The Calculus of Structures

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The Calculus of Structures

Outline of the Course

1 What is the calculus of structures?

2 Classical Logic

atomicity, locality

3 Linear Logic

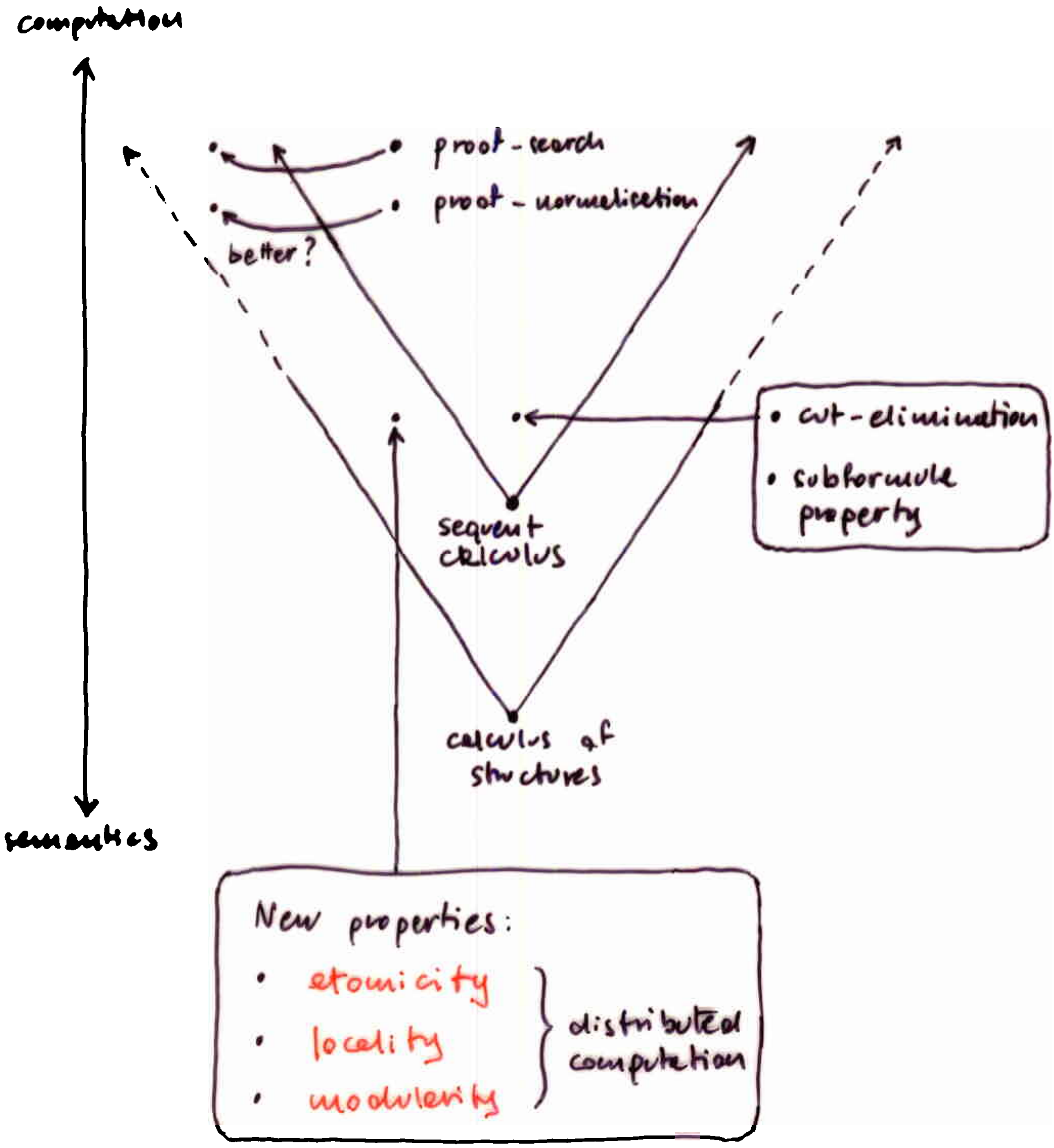
modularity

4 System SBV

seqentialism (and process algebras)

1 What is the calculus of structures?

It's a step back from the sequent calculus

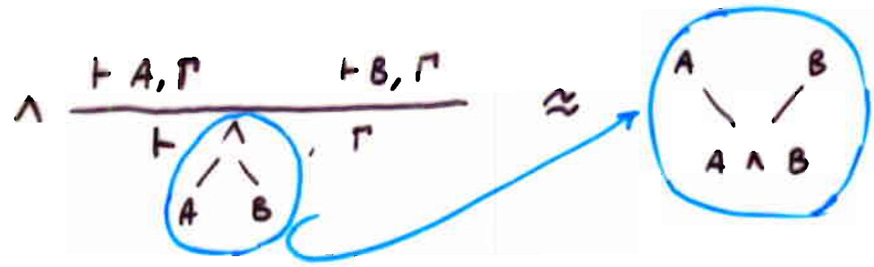


Do we get a better proof theory?

1 What is the calculus of structures? 2 of 16

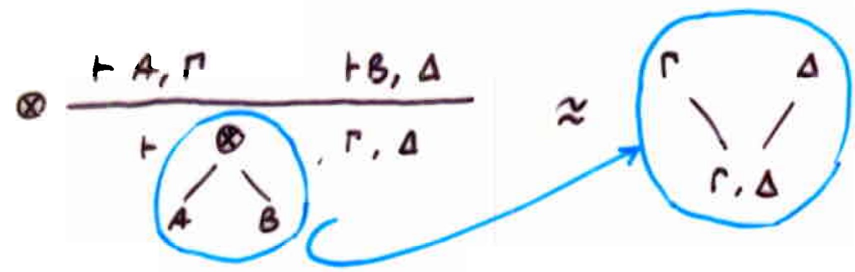
The sequent calculus is very committed to trees 1 of 2

- Example 1 "Additive" conjunction



the formula tree shapes the proof tree

- Example 2 "Multiplicative" conjunction



the formula tree induces an unwanted tree (in proof-search)

The sequent calculus is very committed to trees

- Example 3 Cut elimination

$$
\text{cut}\;\dfrac{\otimes\;\dfrac{\begin{matrix}\Pi_1\\ \vdots\\ \vdash A,\Gamma_1\end{matrix} \qquad \begin{matrix}\Pi_2\\ \vdots\\ \vdash B,\Gamma_2\end{matrix}}{\vdash A\otimes B,\Gamma_1,\Gamma_2} \qquad \parr\;\dfrac{\begin{matrix}\Pi_3\\ \vdots\\ \vdash A^\perp,B^\perp,\Delta\end{matrix}}{\vdash A^\perp \parr B^\perp,\Delta}}{\vdash \Gamma_1,\Gamma_2,\Delta}
$$

$$\downarrow$$

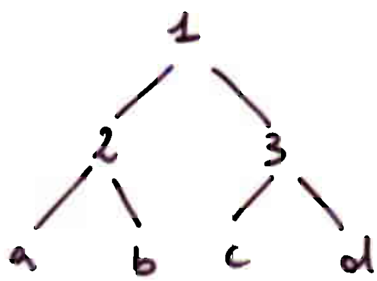
$$
\text{cut}\;\dfrac{\text{cut}\;\dfrac{\begin{matrix}\Pi_1\\ \vdots\\ \vdash A,\Gamma_1\end{matrix} \qquad \begin{matrix}\Pi_3\\ \vdots\\ \vdash A^\perp,B^\perp,\Delta\end{matrix}}{\vdash B^\perp,\Gamma_1,\Delta} \qquad \begin{matrix}\Pi_2\\ \vdots\\ \vdash B,\Gamma_2\end{matrix}}{\vdash \Gamma_1,\Gamma_2,\Delta}
$$

the formula tree decides the order of reductions

1 What is the calculus of structures?

Trees are unfriendly to distributed computation

- Example Suppose that
  - atoms are processors: a, b, c, d
  - communication flows through the tree structure



the communication workload of 1 is four times that of 2 and 3

- Main connectives create an asymmetry

- Step back: in the calculus of structures there are no main connectives

1 What is the calculus of structures?

There are no main connectives

- Example 1 "Additive" conjunction

$$\frac{\vdash (A \vee C) \wedge (B \vee C)}{\vdash (A \wedge B) \vee C}$$

- Example 2 "Multiplicative" conjunction

$$\frac{\vdash (A \mathbin{⅋} C) \otimes B}{\vdash (A \otimes B) \mathbin{⅋} C}$$

- Inference rules can be applied deep inside formulae

- There is a new top-down symmetry

- What happens to the subformula property?

Inference rules can be applied deep inside formulae

- Example 1 "Additive" conjunction

Rule

$$\rho\ \frac{S\{(A \vee C) \wedge (B \vee C)\}}{S\{(A \wedge B) \vee C\}}$$

can be applied as in

$$\rho\ \frac{((A \vee C) \wedge (B \vee C) \wedge D) \vee E}{(((A \wedge B) \vee C) \wedge D) \vee E}$$

- Example 2 "Multiplicative" conjunction

Rule

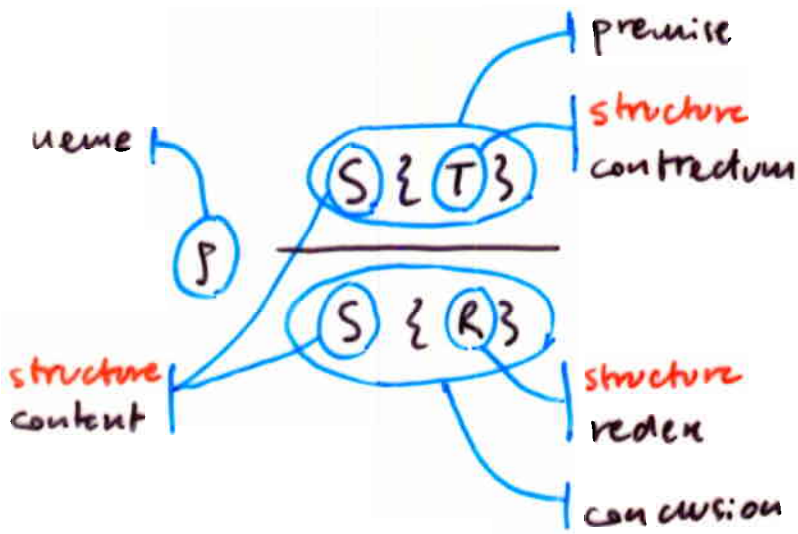
$$\rho\ \frac{S\{(A \parr C) \otimes B\}}{S\{(A \otimes B) \parr C\}}$$

can be applied as in

$$\rho\ \frac{\rho\ \dfrac{(A \parr C) \otimes (B \parr D)}{((A \parr C) \otimes B) \parr D}}{(A \otimes B) \parr C \parr D}$$

Inference rules can be applied deep inside formulae

- Inference rule ρ:



- The hole in $S\{\ \}$ does not appear inside a negation

- Rule ρ corresponds to $T \rightarrow R$

1 What is the calculus of structures?

Structures

- **Atoms** are positive or negative: $a, b, c, \dots, \bar{a}, \bar{b}, \bar{c}, \dots$

- **Structures** $P, Q, R, S, T, U, \dots$ are

$$S ::=$$

atoms	a
disjunctions	$\mid [\underbrace{S, \dots, S}_{>0}]$
conjunctions	$\mid (\underbrace{S, \dots, S}_{>0})$
other relations	$\mid \langle \underbrace{S; \dots; S}_{>0} \rangle \mid \dots$
units	$\mid t \mid f \mid 1 \mid \bot \mid \dots$
modalised structures	$\mid ?S \mid !S \mid \dots$
quantified structures	$\mid \exists x.S \mid \forall x.S \mid \dots$
negated structures	$\mid \bar{S}$

Structures

- Equations are imposed over structures:

Commutativity (not always)	$[R,T] = [T,R]$
associativity (always)	$\langle \vec{R}; \langle \vec{T} \rangle; \vec{U} \rangle = \langle \vec{R}; \vec{T}; \vec{U} \rangle$
De Morgan (always!)	$\overline{[R,T]} = (\bar{R},\bar{T})$
contextual closure	$R = T \Rightarrow S\{R\} = S\{T\}$

- Notation Braces are dropped when unnecessary. Example:

$$S[R,T] \quad \text{instead of} \quad S\{[R,T]\}$$

There is a new top-down symmetry

If

$$\rho{\downarrow}\,\frac{S\{T\}}{S\{R\}}$$

is a rule, corresponding to

$$T \rightarrow R$$

then

$$\rho{\uparrow}\,\frac{S\{\bar{R}\}}{S\{\bar{T}\}}$$

is also a rule, corresponding to

$$\bar{R} \rightarrow \bar{T}$$

There is a new top-down symmetry

Example In linear logic

$$\mathsf{p}{\downarrow}\ \frac{S\{![R,T]\}}{S[!R,?T]}$$

corresponds to

$$!(R \parr T) \multimap (!R \parr ?T)$$

and

$$\mathsf{p}{\uparrow}\ \frac{S(?R,!T)}{S\{?(R,T)\}}$$

corresponds to

$$\overline{(!R \parr ?T)} \multimap \overline{!(R \parr T)}$$

There is a new top-down symmetry

- Derivations (Δ) are chains of instances of inference rules

$$\begin{array}{c} \vdots \\ \pi\,\dfrac{U}{T} \\ \rho\,\dfrac{T}{R} \\ \vdots \end{array}$$

- There is a top-down symmetry. Example

$$\begin{array}{c} \vdots \\ \bar{\rho}\,\dfrac{\bar{R}}{\bar{T}} \\ \bar{\pi}\,\dfrac{\bar{T}}{\bar{U}} \\ \vdots \end{array}$$

is a valid derivation

What happens to the subformula property?

- Morally, it still holds if we design rules carefully. Example: in

$$\frac{S([R,U],T)}{S[(R,T),U]}$$

premise and conclusion are made of the same pieces

- Rules can still be finitary, either upwards, or downwards, or both

- Being finitary does not depend on having main connectives

1 What is the calculus of structure?

Do we get a better proof theory?

- We have some chances because:
  - we abolished the main connective idea
  - we are free to apply rules deeply
  - then we have more freedom
  - we also have a new symmetry!
  - we should see proofs in more detail

- But:
  - we have to be careful in designing systems!
    (we shouldn't abuse freedom)
  - it's still not clear whether we can do
    some good distributed computation

Recipe for a good system

- Choose disjunction and conjunction and make identity and cut.

Example: linear logic

- $[R, T]$ stands for $R \parr T$
  (R, T) stands for $R \otimes T$

- Establish

$$\mathsf{i}{\downarrow}\frac{S\{1\}}{S[R,\bar{R}]} \qquad \mathsf{i}{\uparrow}\frac{S(R,\bar{R})}{S\{\bot\}}$$

interaction down or identity — interaction up or cut

- This is your interaction fragment

Recipe for a good system

- Take each couple of dual logical relations, for example:
  - $\{R,T\}$ stands for $R \bullet T$

    $\langle R,T\rangle$ stands for $R \mathbin{\&} T$
  - and create the rules

$$\rho\!\downarrow \frac{S\langle [R,U],[T,V]\rangle}{S[\langle R,T\rangle,\{U,V\}]} \qquad \rho\!\uparrow \frac{S(\{R,T\},\langle U,V\rangle)}{S\{(R,U),(T,V)\}}$$

  - or, for example

$$\rho\!\downarrow \frac{S\{\forall x.[R,T]\}}{S[\forall x.R,\exists x.T]} \qquad \rho\!\uparrow \frac{S(\exists x.R,\forall x.T)}{S\{\exists x.(R,T)\}}$$

  - This is your core structure fragment

- Add the non-core structure fragment

A one-sided system into the calculus of structures

One-sided (Gentzen–Schütte) system GS1p	A system for classical logic in the calculus of structures (the "neit" system)
$\mathsf{id}\ \dfrac{}{\vdash A, \bar{A}}$	$\mathsf{t}\ \dfrac{}{t}$ $\mathsf{id}\ \dfrac{S}{(S, [A, \bar{A}])}$
$\vee_L\ \dfrac{\vdash \Gamma, A}{\vdash \Gamma, A \vee B}$ $\quad \vee_R\ \dfrac{\vdash \Gamma, B}{\vdash \Gamma, A \vee B}$ $\mathsf{w}\ \dfrac{\vdash \Gamma}{\vdash \Gamma, A}$	$\vee\ \dfrac{(S, \Gamma)}{(S, [\Gamma, A])}$
$\wedge\ \dfrac{\vdash \Gamma, A \quad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B}$	$\wedge\ \dfrac{(S, [\Gamma, A], [\Gamma, B])}{(S, [\Gamma, (A, B)])}$
$\mathsf{c}\ \dfrac{\vdash \Gamma, A, A}{\vdash \Gamma, A}$	$\mathsf{c}\ \dfrac{(S, [\Gamma, A, A])}{(S, [\Gamma, A])}$
$\mathsf{cut}\ \dfrac{\vdash \Gamma, A \quad \vdash \Delta, \bar{A}}{\vdash \Gamma, \Delta}$	$\mathsf{cut}\ \dfrac{(S, [\Gamma, A], [\Delta, \bar{A}])}{(S, [\Gamma, \Delta])}$

- Equations

$$[R] = (R) = R \qquad \overline{[R,T]} = (\bar R, \bar T)$$

$$\overline{(R,T)} = [\bar R, \bar T]$$

$$[\vec R, \vec T] = [\vec T, \vec R] \qquad \bar{\bar R} = R$$

$$(\vec R, \vec T) = (\vec T, \vec R)$$

$$[\vec R, [\vec T], \vec U] = [\vec R, \vec T, \vec U] \qquad \text{if } R = T \text{ then } S\{R\} = S\{T\}$$

$$(\vec R, (\vec T), \vec U) = (\vec R, \vec T, \vec U) \qquad [R, f] = R = (R, t)$$

$$\bar t = f$$

$$\bar f = t$$

- Example Prove $((A \supset B) \supset A) \supset A \equiv \overline{(\overline{(\bar A \vee B)} \vee A)} \vee A$

$$\equiv ((\bar A \vee B) \wedge \bar A) \vee A$$

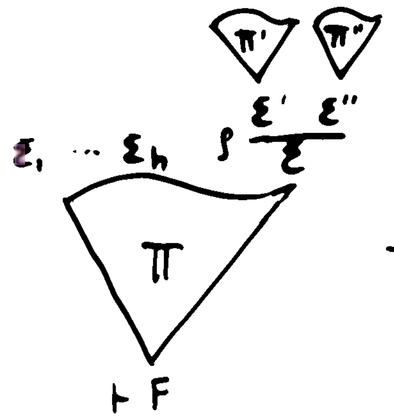
$$
\mathrm{c}\,\dfrac{\vee_R\,\dfrac{\vee_L\,\dfrac{\wedge\,\dfrac{\vee_L\,\dfrac{\mathrm{id}\,\dfrac{}{\vdash \bar A, A}}{\vdash \bar A \vee B, A} \qquad \mathrm{id}\,\dfrac{}{\vdash \bar A, A}}{\vdash (\bar A \vee B) \wedge \bar A, A}}{\vdash ((\bar A \vee B) \wedge \bar A) \vee A, A}}{\vdash ((\bar A \vee B) \wedge \bar A) \vee A, ((\bar A \vee B) \wedge \bar A) \vee A}}{\vdash ((\bar A \vee B) \wedge \bar A) \vee A}
$$

$$
\begin{array}{rc}
\mathrm{t} & \hline t \\
\mathrm{id} & \hline (t, [\bar A, A]) \\
\vee & \hline (t, [\bar A, B, A]) \\
\mathrm{id} & \hline (t, [\bar A, B, A], [\bar A, A]) \\
\wedge & \hline (t, [([\bar A, B], \bar A), A]) \\
\vee & \hline (t, [([\bar A, B], \bar A), A, A]) \\
=;\vee & \hline (t, [f, ([\bar A, B], \bar A), A, ([\bar A, B], \bar A), A]) \\
\mathrm{c} & \hline (t, [f, ([\bar A, B], \bar A), A]) \\
= & \hline [([\bar A, B], \bar A), A]
\end{array}
$$

The calculus of structures generalizes the one-sided sequent calculus

- It is trivial and non-interesting to port a system in the one-sided sequent calculus to the calculus of structures

- The translation works like this:



$$\longrightarrow$$

$$\begin{array}{c} \Pi' + \Pi'' \,\Vert \\ \rho\,\dfrac{(\Sigma_1,\dots,\Sigma_h,\Sigma',\Sigma'')}{(\Sigma_1,\dots,\Sigma_h,\Sigma)} \\ \Pi \,\Vert \\ F \end{array}$$

- Symmetry is not exploited!

- Deepness is not exploited!

- Can we do better than the sequent calculus?

A deep, symmetric system

- Let's apply our recipe!

- We keep the equations we have already

- Interaction

$$\mathrm{i}\!\downarrow \frac{S\{t\}}{S[R,\bar{R}]} \qquad \mathrm{i}\!\uparrow \frac{S(R,\bar{R})}{S\{f\}}$$

- Core structure

$$\mathrm{s}\!\downarrow \frac{S([R,U],[T,V])}{S[(R,T),U,V]} \qquad \mathrm{s}\!\uparrow \frac{S([R,T],U,V)}{S[(R,U),(T,V)]}$$

- Non-core structure (here we have to be creative)

$$\mathrm{w}\!\downarrow \frac{S\{f\}}{S\{R\}} \qquad \mathrm{w}\!\uparrow \frac{S\{R\}}{S\{t\}}$$

$$\mathrm{c}\!\downarrow \frac{S[R,R]}{S\{R\}} \qquad \mathrm{c}\!\uparrow \frac{S\{R\}}{S(R,R)}$$

A deep, symmetric system

- **Definition** A system ($\mathcal{S}$) is a set of inference rules

- **Definition** A rule ρ is strongly admissible for a system $\mathcal{S}$ if $\rho \notin \mathcal{S}$ and for every instance $\rho\,\frac{T}{R}$ there is a derivation $\begin{array}{c} T \\ \Vert\,\mathcal{S} \\ R \end{array}$

- **Definition** This rule is called switch : $\quad s\,\frac{S([R,U],T)}{S[(R,T),U]}$

- **Proposition** $s{\downarrow}$ and $s{\uparrow}$ are strongly admissible for s

  Proof

$$
s\,\frac{s\,\frac{S([R,U],[T,V])}{S[([R,U],T),V]}}{S[(R,T),U,V]}
\qquad
s\,\frac{s\,\frac{S([R,T],U,V)}{S([(R,U),T],V)}}{S[(R,U),(T,V)]}
$$

- **Remark** Switch is self-dual

- **Remark** s is a special case both of $s{\downarrow}$ and $s{\uparrow}$

A deep, symmetric system

- We have a system, let's call it CLC

down | up

$$\mathsf{i}{\downarrow}\ \frac{S\{t\}}{S[R,\bar{R}]} \qquad \mathsf{i}{\uparrow}\ \frac{S(R,\bar{R})}{S\{f\}}$$

interaction

structure

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

core

non-core

$$\mathsf{w}{\downarrow}\ \frac{S\{f\}}{S\{R\}} \qquad \mathsf{w}{\uparrow}\ \frac{S\{R\}}{S\{t\}}$$

$$\mathsf{c}{\downarrow}\ \frac{S[R,R]}{S\{R\}} \qquad \mathsf{c}{\uparrow}\ \frac{S\{R\}}{S(R,R)}$$

- Is this classical logic? Yes: let's see

- Remark $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}\}$ (and $\{\mathsf{i}{\downarrow}, \mathsf{s}\}$) is multiplicative linear logic

A deep, symmetric system

- **Theorem** Every derivation in GS1p can be transformed into a derivation in CLC, and if it is cut-free, it remains cut-free

  Proof CLC is more general than the unit system we saw already.

  (just pay attention to contraction in the rule $\wedge$ and notice that

$$
\dfrac{\dfrac{\dfrac{(S,[\Gamma,A],[\Delta,\bar{A}])}{(S,[\Delta,([\Gamma,A],\bar{A})])}\, s}{(S,[\Gamma,\Delta,(A,\bar{A})])}\, s}{(S,[\Gamma,\Delta])}\, i\uparrow
$$

)

- Then, CLC is classical logic, because every rule is sound

- Is there any use for $w\uparrow$ and $c\uparrow$?

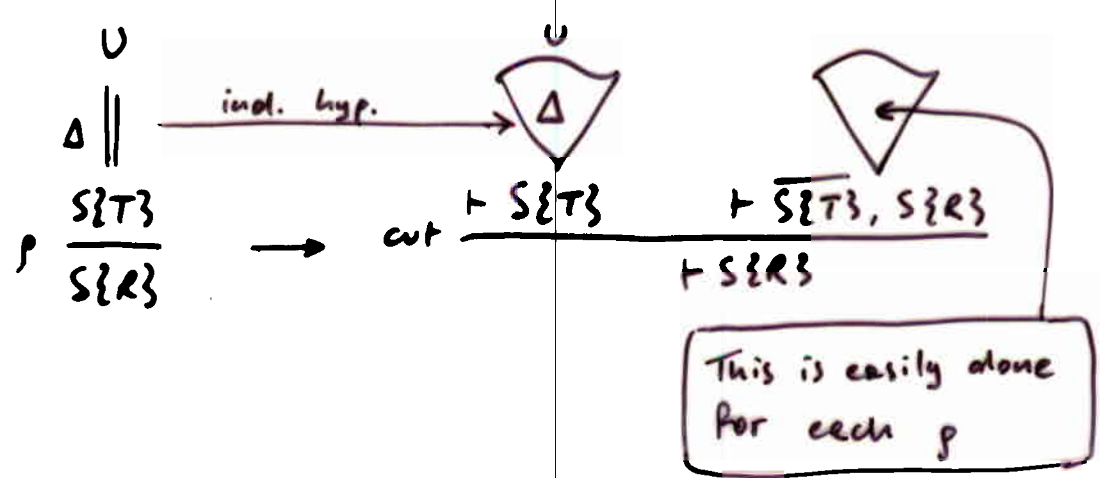
A deep, symmetric system

- What about cut elimination?

- Idea: let's exploit the sequent calculus

- Theorem Every derivation in CLC can be transformed into a derivation in GS1p

Proof



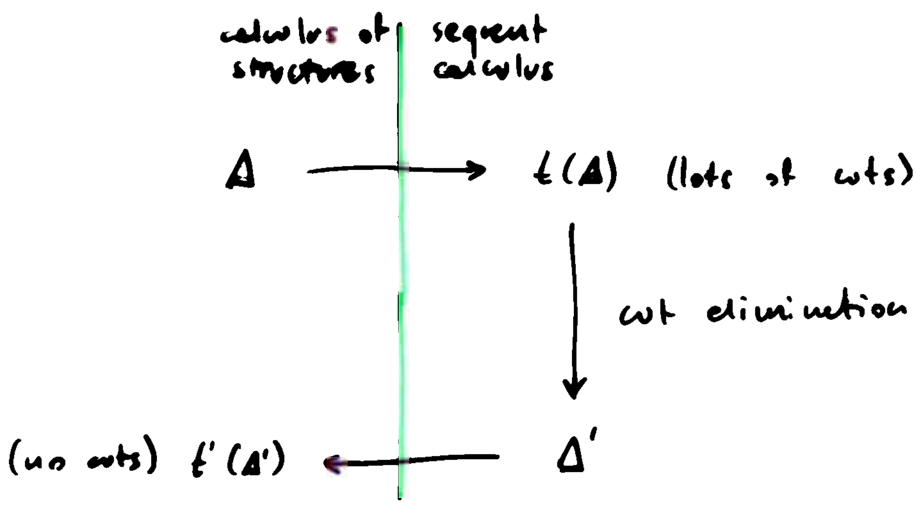
- Let's break the symmetry!

- **Definition** A *proof* is a derivation whose topmost structure is (equivalent to) t

- **Definition** An inference rule ρ is *(weakly) admissible* for a system $\mathcal{S}$ if $\rho \notin \mathcal{S}$ and for every proof $\begin{array}{c} \Pi \,\Vert\, \mathcal{S}\cup\{\rho\} \\ R \end{array}$ there exists a proof $\begin{array}{c} \Pi \,\Vert\, \mathcal{S} \\ R \end{array}$

- **Theorem** $i\uparrow$ is admissible for $\{i\downarrow, s, w\downarrow, c\downarrow\}$ (and there is an algorithmic transformation for it)

Proof



A deep, symmetric system

- Do we have a better system than classical logic in the sequent calculus?

  Perhaps, but still ...

- Do we have a better, or interesting, cut elimination procedure?

  Well ...

- Symmetry still is not fully exploited!

- Deepness still is not fully exploited!

Atomicity

- Consider

$$
\mathsf{i}{\downarrow}\,\frac{S\{t\}}{S[(R,T),\bar{R},\bar{T}]} \quad\longrightarrow\quad \mathsf{s}\,\frac{\mathsf{s}\,\frac{\mathsf{i}{\downarrow}\,\frac{\mathsf{i}{\downarrow}\,\frac{S\{t\}}{S[T,\bar{T}]}}{S([R,\bar{R}],[T,\bar{T}])}}{S[([R,\bar{R}],T),\bar{T}]}}{S[(R,T),\bar{R},\bar{T}]} .
$$

The i↓s become "smaller", so they eventually can be replaced by

$$
\mathsf{ai}{\downarrow}\,\frac{S\{t\}}{S[a,\bar{a}]} .
$$

This rule is called atomic interaction

- Theorem i↓ is strongly admissible for {ai↓, s}

- Nothing unexpected!

Atomicity

- Consider

$$\text{i}{\uparrow}\,\frac{S([R,T],\bar{R},\bar{T})}{S\{f\}} \quad\longrightarrow\quad \text{s}\,\frac{\text{s}\,\dfrac{\text{i}{\uparrow}\,\dfrac{S([R,T],\bar{R},\bar{T})}{S([(R,\bar{R}),T],\bar{T})}}{S[(R,\bar{R}),(T,\bar{T})]}}{\text{i}{\uparrow}\,\dfrac{S(T,\bar{T})}{S\{f\}}} \;.$$

The i↑s, too, become "smaller"; we can replace them by

$$\text{ai}{\uparrow}\,\frac{S(a,\bar{a})}{S\{f\}} \;.$$

This rule is called atomic cointeraction

- Theorem i↑ is strongly admissible for $\{\text{ai}{\uparrow}, \text{s}\}$

- This property, due to symmetry, we can exploit!

Atomicity of cointeraction (cut)

- Consequences:
  - a simpler cut elimination proof
  - decomposition theorems

- Curiosities:
  - a different relation between cut, subformula property, and finitarguers
  - a simple consistency proof

Finitaryness

- In the sequent calculus finitaryness (going up) corresponds to the subformula property.

Example

$$\wedge \frac{\vdash \Gamma, A \qquad \vdash \Gamma, B}{\vdash \Gamma, A \wedge B} \qquad\qquad \text{cut} \frac{\vdash \Gamma, A \qquad \vdash \Delta, \bar{A}}{\vdash \Gamma, \Delta}$$

- finitary
- A and B are subformulae of $A \wedge B$

- non-finitary
- A is not necessarily a subformula of the conclusion

- In the calculus of structures there is no subformula property, but still all inference rules for classical logic are finitary (going up), except for

$$\text{w}\uparrow \frac{S\{R\}}{S\{t\}} \quad \text{and} \quad \text{i}\uparrow \frac{S(R, \bar{R})}{S\{f\}}$$

$$\left(\text{or } \text{ai}\uparrow \frac{S(a, \bar{a})}{S\{f\}} \right)$$

Finitaryness

- Rules in the core are always finitary!
  (They just "reshuffle" logical relations)

- Corules in the non-core up fragment are always strongly admissible for their duals, plus switch and interactions:

$$\rho\uparrow \frac{S\{T\}}{S\{R\}} \quad \longrightarrow \quad \begin{array}{l} \\ \mathsf{i}\downarrow \\ \rho\downarrow \\ \mathsf{s} \\ \mathsf{i}\uparrow \\ \end{array} \begin{array}{c} S\{T\} \\ \hline S(T,[R,\bar{R}]) \\ \hline S(T,[R,\bar{T}]) \\ \hline S[R,(T,\bar{T})] \\ \hline S\{R\} \end{array}$$

- Then the only infinitary rule we are left with is

$$\mathsf{ai}\uparrow \frac{S(a,\bar{a})}{S\{f\}}$$

Finitaryness

- Consider the finitary atomic cointeraction rule:

$$\mathsf{fai}\uparrow \frac{S(a,\bar{a})}{S\{f\}} \quad \text{where } a \text{ or } \bar{a} \text{ appears in } S\{\ \}$$

- It is easy to eliminate all $\mathsf{ai}\uparrow$ instances that are not $\mathsf{fai}\uparrow$ instances, in proofs

$$\begin{array}{c} t \\ \| \\ \mathsf{ai}\uparrow \dfrac{S(a,\bar{a})}{S\{f\}} \\ \| \\ R \end{array}$$

replace here all a's with t and all $\bar{a}$'s with f: the proof remains valid!

bottommost $\mathsf{ai}\uparrow$ which is not an $\mathsf{fai}\uparrow$

proceed inductively upwards in the proof.

- Theorem: Replacing $\mathsf{ai}\uparrow$ by $\mathsf{fai}\uparrow$ does not affect provability

- Finitaryness does not morally depend on full-blown cut elimination!

A simple consistency proof

- **Theorem** Propositional classical logic is consistent

  Proof We cannot get $\begin{array}{c} t \\ \| \\ f \end{array}$ when using fai↑

- **Theorem** If R is provable then $\bar{R}$ is not provable

  Proof Suppose we have

$$\begin{array}{c} t \\ \pi_1 \, \| \\ R \end{array} \quad \text{and} \quad \begin{array}{c} t \\ \pi_2 \, \| \\ \bar{R} \end{array} .$$

Then we make $\begin{array}{c} t \\ \pi_1+\pi_2 \, \| \\ (R, \bar{R}) \end{array}$ and then

we flip it : $\begin{array}{c} [R, \bar{R}] \\ \| \\ f \end{array} .$

Then we can make

$$\mathsf{i}{\downarrow}\,\dfrac{t}{\begin{array}{c} [R, \bar{R}] \\ \| \\ f \end{array}} \; :$$

absurd.

Exploiting deepness

- The following rule is called medial:

$$m \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

- Medial is self-dual

- Look at

$$\mathsf{c}{\downarrow}\frac{\mathsf{c}{\downarrow}\frac{S[P,P,Q,Q]}{S[P,P,Q]}}{S[P,Q]} \quad \text{and} \quad \mathsf{c}{\downarrow}\frac{\mathsf{c}{\downarrow}\frac{m\frac{S[(P,Q),(P,Q)]}{S([P,P],[Q,Q])}}{S([P,P],Q)}}{S(P,Q)} .$$

By medial, contractions get "smaller"

- The following rules are called atomic contraction and atomic cocontraction:

$$\mathsf{ac}{\downarrow}\frac{S[a,a]}{S\{a\}} \quad \text{and} \quad \mathsf{ac}{\uparrow}\frac{S\{a\}}{S(a,a)}$$

- Theorem $\mathsf{c}{\downarrow}$ is strongly admissible for $\{\mathsf{ac}{\downarrow}, m\}$, and dually

Exploiting deepness

- Deepness is essential for getting atomic contraction

- In the sequent calculus, it is impossible to get atomic contraction

- By the way, weakening is easily reduced to atomic form:

$$\mathsf{w}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{S\{f\}}{S[f,a]}}{S[P,a]} \quad \text{and} \quad \mathsf{w}\!\downarrow \frac{\mathsf{w}\!\downarrow \dfrac{\mathsf{ac}\!\uparrow \dfrac{S\{f\}}{S(f,f)}}{S(f,a)}}{S(P,a)} \;;$$

you can treat this with an equation, too

and dually for coweakening

System SKS

This is classical logic

down | up

$$\mathsf{ai}{\downarrow}\ \frac{S\{t\}}{S[a,\bar{a}]} \qquad \mathsf{ai}{\uparrow}\ \frac{S(a,\bar{a})}{S\{f\}}$$

interaction

structure

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

core

non-core

$$\mathsf{m}\ \frac{S[(R,U),(T,V)]}{S([R,T],[U,V])}$$

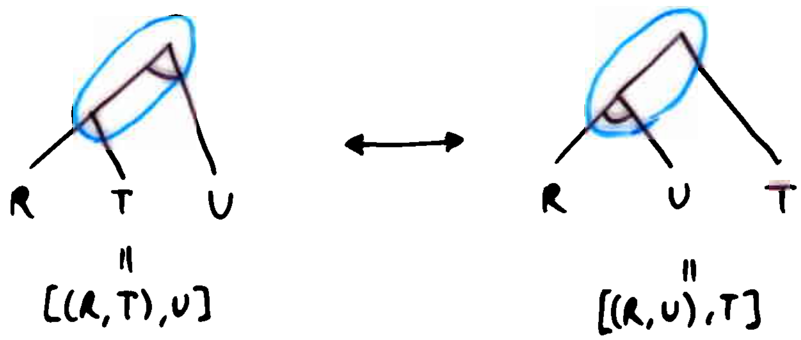
$$\mathsf{aw}{\downarrow}\ \frac{S\{f\}}{S\{a\}} \qquad \mathsf{aw}{\uparrow}\ \frac{S\{a\}}{S\{t\}}$$

$$\mathsf{ac}{\downarrow}\ \frac{S[a,a]}{S\{a\}} \qquad \mathsf{ac}{\uparrow}\ \frac{S\{a\}}{S(a,a)}$$

Locality

- Let's call locality the property of a rule requiring bounded effort to be applied.

  Example: switch



- Locality depends on the representation

- Atomicity can be a special form of locality

- There still is much to do for distributed computation (but look at relational fields)

- Applications in complexity?

Cut elimination

Why cut elimination is different than in the sequent calculus?

Because in the sequent calculus the main connective "drives" the reduction:

$$
\dfrac{\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A} \qquad \dfrac{\Pi_2}{\vdash \Gamma, B}}{\vdash \Gamma, A \wedge B}\,\wedge \qquad \dfrac{\dfrac{\Pi_3}{\vdash \Delta, \bar{A}}}{\vdash \Delta, \bar{A} \vee \bar{B}}\,\vee_L}{\vdash \Gamma, \Delta}\,\text{cut}
$$

$$\downarrow$$

$$
\dfrac{\dfrac{\Pi_1}{\vdash \Gamma, A} \qquad \dfrac{\Pi_3}{\vdash \Delta, \bar{A}}}{\vdash \Gamma, \Delta}\,\text{cut}
$$

2 Classical logic

Cut elimination

- In the calculus of structures:

$$
\begin{array}{c}
\Pi \\
\hline
\\[-1em]
\text{s}\;\dfrac{(a,[d,(b,c,[\bar{a},\bar{b},\bar{c}])])}{[d,(a,b,c,[\bar{a},\bar{b},\bar{c}])]} \\
= \;\overline{\qquad\qquad\qquad\qquad} \\
\text{if}\;\dfrac{S(R,T,[\bar{R},\bar{T}])}{S\{f\}}
\end{array}
\qquad
\begin{array}{l}
R = (a,b) \\
T = c \\
S = [d,\{\ \}]
\end{array}
$$

What are we supposed to do??

- Freedom has a price

- Atomicity helps a lot!

Cut elimination

Theorem ai↑ is admissible

Proof

1 Transform cuts into shallow cuts:

$$\text{ai}{\uparrow}_s \frac{[S, (a, \bar{a})]}{S}$$

2 Permute up super cuts:

$$\text{sai}{\uparrow} \frac{(S_1, S_2)}{[(S_1', m \cdot \bar{a}), (S_2', n \cdot a)]}$$

where $n \cdot a = \underbrace{(a, \ldots, a)}_{n \text{ times}}$

and S_1' is obtained from S_1 by replacing some a's by f;

and S_2' is obtained from S_2 by replacing some $\bar{a}$'s by f

Decompositions

- Theorems
  - For every $\begin{array}{c} T \\ \Vert\ \mathsf{SKS} \\ R \end{array}$ there is a $\begin{array}{c} T \\ \Vert\ \{ai\downarrow\} \\ V \\ \Vert\ \mathsf{SKS}\setminus\{ai\downarrow, ai\uparrow\} \\ U \\ \Vert\ \{ai\uparrow\} \\ R \end{array}$
  - For every $\begin{array}{c} T \\ \Vert\ \mathsf{SKS} \\ R \end{array}$ there is a $\begin{array}{c} T \\ \Vert\ \{ac\uparrow\} \\ V \\ \Vert\ \mathsf{SKS}\setminus\{ac\downarrow, ac\uparrow\} \\ U \\ \Vert\ \{ac\downarrow\} \\ R \end{array}$

- One cannot do these things in the sequent calculus

- We start seeing some modularity

Is there any use for coweakening and cocontraction?

Yes:

- We saw ac↑ already for getting aw↓ (but that use was trivial)

- In interpolation theorems!:

  It is always possible to generate derivations such that, if $\begin{array}{c} T \\ \| \\ R \end{array}$, then

$$\begin{array}{c l} T & \\ \| & \uparrow \ \text{growing} \\ U & \leftarrow \ \text{interpolant:} \quad T \Rightarrow U \Rightarrow R \\ \| & \downarrow \ \text{growing} \\ R & \end{array}$$

3 Linear logic

Multiplicative exponential linear logic

System SELS

down	up
$\mathsf{ai}\!\downarrow \dfrac{S\{1\}}{S[a,\bar{a}]}$	$\mathsf{ai}\!\uparrow \dfrac{S(a,\bar{a})}{S\{\bot\}}$

interaction

structure

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

down	up
$\mathsf{p}\!\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]}$	$\mathsf{p}\!\uparrow \dfrac{S(?R,!T)}{S\{?(R,T)\}}$

core

non-core

down	up
$\mathsf{w}\!\downarrow \dfrac{S\{\bot\}}{S\{?R\}}$	$\mathsf{w}\!\uparrow \dfrac{S\{!R\}}{S\{1\}}$
$\mathsf{b}\!\downarrow \dfrac{S[?R,R]}{S\{?R\}}$	$\mathsf{b}\!\uparrow \dfrac{S\{!R\}}{S(!R,R)}$

+ decidable equations, especially $\begin{cases} ??R = ?R \\ !!R = !R \end{cases}$

Multiplicative exponential linear logic

- Interactions are atomic

- Promotion is local!

- Absorption (i.e., contraction) is not atomic

- Modularity starts to manifest itself: each of $ai{\uparrow}$, $p{\uparrow}$, $w{\uparrow}$ and $b{\uparrow}$ is admissible for the down fragment and can be shown admissible independently (to a certain extent)

- So, there are $2^4 = 16$ equivalent systems whose properties are known

Modularity: decompositions

Theorem For every $\begin{array}{c} T \\ \| \\ R \end{array}$

there is
$$\begin{array}{c} T \\ \| \; \{b\uparrow\} \\ T_1 \\ \| \; \{w\downarrow\} \\ T_2 \\ \| \; \{ai\downarrow\} \\ T_3 \\ \| \; \text{core of SELS} \\ R_3 \\ \| \; \{ai\uparrow\} \\ R_2 \\ \| \; \{w\uparrow\} \\ R_1 \\ \| \; \{b\downarrow\} \\ R \end{array}$$
and
$$\begin{array}{c} T \\ \| \; \{b\uparrow\} \\ T_1 \\ \| \; \{w\uparrow\} \\ T_2 \\ \| \; \{ai\downarrow\} \\ T_3 \\ \| \; \text{core of SELS} \\ R_3 \\ \| \; \{ai\uparrow\} \\ R_2 \\ \| \; \{w\downarrow\} \\ R_1 \\ \| \; \{b\downarrow\} \\ R \end{array}$$

Proof Difficult!

3 Linear logic

Full linear logic

- We apply all our techniques and get:

- A system, called SELS, with 34 rules, 16 of which in the up-non-down fragment (all admissible, of course): so we have $2^{16} = 65,536$ equivalent systems

- All rules are local (or atomic), including contractions

- All rules follow our recipe + medial + contraction + weakening, so the system is big but very uniform

Full linear logic

up | down

$$\mathsf{ai}\!\downarrow \frac{S\{1\}}{S[a,\bar a]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar a)}{S\{\bot\}}$$

interaction

structure

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{d}\!\downarrow \frac{S\bullet([R,U],[T,V])\bullet}{S[\bullet(R,T)\bullet,\bullet[U,V]\bullet]} \qquad \mathsf{d}\!\uparrow \frac{S(\bullet[R,U]\bullet,\bullet(T,V)\bullet)}{S\bullet[(R,T),(U,V)]\bullet}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}$$

core

non-core

$$\mathsf{aw}\!\downarrow \frac{S\{0\}}{S\{a\}} \qquad \mathsf{ac}\!\downarrow \frac{S\bullet[a,a]\bullet}{S\{a\}} \qquad \mathsf{ac}\!\uparrow \frac{S\{a\}}{S\bullet(a,a)\bullet} \qquad \mathsf{aw}\!\uparrow \frac{S\{a\}}{S\{\top\}}$$

$$\mathsf{l_0}\!\downarrow \frac{S\{0\}}{S[0,0]} \qquad \mathsf{l}\!\downarrow \frac{S\bullet[[R,U],[T,V]]\bullet}{S[\bullet[R,T]\bullet,\bullet[U,V]\bullet]} \qquad \mathsf{l}\!\uparrow \frac{S(\bullet(R,U)\bullet,\bullet(T,V)\bullet)}{S\bullet((R,T),(U,V))\bullet} \qquad \mathsf{l_0}\!\uparrow \frac{S(\top,\top)}{S\{\top\}}$$

$$\mathsf{k_0}\!\downarrow \frac{S\{0\}}{S(0,0)} \qquad \mathsf{k}\!\downarrow \frac{S\bullet[(R,U),(T,V)]\bullet}{S(\bullet[R,T]\bullet,\bullet[U,V]\bullet)} \qquad \mathsf{k}\!\uparrow \frac{S[\bullet(R,U)\bullet,\bullet(T,V)\bullet]}{S\bullet([R,T],[U,V])\bullet} \qquad \mathsf{k_0}\!\uparrow \frac{S[\top,\top]}{S\{\top\}}$$

$$\mathsf{m_0}\!\downarrow \frac{S\{0\}}{S\bullet(0,0)\bullet} \qquad \mathsf{m}\,\frac{S\bullet[\bullet(R,U)\bullet,\bullet(T,V)\bullet]\bullet}{S\bullet(\bullet[R,T]\bullet,\bullet[U,V]\bullet)\bullet} \qquad \mathsf{m_0}\!\uparrow \frac{S\bullet[\top,\top]\bullet}{S\{\top\}}$$

$$\mathsf{x_0}\!\downarrow \frac{S\{0\}}{S\{?0\}} \qquad \mathsf{x}\!\downarrow \frac{S\bullet[?R,?T]\bullet}{S\{?\bullet[R,T]\bullet\}} \qquad \mathsf{x}\!\uparrow \frac{S\{!\bullet(R,T)\bullet\}}{S\bullet(!R,!T)\bullet} \qquad \mathsf{x_0}\!\uparrow \frac{S\{!\top\}}{S\{\top\}}$$

$$\mathsf{y_0}\!\downarrow \frac{S\{0\}}{S\{!0\}} \qquad \mathsf{y}\!\downarrow \frac{S\bullet[!R,!T]\bullet}{S\{!\bullet[R,T]\bullet\}} \qquad \mathsf{y}\!\uparrow \frac{S\{?\bullet(R,T)\bullet\}}{S\bullet(?R,?T)\bullet} \qquad \mathsf{y_0}\!\uparrow \frac{S\{?\top\}}{S\{\top\}}$$

$$\mathsf{z_0}\!\downarrow \frac{S\{\bot\}}{S\{?0\}} \qquad \mathsf{z}\!\downarrow \frac{S[?R,T]}{S\{?\bullet[R,T]\bullet\}} \qquad \mathsf{z}\!\uparrow \frac{S\{!\bullet(R,T)\bullet\}}{S(!R,T)} \qquad \mathsf{z_0}\!\uparrow \frac{S\{!\top\}}{S\{1\}}$$

System SLLS

Cut elimination

It always holds. How do we prove it?

MLL: splitting

MELL: decomposition + splitting

MALL: splitting

LL: by translation to the sequent calculus

4 System SBV

Idea

- CCS is a language for distributed computation where

$$a.b \mid \bar{a}.\bar{b} \to 0$$

- Can we make a logic out of this?

- If so, we want $\overline{a.b} = \bar{a}.\bar{b}$

- Then "." is a non-commutative self-dual logical relation

- Problem: getting this in the sequent calculus is very difficult (let's say impossible, see later)

Recipe!

- Ingredients:

  2 commutative dual logical relations

  1 non-commutative self-dual logical relation

  1 self-dual unit common to all relations

- Recipe:

  Just create an interaction and a core structure fragment (everything is multiplicative, for now)

- We get a very simple system whose proof theory is extremely intricate

- The system is atomic and local

4 System SBV

The system

- Rules:

down | up

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[R,\bar{R}]} \qquad \mathsf{ai}\!\uparrow \frac{S(R,\bar{R})}{S\{\circ\}}$$

interaction

structure (core)

$$\mathsf{s}\ \frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R;T\rangle,\langle U;V\rangle)}{S\langle (R,U);(T,V)\rangle}$$

- Equations:

Commutativity:

$$[\vec{R},\vec{T}] = [\vec{T},\vec{R}]$$
$$(\vec{R},\vec{T}) = (\vec{T},\vec{R})$$

Associativity:

$$[\vec{R},[\vec{T}]] = [\vec{R},\vec{T}]$$
$$(\vec{R},(\vec{T})) = (\vec{R},\vec{T})$$
$$\langle \vec{R};\langle \vec{T}\rangle;\vec{U}\rangle = \langle \vec{R};\vec{T};\vec{U}\rangle$$

Context closure:

if $R=T$ then $S\{R\}=S\{T\}$

Unit:

$$R = [R,\circ] = (R,\circ) = \langle R;\circ\rangle = \langle \circ;R\rangle$$

Negation:

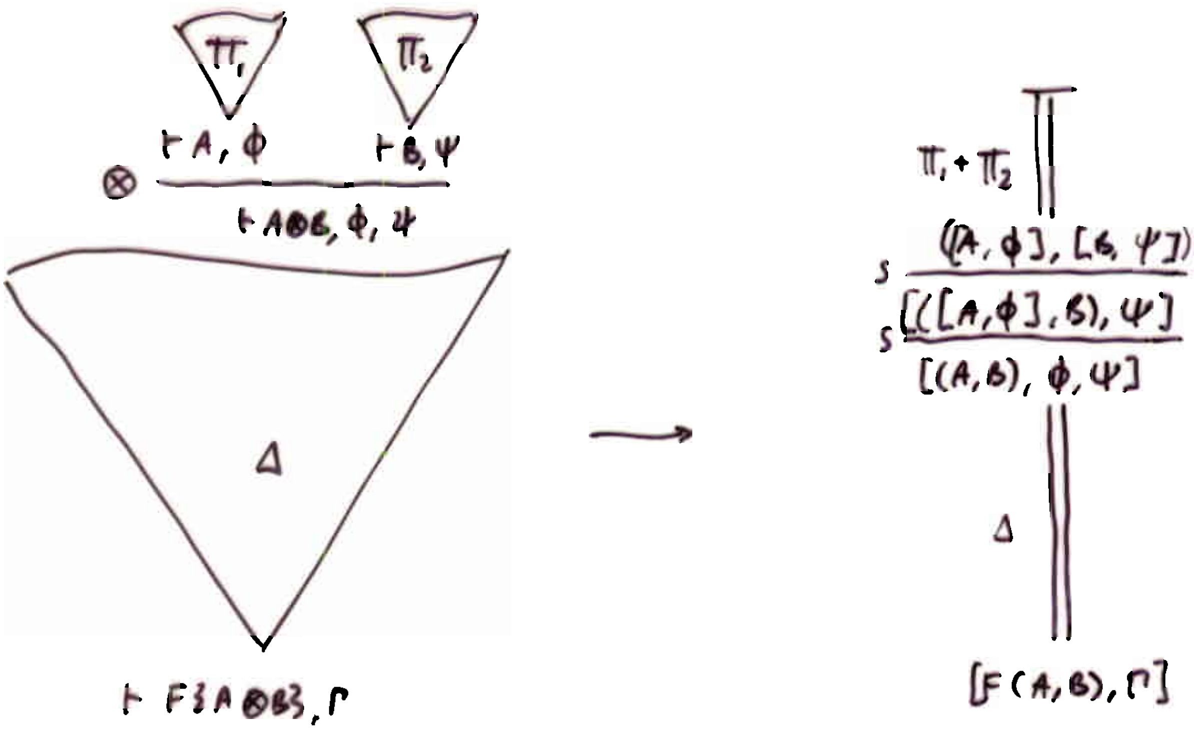
$$\bar{\bar{R}} = R$$
$$\overline{[R,T]} = (\bar{R},\bar{T})$$
$$\overline{(R,T)} = [\bar{R},\bar{T}]$$
$$\overline{\langle R;T\rangle} = \langle \bar{R};\bar{T}\rangle$$
$$\bar{\circ} = \circ$$

Singleton:

$$[R] = (R) = \langle R\rangle = R$$

Cut elimination by splitting

The idea comes from the sequent calculus:



Cut Elimination by splitting

- **Definition** $BV = \{ai\downarrow, s, q\downarrow\}$

- **Theorem** (Splitting)

  - If $\begin{array}{c} \Pi\, BV \\ S\langle R;T\rangle \end{array}$ then $\begin{array}{c} [\{\ \},\langle S_1;S_2\rangle] \\ \Vert\, BV \\ S\{\ \} \end{array}$, $\begin{array}{c} \Pi\, BV \\ [R,S_1] \end{array}$ and $\begin{array}{c} \Pi\, BV \\ [T,S_2] \end{array}$

  - If $\begin{array}{c} \Pi\, BV \\ S(R,T) \end{array}$ then $\begin{array}{c} [\{\ \},S_1,S_2] \\ \Vert\, BV \\ S\{\ \} \end{array}$, $\begin{array}{c} \Pi\, BV \\ [R,S_1] \end{array}$ and $\begin{array}{c} \Pi\, BV \\ [T,S_2] \end{array}$

Proof Complex, but uniform

4 System SBV

Cut elimination by splitting

- **Theorem** $\mathsf{ai}\uparrow$ is admissible for BV

  Proof Splitting

- **Theorem** $\mathsf{q}\uparrow$ is admissible for BV

  Proof Splitting

- SBV and BV (and BV $\cup \{\mathsf{ai}\uparrow\}$ and BV $\cup \{\mathsf{q}\uparrow\}$) are equivalent

4 System SBV

Decomposition

Theorem

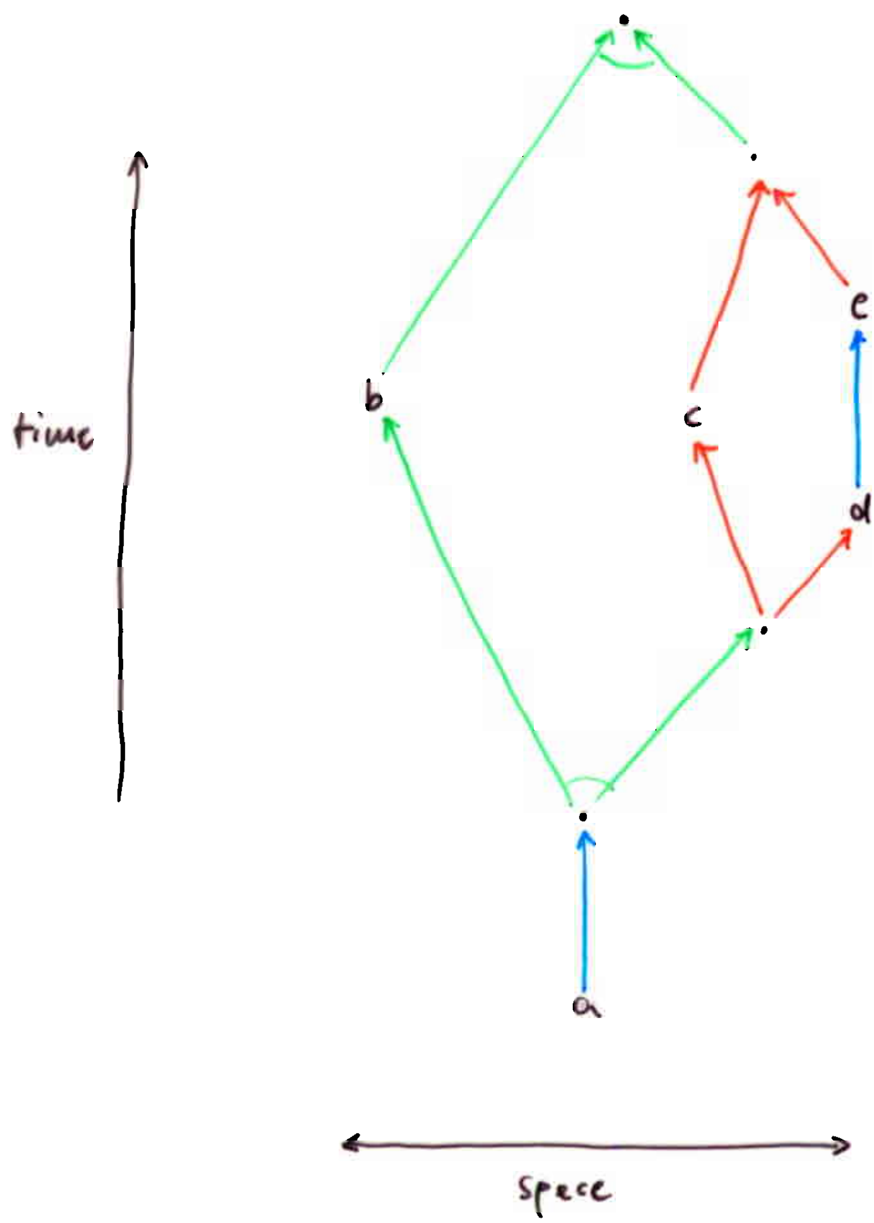
If $\begin{array}{c} T \\ \| \,\mathsf{SBV} \\ R \end{array}$ then $\begin{array}{l} T \\ \| \,\{ai\downarrow\} \\ T' \\ \| \,\text{core of } \mathsf{SBV} = \{s, q\downarrow, q\uparrow\} \\ R' \\ \| \,\{ai\uparrow\} \\ R \end{array}$

Proof Permutations

4 System SBV

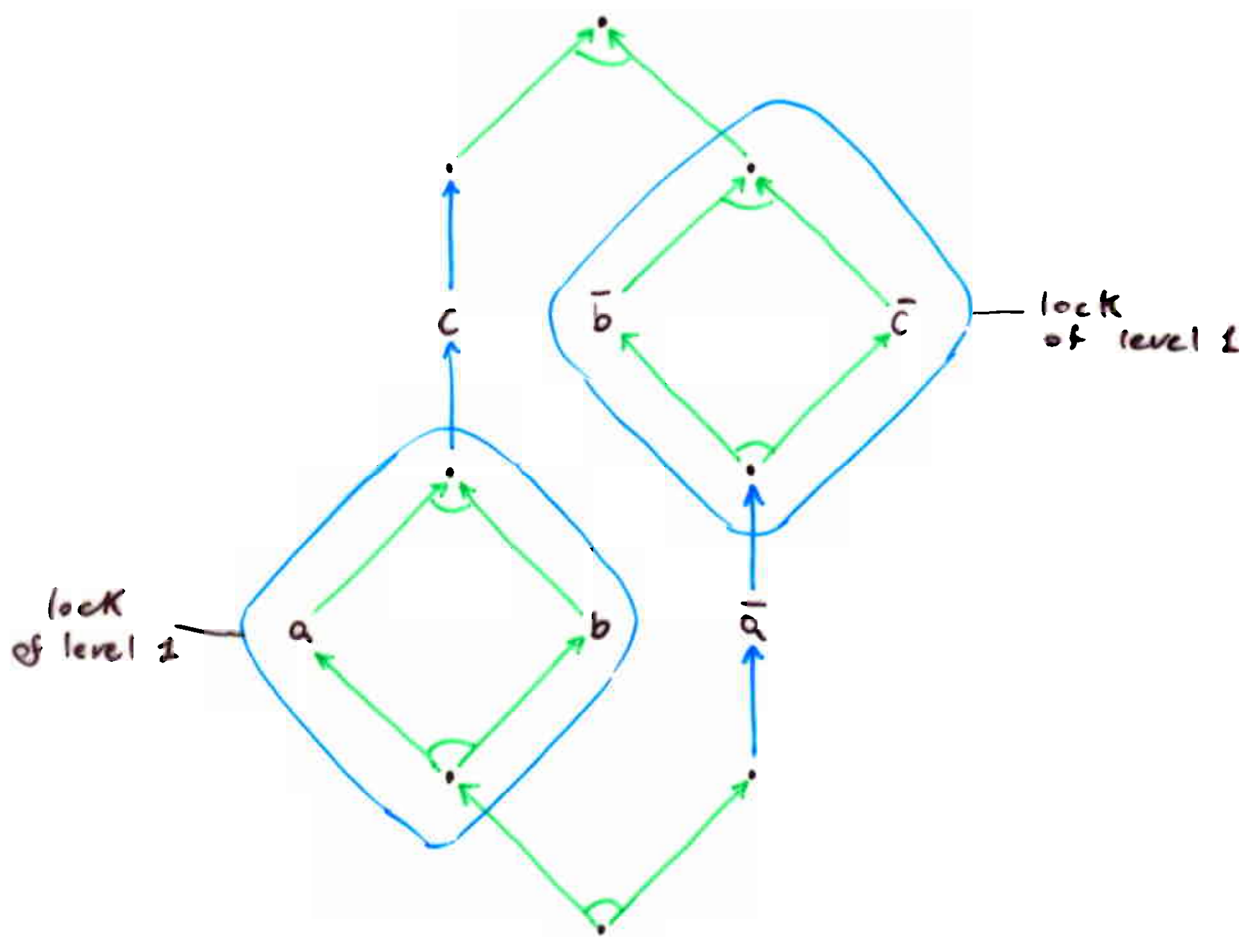
Intuitive representation of SBV structures

$\langle a ; [b , (c , \langle d ; e \rangle)] \rangle$



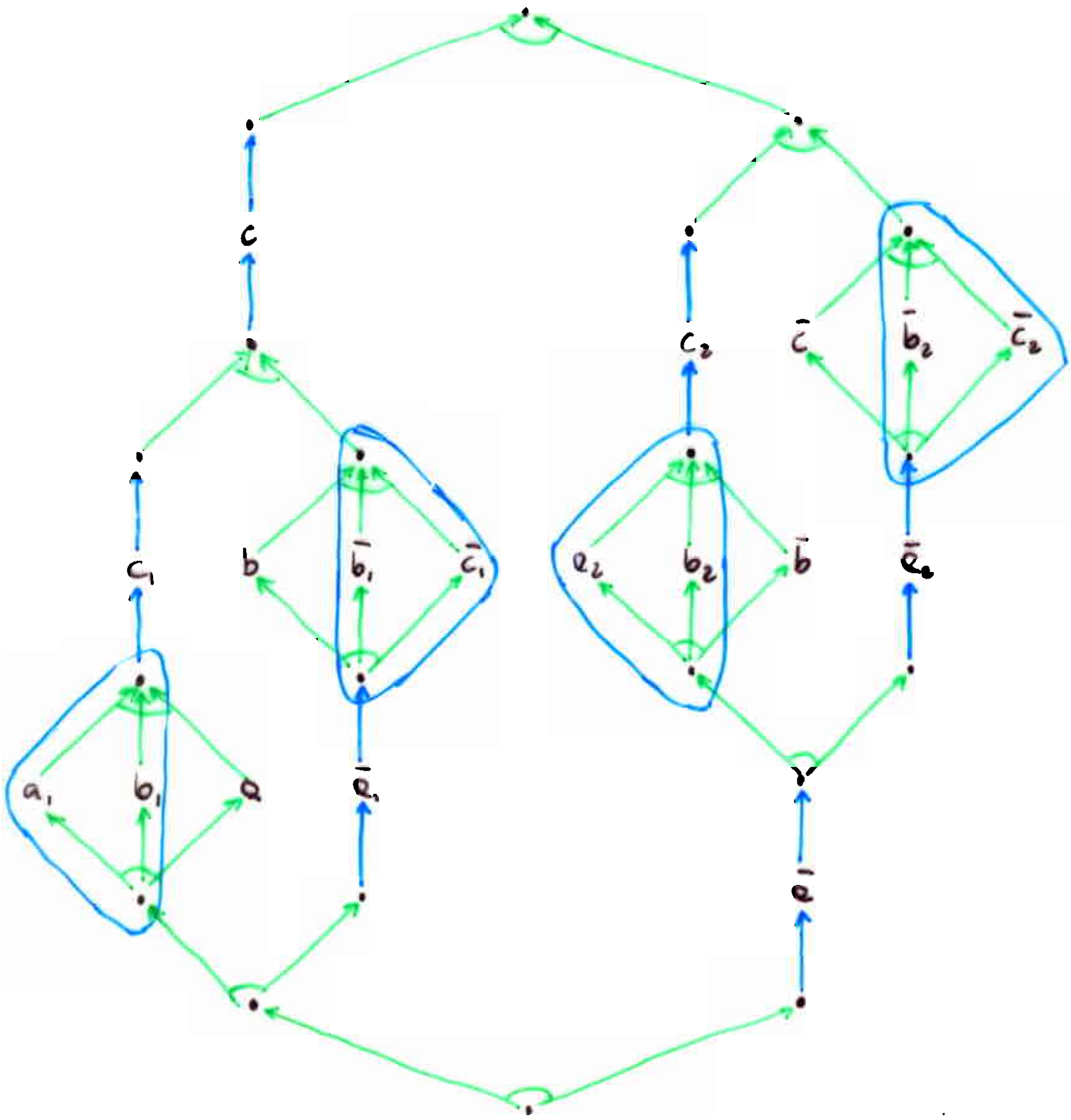
4 System SBV

SBV cannot be expressed in the sequent calculus

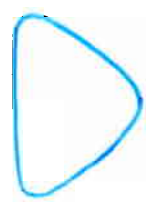


S_1

SBV cannot be expressed in the sequent calculus 2 of 3



S_2



= lock of level 2

4 System SBV

SBV cannot be expressed in the sequent calculus

- **Theorem** $S_1, S_2, \ldots$ are all provable in SBV if and only if one starts reducing from the locks

  Proof Use relational fields semantics

- **Theorem** There is no system in the (normal) sequent calculus which is equivalent to SBV

  Proof Given any sequent system, produce a structure S_K whose lock is deeper than the depth of the sequent system

The calculus of structures

Do we get a better proof theory?

Can we do better than the sequent calculus?

We observe:

- atomicity

- locality

- modularity:

  - in the rules

  - in decompositions

  - in cut elimination arguments

- we easily define logics that 'challenge' the sequent calculus