is *derivable* for a system $\mathcal{S}$ if for every instance $\rho\,\dfrac{T}{R}$ there is a derivation $\begin{array}{c} T \\ \Vert\,\mathcal{S} \\ R \end{array}$

- **Definition** This rule is called *switch*: $s\,\dfrac{S([R,U],T)}{S[(R,T),U]}$

- **Proposition** $s{\downarrow}$ and $s{\uparrow}$ are derivable for $\{s\}$

  Proof

$$\begin{array}{rc} & S([R,U],[T,V]) \\ \cline{2-2} s & S[([R,U],T),V] \\ \cline{2-2} s & S[(R,T),U,V] \end{array} \qquad \begin{array}{rc} & S([R,T],U,V) \\ \cline{2-2} s & S([(R,U),T],V) \\ \cline{2-2} s & S[(R,U),(T,V)] \end{array}$$

- **Remark** Switch is self-dual

- **Remark** s is a special case both of $s{\downarrow}$ and $s{\uparrow}$

A deep, symmetric system

- We have a system, let's call it SKSg

down | up

$$\mathsf{i}{\downarrow}\,\frac{S\{t\}}{S[R,\bar{R}]} \qquad \mathsf{i}{\uparrow}\,\frac{S(R,\bar{R})}{S\{f\}}$$

interaction

structure

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

core

non-core

$$\mathsf{w}{\downarrow}\,\frac{S\{f\}}{S\{R\}} \qquad \mathsf{w}{\uparrow}\,\frac{S\{R\}}{S\{t\}}$$

$$\mathsf{c}{\downarrow}\,\frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}{\uparrow}\,\frac{S\{R\}}{S(R,R)}$$

- Is this classical logic? Yes: let's see

- Remark $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}\}$ (and $\{\mathsf{i}{\downarrow}, \mathsf{s}\}$) is multiplicative linear logic

A deep, symmetric system

- **Theorem** Every derivation in GS1p can be transformed into a derivation in SKSg, and if it is cut-free, it remains cut-free

  Proof SKSg is more general than the unit system we saw already.
  (just pay attention to contraction in the rule ∧ and notice that

$$
\text{s}\,\frac{\text{s}\,\dfrac{(S,[\Gamma,A],[\Delta,\bar{A}])}{(S,[\Delta,([\Gamma,A],\bar{A})])}}{\text{i}\uparrow\,\dfrac{(S,[\Gamma,\Delta,(A,\bar{A})])}{(S,[\Gamma,\Delta])}}
$$

)

- Then, SKSg is classical logic, because every rule is sound

- Is there any use for w↑ and c↑?

3 Classical logic

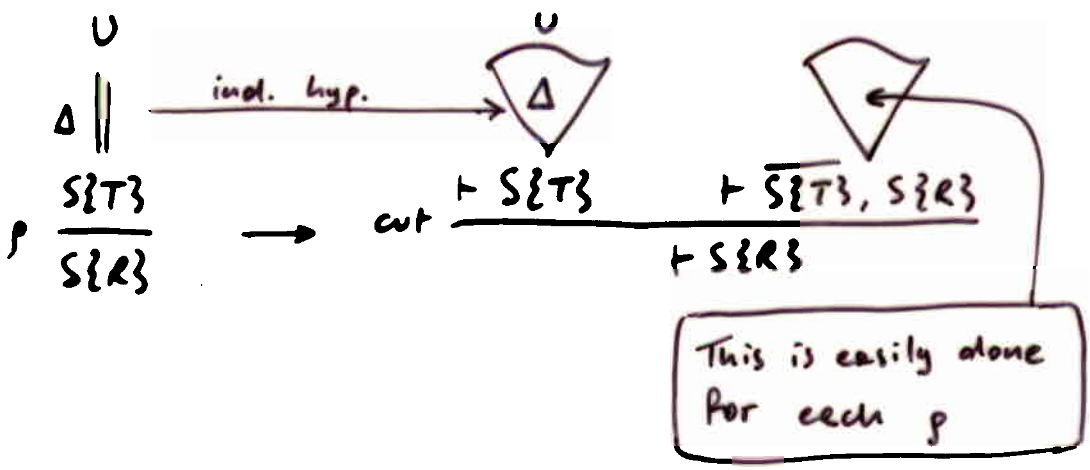
A deep, symmetric system

- What about cut elimination?

- Idea: let's exploit the sequent calculus

- **Theorem** Every derivation in SKSg can be transformed into a derivation in GS1p

Proof



4 deep, symmetric system

- Let's break the symmetry!

- **Definition** A **proof** is a derivation whose topmost structure is (equivalent to) t

---

- **Definition** An inference rule ρ is **admissible** for a system $\mathcal{S}$ if $\rho \notin \mathcal{S}$ and for every proof $\begin{array}{c}\Pi \Vert \mathcal{S} \cup \{\rho\} \\ R\end{array}$ there exists a proof $\begin{array}{c}\Pi \Vert \mathcal{S} \\ R\end{array}$

- **Theorem** $i\uparrow$ is admissible for $\{i\downarrow, s, w\downarrow, c\downarrow\}$ (and there is an algorithmic transformation for it)

Proof

calculus of structures | sequent calculus

$$\begin{tikzcd}
\Delta \arrow[rr] & & t(\Delta) \text{ (lots of cuts)} \arrow[d, "\text{cut elimination}"] \\
\text{(no cuts) } t'(\Delta') & & \Delta' \arrow[ll]
\end{tikzcd}$$

A deep, symmetric system

- Do we have a better system than classical logic in the sequent calculus?

  Perhaps, but still ...

- Do we have a better, or interesting, cut elimination procedure?

  Well ...

- Symmetry still is not fully exploited!

- Deepness still is not fully exploited!

Atomicity

- Consider

$$\mathrm{i}{\downarrow}\,\frac{S\{t\}}{S[(R,T),\bar{R},\bar{T}]} \longrightarrow \mathrm{s}\,\frac{\mathrm{s}\,\frac{\mathrm{i}{\downarrow}\,\frac{\mathrm{i}{\downarrow}\,\frac{S\{t\}}{S[T,\bar{T}]}}{S([R,\bar{R}],[T,\bar{T}])}}{S[([R,\bar{R}],T),\bar{T}]}}{S[(R,T),\bar{R},\bar{T}]} .$$

The i↓s become "smaller", so they eventually can be replaced by

$$\mathrm{ai}{\downarrow}\,\frac{S\{t\}}{S[a,\bar{a}]} .$$

This rule is called atomic interaction

- Theorem i↓ is derivable for {ai↓, s}.

- Nothing unexpected!

Atomicity

- Consider

$$
\text{i}\uparrow \frac{S([R,T],\bar{R},\bar{T})}{S\{f\}} \longrightarrow \text{i}\uparrow \frac{\text{i}\uparrow \frac{\text{s}\frac{\text{s}\frac{S([R,T],\bar{R},\bar{T})}{S([(R,\bar{R}),T],\bar{T})}}{S[(R,\bar{R}),(T,\bar{T})]}}{S(T,\bar{T})}}{S\{f\}} .
$$

The i↑s, too, become "smaller"; we can replace them by

$$
\text{ai}\uparrow \frac{S(a,\bar{a})}{S\{f\}} .
$$

This rule is called *atomic cointeraction*

- **Theorem** i↑ is derivable for {ai↑, s}.

- This property, due to symmetry, we can exploit!

Atomicity of cointeraction (cut)

- Consequences:
  - a simpler cut elimination proof
  - decomposition theorems

- Curiosities:
  - a different relation between cut, subformula property, and Kinitarguers
  - a simple consistency proof

Finitaryness

- In the sequent calculus finitaryness (going up) corresponds to the subformula property.

Example

$$\wedge\ \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \qquad\qquad \mathsf{cut}\ \frac{\vdash \Gamma, A \qquad \vdash \Delta, \bar{A}}{\vdash \Gamma, \Delta}$$

For $\wedge$:

- finitary
- A and B are subformulae of $A \wedge B$

For cut:

- non-finitary
- A is not necessarily a subformule of the conclusion

- In the calculus of structures there is no subformule property, but still all inference rules for classical logic are finitary (going up), except for

$$\mathsf{w}{\uparrow}\ \frac{S\{R\}}{S\{t\}} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\ \frac{S(R, \bar{R})}{S\{f\}}$$

$$\left(\text{or } \mathsf{ai}{\uparrow}\ \frac{S(a, \bar{a})}{S\{f\}}\right)$$

Finitaryness

- Rules in the core are always finitary! (They just "reshuffle" logical relations)

- Corules in the non-core up fragment are always strongly admissible for their duals, plus switch and interactions:

$$\rho\uparrow \frac{S\{T\}}{S\{R\}} \longrightarrow \dfrac{\dfrac{\dfrac{\dfrac{S\{T\}}{S(T,[R,\bar{R}])}\,\mathsf{i}\downarrow}{S(T,[R,\bar{T}])}\,\rho\downarrow}{S[R,(T,\bar{T})]}\,\mathsf{s}}{S\{R\}}\,\mathsf{i}\uparrow$$

- Then the only infinitary rule we are left with is

$$\mathsf{ai}\uparrow \frac{S(a,\bar{a})}{S\{f\}}$$

Finitaryness

- Consider the finitary atomic cointeraction rule:

$$\mathsf{fai}\uparrow \frac{S(a,\bar{a})}{S\{f\}} \quad \text{where } a \text{ or } \bar{a} \text{ appears in } S\{\ \}$$

- It is easy to eliminate all $\mathsf{ai}\uparrow$ instances that are not $\mathsf{fai}\uparrow$ instances, in proofs

replace here all a's with t and all $\bar{a}$'s with f: the proof remains valid!

$$\begin{array}{c} t \\ \| \\ \mathsf{ai}\uparrow \dfrac{S(a,\bar{a})}{S\{f\}} \\ \| \\ R \end{array}$$

bottommost $\mathsf{ai}\uparrow$ which is not an $\mathsf{fai}\uparrow$

proceed inductively upwards in the proof.

- Theorem Replacing $\mathsf{ai}\uparrow$ by $\mathsf{fai}\uparrow$ does not affect provability

- Finitaryness does not morally depend on full-blown cut elimination!

A simple consistency proof

- **Theorem** Propositional classical logic is consistent

  Proof We cannot get $\begin{array}{c} t \\ \| \\ f \end{array}$ when using fait

---

- **Theorem** If R is provable then $\bar{R}$ is not provable

  Proof Suppose we have

$$\begin{array}{c} t \\ \Pi_1 \, \| \\ R \end{array} \quad \text{and} \quad \begin{array}{c} t \\ \Pi_2 \, \| \\ \bar{R} \end{array} .$$

Then

$$\begin{array}{c} t \\ \Pi' \, \| \\ R \\ \| \\ f \end{array}$$

flip!

absurd.

Exploiting deepness

- The following rule is called medial:

$$m \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

- Medial is self-dual

- Look at

$$c\downarrow \frac{c\downarrow \dfrac{S[P,P,Q,Q]}{S[P,P,Q]}}{S[P,Q]} \quad \text{and} \quad c\downarrow \frac{c\downarrow \dfrac{m \dfrac{S[(P,Q),(P,Q)]}{S([P,P],[Q,Q])}}{S([P,P],Q)}}{S(P,Q)} .$$

By medial, contractions get "smaller"

- The following rules are called atomic contraction and atomic cocontraction:

$$ac\downarrow \frac{S[a,a]}{S\{a\}} \quad \text{and} \quad ac\uparrow \frac{S\{a\}}{S(a,a)}$$

- Theorem $c\downarrow$ is derivable for $\{ac\downarrow, m\}$, and dually

Exploiting deepness

- Deepness is essential for getting atomic contraction

- In the sequent calculus, it is impossible to get atomic contraction

- By the way, weakening is easily reduced to atomic form:

$$\mathsf{w}\downarrow\frac{\mathsf{w}\downarrow\dfrac{S\{f\}}{S[f,Q]}}{S[P,Q]} \quad \text{and} \quad \mathsf{w}\downarrow\frac{\mathsf{w}\downarrow\dfrac{\mathsf{ac}\uparrow\dfrac{S\{f\}}{S(f,f)}}{S(f,Q)}}{S(P,Q)} \;;$$

you can treat this with an equation, too

and dually for coweakening

3 Classical logic

System SKS

This is classical logic

down | up

$$\mathsf{ai}{\downarrow}\ \frac{S\{t\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\ \frac{S(a,\bar{a})}{S\{f\}}$$

interaction

structure

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

core

non-core

$$\mathsf{m}\ \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

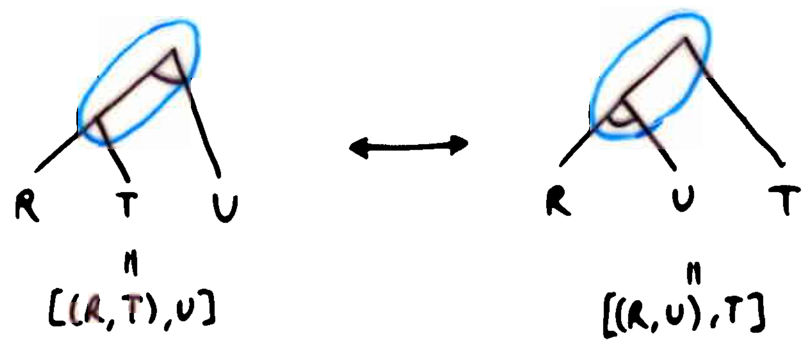
$$\mathsf{aw}{\downarrow}\ \frac{S\{f\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\ \frac{S\{a\}}{S\{t\}}$$

$$\mathsf{ac}{\downarrow}\ \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\ \frac{S\{a\}}{S(a,a)}$$

Locality

- Let's call *locality* the property of a rule requiring bounded effort to be applied.

  Example: switch



- Locality depends on the representation

- Atomicity can be a special form of locality

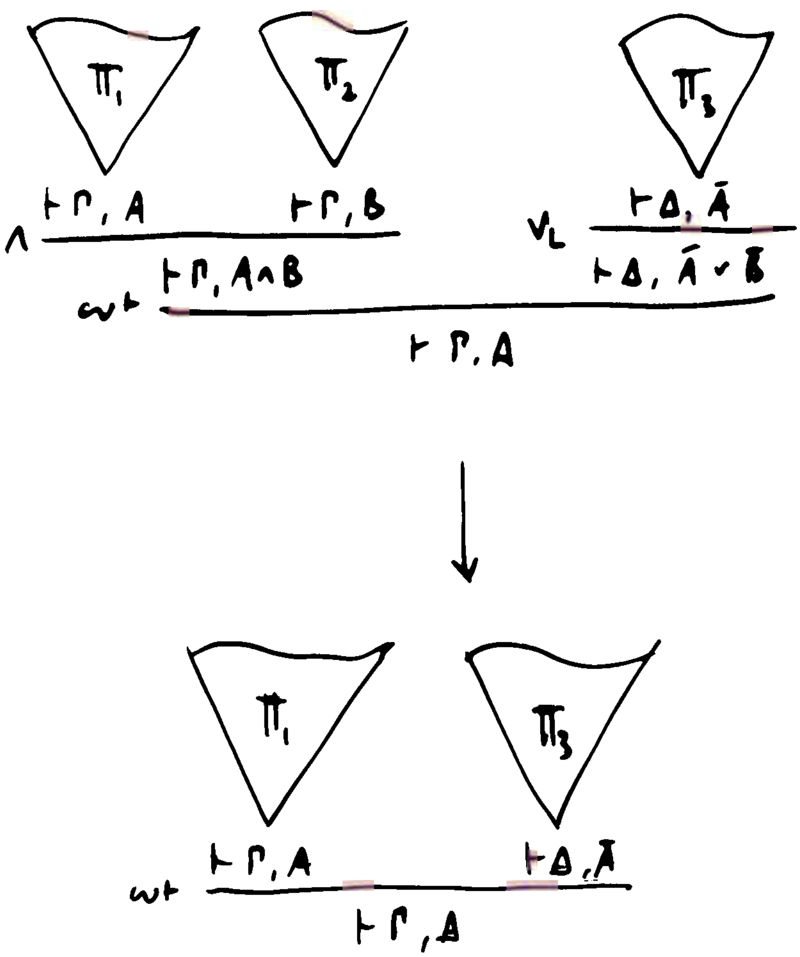
- There still is much to do for distributed computation (but look at relation webs)

- Applications in complexity?

Cut elimination

Why cut elimination is different than in the sequent calculus?

Because in the sequent calculus the main connective "drives" the reduction:



Cut elimination

- In the calculus of structures:

$$\mathrm{s}\,\frac{\dfrac{\|}{(a,[d,(b,c,[\bar{a},\bar{b},\bar{c}])])}}{[d,(a,b,c,[\bar{a},\bar{b},\bar{c}])]} \quad = \quad \text{if} \quad \frac{S(R,T,[\bar{R},\bar{T}])}{S\{f\}}$$

$R = (a,b)$

$T = c$

$S = [d,\{\ \}]$

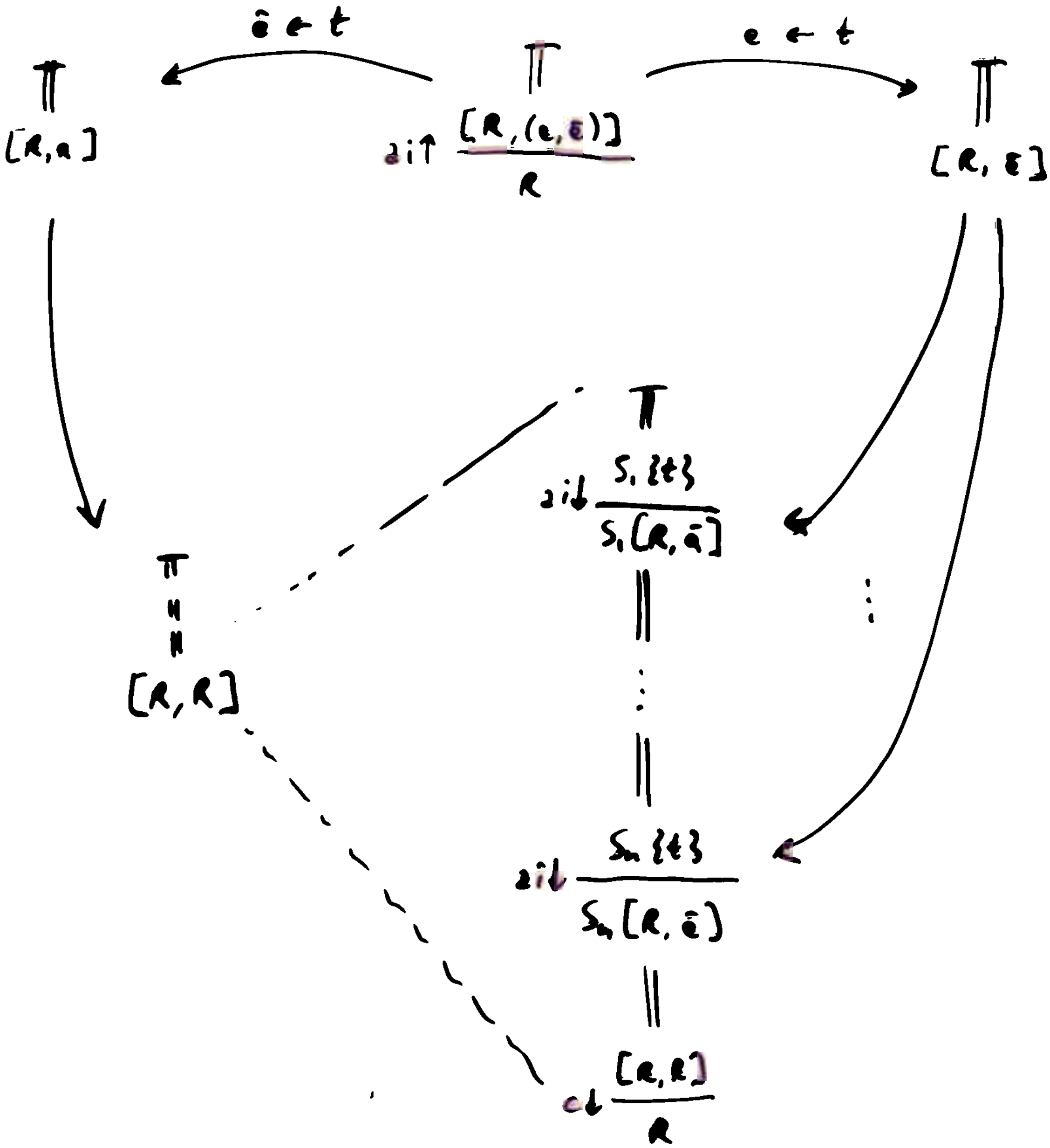
What are we supposed to do??

- Freedom has a price

- Atomicity helps a lot!

Theorem ai↑ is admissible

Proof



The simplest cut-elimination proof ever!

Decompositions

- Theorems

  - For every $\begin{array}{c} \mathsf{T} \\ \Vert\, \mathsf{SKS} \\ \mathsf{R} \end{array}$ there is a $\begin{array}{c} \mathsf{T} \\ \Vert\, \{ai\downarrow\} \\ \mathsf{V} \\ \Vert\, \mathsf{SKS} \setminus \{ai\downarrow, ai\uparrow\} \\ \mathsf{U} \\ \Vert\, \{ai\uparrow\} \\ \mathsf{R} \end{array}$

  - For every $\begin{array}{c} \mathsf{T} \\ \Vert\, \mathsf{SKS} \\ \mathsf{R} \end{array}$ there is a $\begin{array}{c} \mathsf{T} \\ \Vert\, \{ac\uparrow\} \\ \mathsf{V} \\ \Vert\, \mathsf{SKS} \setminus \{ac\downarrow, ac\uparrow\} \\ \mathsf{U} \\ \Vert\, \{ac\downarrow\} \\ \mathsf{R} \end{array}$

- One cannot do these things in the sequent calculus

- We start seeing some modularity

Is there any use for coweakening and cocontraction?

Yes:

- We saw ac↑ already for getting and (but that use was trivial)

- In interpolation theorems!:

  It is always possible to generate derivations such that, if $\begin{array}{c} T \\ \| \\ R \end{array}$, then

$$\begin{array}{lll} T & & \\ \| & \uparrow & \text{growing} \\ U & \leftarrow & \text{interpolant:} \quad T \to U \to R \\ \| & \downarrow & \text{growing} \\ R & & \end{array}$$

4 Linear logic

Multiplicative exponential linear logic

System SELS

down | up

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar a]} \qquad\qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar a)}{S\{\bot\}}$$

interaction

structure

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad\qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}$$

core

non-core

$$\mathsf{w}\!\downarrow \frac{S\{\bot\}}{S\{?R\}} \qquad\qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{1\}}$$

$$\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad\qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R,R)}$$

+ decidable equations, especially $\begin{cases} ??R = ?R \\ !!R = !R \end{cases}$

Multiplicative exponential linear logic

- Interactions are atomic

- Promotion is local!

- Absorption (i.e., contraction) is not atomic

- Modularity starts to manifest itself: each of $ai\uparrow$, $p\uparrow$, $w\uparrow$ and $b\uparrow$ is admissible for the down fragment and can be shown admissible independently (to a certain extent)

- So, there are $2^4 = 16$ equivalent systems whose properties are known

Modularity: decompositions

Theorem For every $\begin{array}{c} T \\ \Vert \\ R \end{array}$

there is

$$
\begin{array}{c}
T \\
\Vert \; \{b\uparrow\} \\
T_1 \\
\Vert \; \{w\downarrow\} \\
T_2 \\
\Vert \; \{ai\downarrow\} \\
T_3 \\
\Vert \; \text{core of SELS} \\
R_3 \\
\Vert \; \{ai\uparrow\} \\
R_2 \\
\Vert \; \{w\uparrow\} \\
R_1 \\
\Vert \; \{b\downarrow\} \\
R
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
T \\
\Vert \; \{b\uparrow\} \\
T_1 \\
\Vert \; \{w\uparrow\} \\
T_2 \\
\Vert \; \{ai\downarrow\} \\
T_3 \\
\Vert \; \text{core of SELS} \\
R_3 \\
\Vert \; \{ai\uparrow\} \\
R_2 \\
\Vert \; \{w\downarrow\} \\
R_1 \\
\Vert \; \{b\downarrow\} \\
R
\end{array}
$$

Proof Difficult!

Full linear logic

- We apply all our techniques and get:

- A system, called SLLS, with 34 rules, 16 of which in the up-non-down fragment (all admissible, of course): so we have $2^{16} = 65{,}536$ equivalent systems

- All rules are local (or atomic), including contractions

- All rules follow our recipe + medial + contraction + weakening, so the system is big but very uniform

Full linear logic

up | down

$$\mathsf{ai}\!\downarrow\frac{S\{1\}}{S[a,\bar a]}\qquad\qquad \mathsf{ai}\!\uparrow\frac{S(a,\bar a)}{S\{\bot\}}$$

interaction

structure

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}\!\downarrow\frac{S(\bullet[R,U],[T,V]\bullet)}{S[(\bullet R,T\bullet),[\bullet U,V\bullet]]}\qquad \mathsf{d}\!\uparrow\frac{S((\bullet R,U\bullet),[\bullet T,V\bullet])}{S[\bullet(R,T),(U,V)\bullet]}$$

$$\mathsf{p}\!\downarrow\frac{S\{![R,T]\}}{S[!R,?T]}\qquad \mathsf{p}\!\uparrow\frac{S(?R,!T)}{S\{?(R,T)\}}$$

up one

$$\mathsf{aw}\!\downarrow\frac{S\{0\}}{S\{a\}}\qquad \mathsf{ac}\!\downarrow\frac{S[\bullet a,a\bullet]}{S\{a\}}\qquad \mathsf{ac}\!\uparrow\frac{S\{a\}}{S(\bullet a,a\bullet)}\qquad \mathsf{aw}\!\uparrow\frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{l_0}\!\downarrow\frac{S\{0\}}{S[0,0]}\qquad \mathsf{l}\!\downarrow\frac{S[\bullet[R,U],[T,V]\bullet]}{S[[\bullet R,T\bullet],[\bullet U,V\bullet]]}\qquad \mathsf{l}\!\uparrow\frac{S((\bullet R,U\bullet),(\bullet T,V\bullet))}{S(\bullet(R,T),(U,V)\bullet)}\qquad \mathsf{l_0}\!\uparrow\frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{k_0}\!\downarrow\frac{S\{0\}}{S(0,0)}\qquad \mathsf{k}\!\downarrow\frac{S[\bullet(R,U),(T,V)\bullet]}{S([\bullet R,T\bullet],[\bullet U,V\bullet])}\qquad \mathsf{k}\!\uparrow\frac{S[(\bullet R,U\bullet),(\bullet T,V\bullet)]}{S(\bullet[R,T],[U,V]\bullet)}\qquad \mathsf{k_0}\!\uparrow\frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{m_0}\!\downarrow\frac{S\{0\}}{S(\bullet 0,0\bullet)}\qquad \mathsf{m}\,\frac{S[\bullet(\bullet R,U\bullet),(\bullet T,V\bullet)\bullet]}{S(\bullet[\bullet R,T\bullet],[\bullet U,V\bullet]\bullet)}\qquad \mathsf{m_0}\!\uparrow\frac{S[\bullet\top,\top\bullet]}{S\{\top\}}$$

$$\mathsf{x_0}\!\downarrow\frac{S\{0\}}{S\{?0\}}\qquad \mathsf{x}\!\downarrow\frac{S[\bullet ?R,?T\bullet]}{S\{?[\bullet R,T\bullet]\}}\qquad \mathsf{x}\!\uparrow\frac{S\{!(\bullet R,T\bullet)\}}{S(\bullet !R,!T\bullet)}\qquad \mathsf{x_0}\!\uparrow\frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{y_0}\!\downarrow\frac{S\{0\}}{S\{!0\}}\qquad \mathsf{y}\!\downarrow\frac{S[\bullet !R,!T\bullet]}{S\{![\bullet R,T\bullet]\}}\qquad \mathsf{y}\!\uparrow\frac{S\{?(\bullet R,T\bullet)\}}{S(\bullet ?R,?T\bullet)}\qquad \mathsf{y_0}\!\uparrow\frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{z_0}\!\downarrow\frac{S\{\bot\}}{S\{?0\}}\qquad \mathsf{z}\!\downarrow\frac{S[?R,T]}{S\{?[\bullet R,T\bullet]\}}\qquad \mathsf{z}\!\uparrow\frac{S\{!(\bullet R,T\bullet)\}}{S(!R,T)}\qquad \mathsf{z_0}\!\uparrow\frac{S\{!\top\}}{S\{1\}}$$

System SLLS

Cut elimination

It alway holds. How do we prove it?

MLL: splitting

MELL: decomposition + splitting

MALL: splitting

LL: by translation to the sequent calculus

5 System SBV

Idea

- CCS is a language for distributed computation where

$$a.b \mid \bar{a}.\bar{b} \to 0$$

- Can we make a logic out of this?

- If so, we want $\overline{a.b} = \bar{a}.\bar{b}$

- Then "." is a non-commutative self-dual logical relation

- Problem: getting this in the sequent calculus is very difficult (let's say impossible, see later)

Recipe!

- Ingredients:

  2 commutative dual logical relations

  1 non-commutative self-dual logical relation

  1 self-dual unit common to all relations

- Recipe:

  Just create an interaction and a core structure fragment (everything is multiplicative, for now)

- We get a very simple system whose proof theory is extremely intricate

- The system is atomic and local

5 System SBV

The system

- Rules:

down | up

interaction

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[R,\bar{R}]} \qquad \mathsf{ai}\!\uparrow \frac{S(R,\bar{R})}{S\{\circ\}}$$

structure (core)

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R;T\rangle,\langle U;V\rangle)}{S\langle (R,U);(T,V)\rangle}$$

- Equations:

Commutativity:

$$[\vec{R},\vec{T}] = [\vec{T},\vec{R}]$$
$$(\vec{R},\vec{T}) = (\vec{T},\vec{R})$$

Associativity:

$$[\vec{R},[\vec{T}]] = [\vec{R},\vec{T}]$$
$$(\vec{R},(\vec{T})) = (\vec{R},\vec{T})$$
$$\langle \vec{R};\langle \vec{T}\rangle;\vec{U}\rangle = \langle \vec{R};\vec{T};\vec{U}\rangle$$

Context closure:

if $R=T$ then $S\{R\}=S\{T\}$

Unit:

$$R = [R,\circ] = (R,\circ) = \langle R;\circ\rangle = \langle \circ;R\rangle$$

Negation:

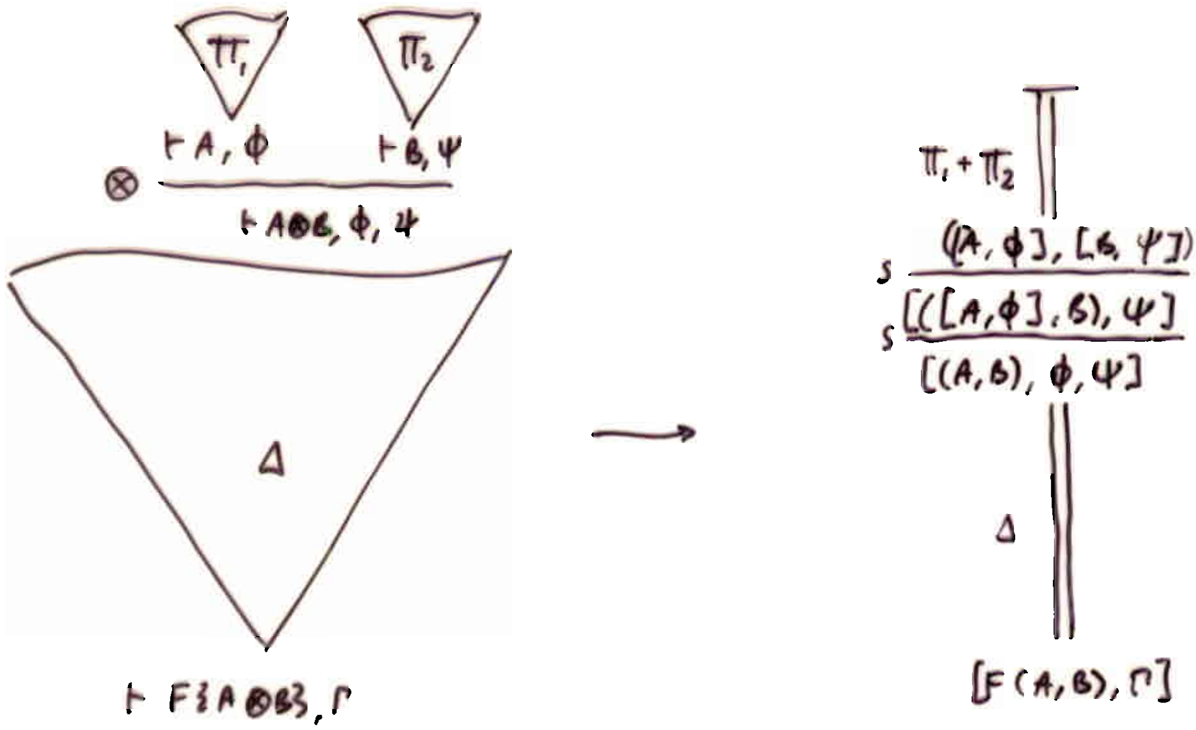
$$\bar{\bar{R}} = R$$
$$\overline{[R,T]} = (\bar{R},\bar{T})$$
$$\overline{(R,T)} = [\bar{R},\bar{T}]$$
$$\overline{\langle R;T\rangle} = \langle \bar{R};\bar{T}\rangle$$
$$\bar{\circ} = \circ$$

Singleton:

$$[R] = (R) = \langle R\rangle = R$$

Cut elimination by splitting

The idea comes from the sequent calculus:



Cut Elimination by splitting

- **Definition** $BV = \{ ai{\downarrow}, s, q{\downarrow} \}$

- **Theorem** (Splitting)

  - If $\begin{array}{c} \Pi\, BV \\ S\langle R;T\rangle \end{array}$ then $\begin{array}{c} [\{\ \}, \langle S_1;S_2\rangle] \\ \Vert\, BV \\ S\{\ \} \end{array}$, $\begin{array}{c} \Pi\, BV \\ [R,S_1] \end{array}$ and $\begin{array}{c} \Pi\, BV \\ [T,S_2] \end{array}$

  - If $\begin{array}{c} \Pi\, BV \\ S(R,T) \end{array}$ then $\begin{array}{c} [\{\ \}, S_1, S_2] \\ \Vert\, BV \\ S\{\ \} \end{array}$, $\begin{array}{c} \Pi\, BV \\ [R,S_1] \end{array}$ and $\begin{array}{c} \Pi\, BV \\ [T,S_2] \end{array}$

Proof Complex, but uniform

Cut elimination by splitting

- **Theorem** $ai\uparrow$ is admissible for BV

  Proof Splitting

- **Theorem** $q\uparrow$ is admissible for BV

  Proof Splitting

- SBV and BV (and $BV \cup \{ai\uparrow\}$ and $BV \cup \{q\uparrow\}$) are equivalent

5 System SBV

Decomposition

Theorem

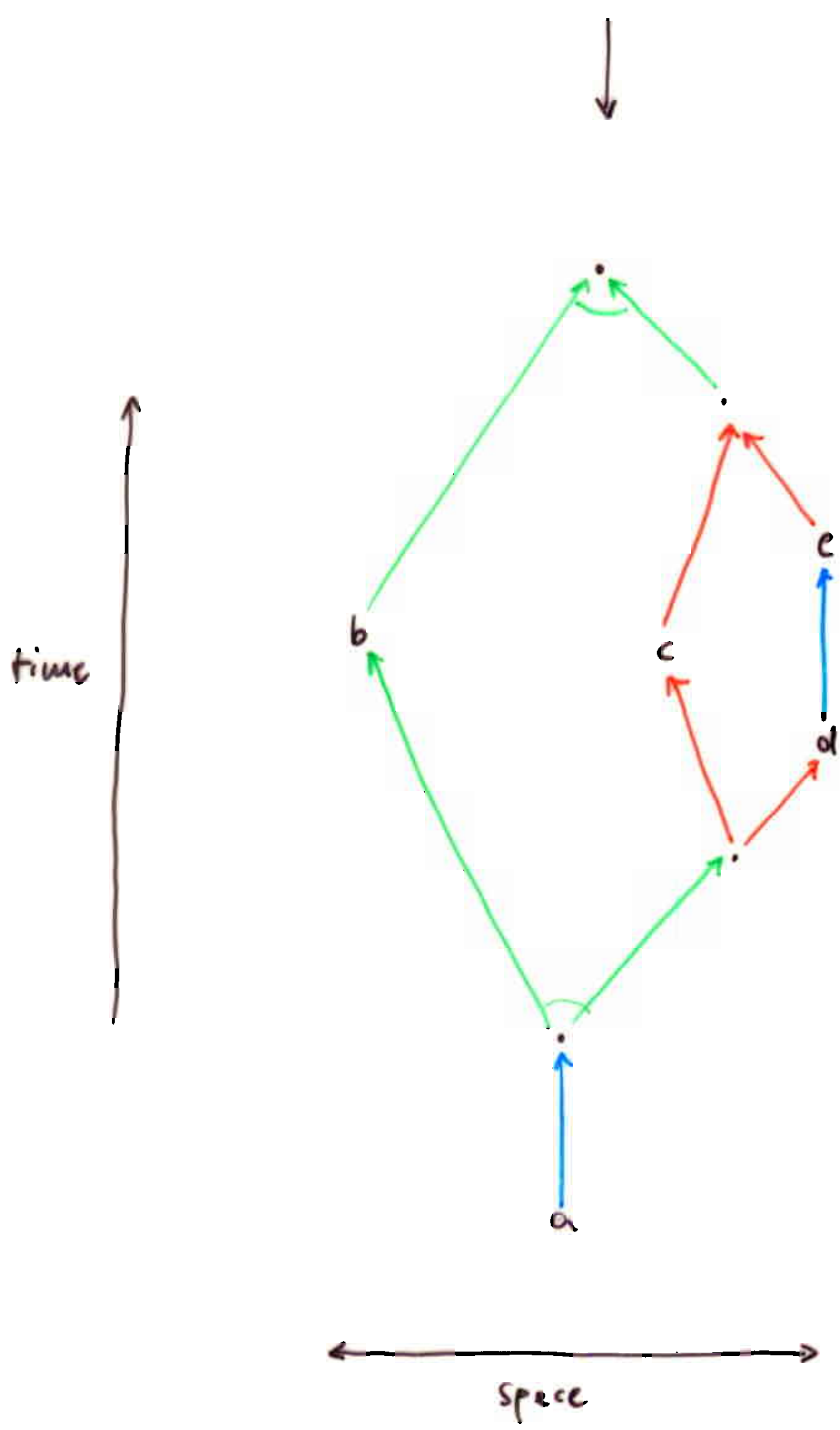
If $\begin{array}{c} T \\ \| \, \mathsf{SBV} \\ R \end{array}$ then $\begin{array}{l} T \\ \| \; \{ai\downarrow\} \\ T' \\ \| \; \text{core of } \mathsf{SBV} = \{s, q\downarrow, q\uparrow\} \\ R' \\ \| \; \{ai\uparrow\} \\ R \end{array}$

Proof Permutations

5 System SBV

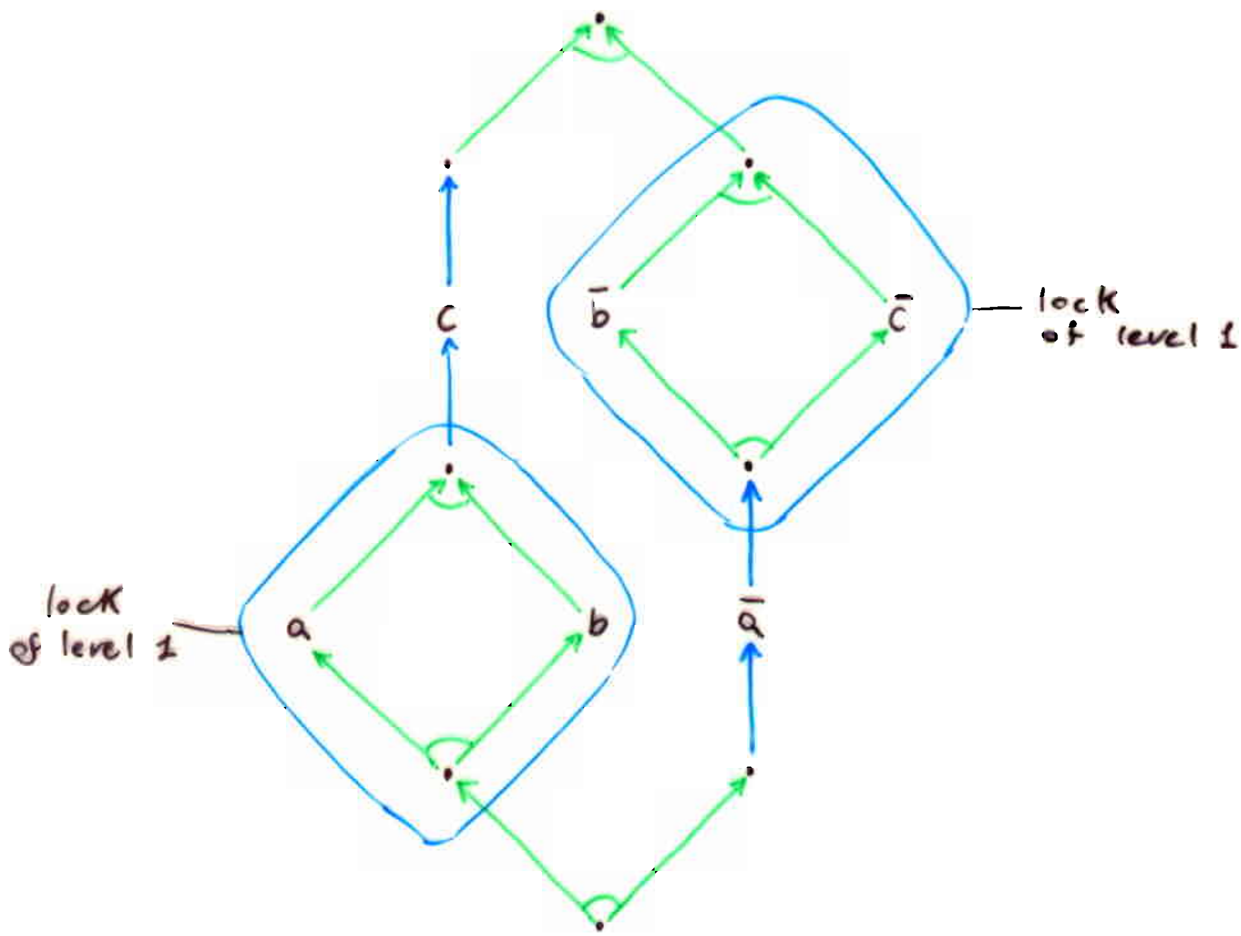
Intuitive representation of SBV structures

$\langle a ; [b, (c, \langle d ; e \rangle)] \rangle$



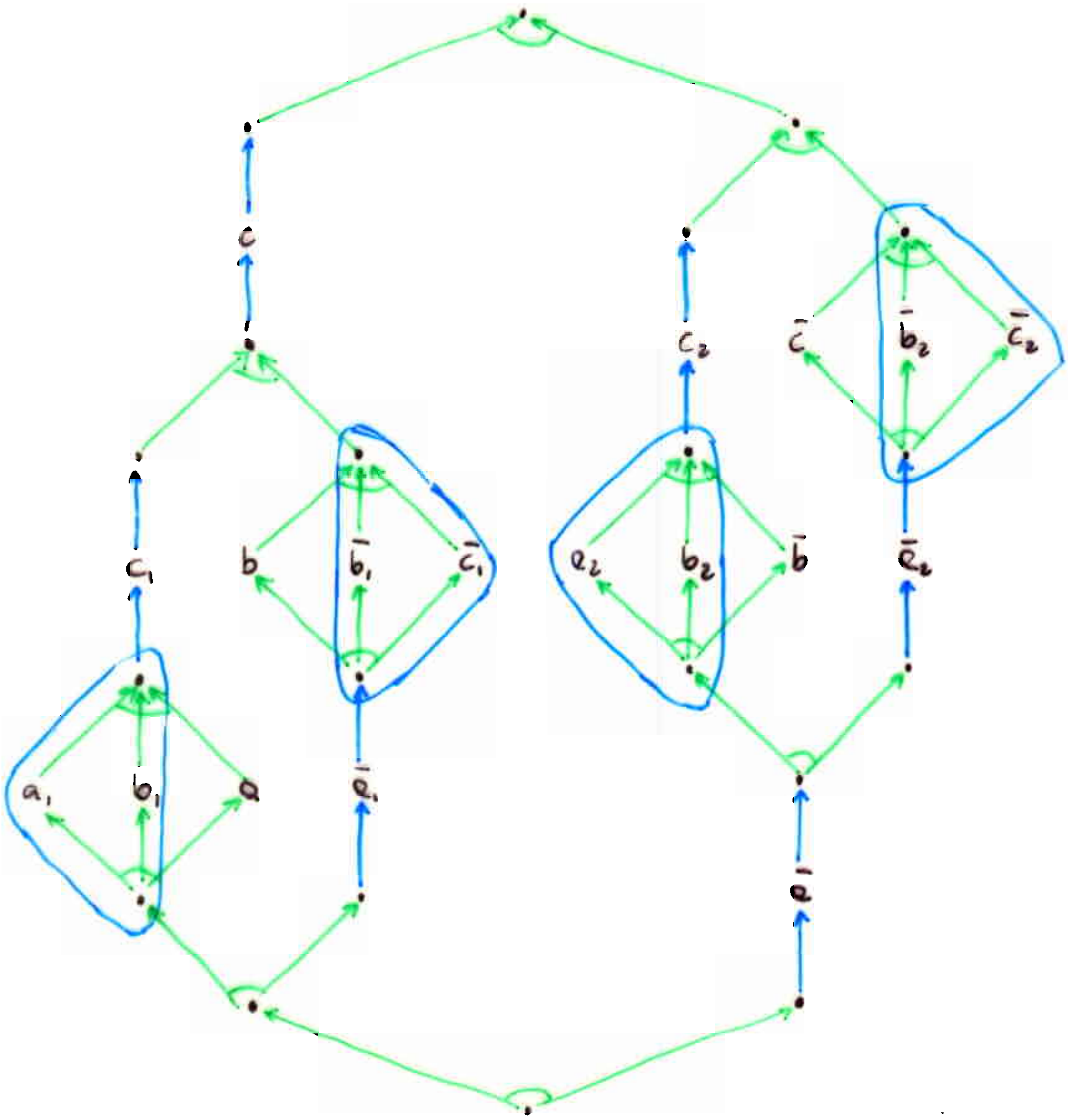
5 System SBV

SBV cannot be expressed in the sequent calculus



S_1

SBV cannot be expressed in the sequent calculus 2 of 3



S_2

 = lock of level 2

SBV cannot be expressed in the sequent calculus

- Theorem $S_1, S_2, \ldots$ are all provable in SBV if and only if one starts reducing from the locks

  Proof Use relation webs semantics

- Theorem There is no system in the (normal) sequent calculus which is equivalent to SBV

  Proof Given any sequent system, produce a structure S_K whose lock is deeper than the depth of the sequent system

The calculus of structures

Do we get a better proof theory?
Can we do better than the sequent calculus?

We observe:

- atomicity

- locality

- modularity:
  - in the rules
  - in decompositions
  - in cut elimination arguments

- we easily define logics that 'challenge' the sequent calculus