Course notes on geodesics in first-passage percolation

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Abstract

These notes accompany courses given in University of Bath and Northwestern University during their summer schools in 2016. The main topic is geodesics in first-passage percolation, specifically recent developments by Hoffman and by Damron-Hanson, on directional properties of infinite geodesics. As is done in our recent survey [6], we outline how many of the results from Damron-Hanson '14 [10] can be extended from two dimensions to general dimensions.

Contents

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1 Introduction

We will speak about geodesics in first-passage percolation (FPP), specifically directions and existence. All of this material is contained in the recent survey [6] by Auffinger-Damron-Hanson. Main outline:

1. Intro and Hoffman '05
2. Newman's conjectures and curvature
3. Busemann functions
4. Busemann gradient fields: Damron-Hanson '13
5. Bigeodesics: Damron-Hanson '15

Consider $\mathbb{Z}^d$ with nearest neighbor edges $\mathcal{E}^d$. We begin with a list of definitions. FPP consists of:

- (t_e), a collection of nonnegative weights assigned to the edges,
- $T(\gamma) = \sum_{e \in \gamma} t_e$, where γ is a lattice path,
- $T(x, y) = \inf_{\gamma : x \to y} T(\gamma)$ for $x, y \in \mathbb{Z}^d$, $T(A, B) = \inf\{T(x, y) : x \in A, y \in B\}$ for $A, B \subset \mathbb{Z}^d$, and
- for $x, y \in \mathbb{R}^d$, $T(x, y) = T([x], [y])$, where $x \in [x] + [0, 1)^d$, $y \in [y] + [0, 1)^d$.
- a geodesic from x to y is a lattice path γ from x to y with $T(\gamma) = T(x, y)$. [Might not exist.]

Note the triangle inequality of T:

$$T(x, y) \leq T(x, z) + T(z, y).$$

- So a subpath of a geodesic is a geodesic: if γ is a geodesic from x to y and γ' is the section from w to z, if it is not a geodesic, replace γ' by the geodesic to get a path of lower weight.
- A geodesics can be taken to be (vertex) self-avoiding.

1.1 Existence of finite geodesics

Definition 1.1 (Zhang [36])**.** *Let ρ be the random variable*

$$\rho(x) = \lim_n T(x, \partial B(n)),$$

where $B(n) = [-n, n]^d$ and $\partial B(n) = \{x \in B(n) : \exists y \in B(n)^c \text{ s.t. } x \sim y\}$.

(Exercise.) Show that

$$\rho(x) = \inf \{T(\gamma) : \gamma \text{ is an infinite self-avoiding path from } x\}.$$

Lemma 1.2. 1. *If $\rho(0) = \infty$ then geodesics exist: for all x, y, there is a geodesic from x to y.*

2. *If (t_e) are i.i.d. with $\mathbb{P}(t_e = 0) < p_c$, then a.s., geodesics exist (between all vertices).*

3. **(Exercise.)** *if (t_e) are i.i.d. and continuous, then a.s., geodesics are unique.*

In item 2, we are using

$$p_c = \sup\{p : \mathbb{P}_p(\exists \text{ infinite path of 0-edges}) = 0)\},$$

in a model where $\mathbb{P}_p(t_e = 0) = p = 1 - \mathbb{P}_p(t_e = 1)$. (It is known that $p_c(d) \in (0, 1)$ for all d and $p_c(d) \sim 1/(2d)$.)

Proof. 1. If $\rho(0) = \infty$, also $\rho(x) = \infty$ for all x, since

$$|T(x, \partial B(n)) - T(0, \partial B(n))| \leq T(0, x) < \infty.$$

So pick γ any deterministic path from x to y and choose n large enough so that $x \in B(n)$ and $T(x, \partial B(n)) > T(\gamma)$. Then if γ' is a path from x to y touching $B(n)^c$, one has

$$T(\gamma') \geq T(x, \partial B(n)) > T(\gamma),$$

so it cannot be a geodesic, and the infimum is over a finite set.

2. Choose $\delta > 0$ such that $\mathbb{P}(t_e < \delta) < p_c$, and there is no infinite path of edges with weight $< \delta$. Thus $\rho(0) = \infty$ almost surely since every infinite path has infinite passage time.

□

In fact, (follows from Kesten [24, Prop. 5.8]) a stronger version of item 2 holds. If (t_e) are i.i.d. with $\mathbb{P}(t_e = 0) < p_c$, then

$$\mathbb{P}\left(\liminf_{x\to\infty} \frac{T(0,x)}{\|x\|_1} > 0\right) = 1. \tag{1.1}$$

(Exercise.) Show this for $\mathbb{P}(t_e = 0) = 0$.

Open question. Do geodesics exist in the i.i.d. case when $\mathbb{P}(t_e = 0) = p_c$? True in $2d$ by Wierman-Reh [31, Cor. 1.3]. In general dimensions if $> p_c$ by Zhang [37, Theorem 2].

Under which conditions is $\rho(0) < \infty$? In $2d$, Damron-Lam-Wang '15 [14, Cor. 1.3] showed that $\rho(0) < \infty$ a.s. if and only if $\sum_n F^{-1}(p_c + 1/2^n) < \infty$. Here F is the distribution function of a single t_e, and F^{-1} is the generalized inverse:

$$F^{-1}(t) = \inf\{x : F(x) \geq t\} \text{ for } t > 0.$$

For general dimensions, I don't know better conditions for $\rho(0) = \infty$ than $\mathbb{P}(t_e = 0) < p_c$. (But see also [36, Theorem 8.1.7] for $\rho(0) = \infty$ in high dimensions for Bernoulli distributions.)

2 Infinite geodesics: Hoffman

Definition 2.1. *An infinite path γ is an infinite geodesic if each segment is a finite geodesic. Let $\mathcal{N} = \mathcal{N}(\omega)$ be the maximal k such that there are k disjoint infinite geodesics.*

Note: we may remove loops from γ to ensure it is self-avoiding and still an infinite geodesic.

Proposition 2.2. *The variable $\mathcal{N}$ is translation-invariant, so if the distribution of (t_e) is ergodic, $\mathcal{N}$ is almost surely constant. If geodesics exist in ω, then $\mathcal{N}(\omega) \geq 1$.*

Proof. You can check $\mathcal{N}$ is measurable, so the first statement is clear. Infinite geodesics exist by a subsequence argument. Let γ_n be a geodesic from 0 to ne_1 and take a subsequential limit as follows. All γ_n's must take one of the $2d$ edges incident to 0 first. Choose f_1 such that it is the first edge of infinitely many γ_n's (say γ_{n_k}'s). Then f_2 which is the second edge of infinitely many of the γ_{n_k}'s, and so on. If γ is the path following edges $f_1, f_2, \ldots$, then each segment of γ is a segment of a finite geodesic, so is a geodesic. □

2.1 Properties of Busemann functions

The main question today: what is the value of $\mathcal{N}$? For this we need Busemann functions.

Definition 2.3. *Let γ be an infinite geodesic with starting point x_0 and vertices $x_1, x_2, \ldots$. The Busemann function for γ is defined as*

$$B_\gamma(x) = \lim_n \left[T(x, x_n) - T(x_0, x_n)\right].$$

This limit exists by monotonicity:

$$T(x,x_n)-T(x_0,x_n) = T(x,x_n)+T(x_n,x_{n+1})-T(x_0,x_{n+1}) \geq T(x,x_{n+1})-T(x_0,x_{n+1})$$

and the bound due to the triangle inequality:

$$|T(x, x_n) - T(x_0, x_n)| \leq T(x, x_0).$$

Sometimes we write

$$B_\gamma(x, y) = B_\gamma(x) - B_\gamma(y).$$

Here are some properties.

1. For $m < n$, $B_\gamma(x_m) - B_\gamma(x_n) = T(x_m, x_n)$.
2. **(Exercise.)** If γ_1, γ_2 are infinite geodesics that coalesce (they have finite symmetric difference) and have initial points x_0, y_0, then

   $$B_{\gamma_1}(x) = B_{\gamma_2}(x) - B_{\gamma_2}(x_0) \text{ for all } x.$$

   So

   $$B_{\gamma_1}(x, y) = B_{\gamma_2}(x, y) \text{ for all } x, y.$$

3. $B_\gamma(x,y) = B_\gamma(x,z) + B_\gamma(z,y)$ for all x, y, z.
4. Let θ be a translation of the lattice by an integer vector. Then

$$B_\gamma(x,y)(\omega) = B_{\theta\gamma}(\theta(x), \theta(y))(\theta(\omega)),$$

where the translated weight configuration $\theta(\omega)$ is defined as $t_e(\theta(\omega)) = t_{\theta^{-1}e}(\omega)$. Furthermore $\theta\gamma$ is the translated geodesic ray.

2.2 Hoffman's argument

The following was shown also by Garet-Marchand [15] (and Haggstrom-Pemantle [20] for exponential).

Theorem 2.4 (Hoffman '05 [21])**.** *Let (t_e) be i.i.d., continuous, with finite mean. Then $\mathcal{N} \geq 2$.*

Holds under much more general assumptions.

Proof. We know $\mathcal{N} \geq 1$. Suppose for a contradiction that $\mathcal{N} = 1$. Then if γ, γ' are infinite geodesics, they must touch infinitely many times. By unique geodesics, they must coalesce. So we can define the function

$$f(x,y) = B_\gamma(x,y),$$

where $\gamma = \gamma(\omega)$ is any infinite geodesic.

(Exercise.) Show this equals $\lim_n [T(x, ne_1) - T(y, ne_1)]$ when we assume $\mathcal{N} = 1$, so it is measurable.

Also it has finite mean and is translation covariant:

$$f(x,y)(\omega) = f(\theta(x), \theta(y))(\theta(\omega))$$

for any integer translation. Additivity and the ergodic theorem give for any x,

$$\frac{1}{n} f(0, nx) = \frac{1}{n} \sum_{k=1}^{n} f((k-1)x, kx) = \frac{1}{n} \sum_{k=1}^{n} f(0,x)(\theta^{k-1}\omega) \to \mathbb{E} f(0,x) \text{ a.s. and in } L^1,$$

where θ is translation by $-x$.

What is this limit? By translation invariance, $x \mapsto \mathbb{E} f(0,x)$ is additive for $x \in \mathbb{Z}^d$:

$$\mathbb{E} f(0, x+y) = \mathbb{E} f(0,x) + \mathbb{E} f(x, x+y) = \mathbb{E} f(0,x) + \mathbb{E} f(0,y),$$

so it is determined by its values for $x = \pm e_i$. By symmetry for all i, j,

$$\mathbb{E} f(0, \pm e_i) = \mathbb{E} f(0, \pm e_j),$$

and

$$0 = \mathbb{E} f(0, e_1) + \mathbb{E} f(e_1, 0) = \mathbb{E} f(0, e_1) + \mathbb{E} f(0, -e_1) = 2\mathbb{E} f(0, e_1),$$

giving $\mathbb{E}f(0,x) = 0$ for all x. Thus for all x,

$$\frac{1}{n}f(0,nx) \to 0 \text{ a.s. and in } L^1.$$

In fact one can show a stronger statement (by an argument similar that given in the next section for the Shape Theorem): for each $\epsilon > 0$,

$$\mathbb{P}(|f(0,x)| \geq \epsilon\|x\|_1 \text{ for infinitely many } x \in \mathbb{Z}^d) = 0,$$

so that if γ is the limit of geodesics from 0 to ne_1, with vertices $0 = x_0, x_1, x_2, \ldots$, then

$$\frac{B_\gamma(0,x_n)}{\|x_n\|_1} = \frac{f(0,x_n)}{\|x_n\|_1} \to 0 \text{ almost surely.}$$

On the other hand, since $0, x_n \in \gamma$, $B(0,x_n) = T(0,x_n)$, so this implies

$$\frac{T(0,x_n)}{\|x_n\|_1} \to 0 \text{ almost surely },$$

which contradicts (1.1). □

What else is known?

- (Hoffman '08 [22, Theorem 1.3]) $\mathcal{N} \geq 4$ in $2d$.
- (Damron-Hochman '13 [13, Theorem 1.4]) Given K, there exist continuous distributions in $2d$ with $\mathcal{N} \geq K$.
- In the discrete case (no unique geodesics), one considers the graph of infection of the origin. $\mathcal{T}$ is the union of all geodesics from 0 to all other vertices. Let $\mathcal{K}$ be the number of ends. (A graph has $\geq k$ ends if removing a finite set splits it into $\geq k$ components.) Then: (Auffinger-Damron '14 [4, Theorem 2.3]) if $I = \inf \operatorname{supp} t_e > 0$ and $\mathbb{P}(t_e = I)$ is large enough (with the distribution of t_e not purely atomic), then $\mathcal{K} = \infty$ almost surely.

3 Interlude: the shape theorem

(This section is adapted from tutorial lectures of Jack Hanson from the Bath summer school.)

We will need the shape theorem for all the results beyond, and particularly for Newman's results. It is a type of law of large numbers for the set

$$B(t) = \{x \in \mathbb{R}^d : T(0,x) \leq t\}.$$

Theorem 3.1 (Richardson [32], Cox-Durrett [9], Kesten [24]). *Assume* $\mathbb{E}\min\{t_1, \ldots, t_{2d}\}^d < \infty$*, where* t_i *are i.i.d. copies of* t_e *and* $\mathbb{P}(t_e = 0) < p_c$*. There exists a deterministic, convex set in* $\mathbb{R}^d$*, symmetric about the axes and with nonempty interior, such that for any* $\epsilon > 0$,

$$\mathbb{P}\left((1-\epsilon)\mathcal{B} \subset B(t)/t \subset (1+\epsilon)\mathcal{B} \text{ for all large } t\right) = 1.$$

There is also a version for ergodic distributions by Boivin [8].
(Exercise.) Show that $\mathcal{B}$ is the unit ball of a norm, called g, and

$$\limsup_{x\to\infty} \frac{|T(0,x) - g(x)|}{\|x\|_1} = 0 \text{ almost surely.}$$

This can be thought of as

$$T(0,x) = g(x) + o(\|x\|_1) \text{ as } x \in \mathbb{R}^d \to \infty.$$

Proof of shape theorem. The idea of the proof is to first show "radial" convergence; that is, for a fixed $x \in \mathbb{Z}^d$, to show that

$$g(x) := \lim_n \frac{T(0,nx)}{n} \text{ exists.}$$

To do this, we appeal to the subadditive ergodic theorem. Then we "patch" together convergence in many different directions x to get a uniform convergence.

Liggett's version [28, Theorem 1.10] of Kingman's subadditive ergodic theorem states:

Theorem 3.2. *Let $\{X_{m,n} : 0 \le m < n\}$ is an array of random variables satisfying the following assumptions:*

1. *for each n, $\mathbb{E}|X_{0,n}| < \infty$ and $\mathbb{E}X_{0,n} \ge -cn$ for some constant $c > 0$,*
2. *$X_{0,n} \le X_{0,m} + X_{m,n}$ for $0 < m < n$,*
3. *for each $m \ge 0$, the sequence $\{X_{m+1,m+k+1} : k \ge 1\}$ is equal in distribution to the sequence $\{X_{m,m+k} : k \ge 1\}$, and*
4. *for each $k \ge 1$, $\{X_{nk,(n+1)k} : n \ge 1\}$ is a stationary ergodic process.*

Then

$$g := \lim_n \frac{1}{n}\mathbb{E}X_{0,n} = \inf_n \frac{1}{n}\mathbb{E}X_{0,n} \text{ exists,}$$

and

$$\lim_n \frac{1}{n} X_{0,n} = g \text{ a.s. and in } L^1.$$

We apply this theorem for a fixed $x \in \mathbb{Z}^d$ to the sequence

$$X_{m,n} = T(mx, nx).$$

Item 2 holds by the triangle inequality, whereas 3 and 4 hold by stationarity of the environment under integer translations. In item 1 we can take any $c > 0$, since $T \ge 0$ a.s. The only thing to check is that $\mathbb{E}T(0,nx) < \infty$ for each n. By subadditivity, and symmetry, it suffices to check that $\mathbb{E}T(0,e_1) < \infty$. To do this, we construct $2d$ edge-disjoint deterministic paths $\gamma_1, \dots, \gamma_{2d}$ from 0 to e_1 and note

$$\mathbb{E}T(0,e_1) \le \mathbb{E}\min\{T(\gamma_1), \dots, T(\gamma_{2d})\}.$$

Now you can check:

(Exercise.) If $\mathbb{E}\min\{t_1,\ldots,t_{2d}\} < \infty$ for i.i.d. edge-weights t_i, then the right side above is finite.

By the theorem, then, we define

$$g(x) = \lim_n \frac{T(0,nx)}{n},$$

which is a.s. constant. We next extend g to $\mathbb{Q}^d$ by taking such a rational x and letting $m \in \mathbb{N}$ be such that mx is an integer point. Then set

$$g(x) = \lim_n \frac{T(0,mnx)}{mn} = \frac{1}{m}\lim_n \frac{T(0,n(mx))}{n} = \frac{1}{m}g(mx).$$

g thus defined on $\mathbb{Q}^d$ satisfies the following properties: for $x,y \in \mathbb{Q}^d$,

1. $g(x+y) \le g(x) + g(y)$,
2. g is uniformly continuous on bounded sets,
3. for $q \in \mathbb{Q}$, $g(qx) = |q|g(x)$,
4. g is symmetric about the axes.

Item 1 follows from the triangle inequality for T, and 4 follows from symmetries of the edge weights. Item 3 is an easy exercise, and 2 follows from 1: for $h = (h_1,\ldots,h_d)$,

$$|g(z)-g(z+h)| \le g(h) = g(h_1e_1+\cdots+h_de_d) \le g(e_1)(|h_1|+\cdots+|h_d|) \le \|h\|_1\mathbb{E}T(0,e_1).$$

Then g has a continuous extension to $\mathbb{R}^d$. The above properties extend to real arguments, so g is a seminorm. (A norm, except it could have $g(x)=0$ for some $x \neq 0$.) It is a result of Kesten [25, Theorem 6.1] that g is a norm when $\mathbb{P}(t_e = 0) < p_c$.

Now that we have "radial" convergence to a norm g, we need to patch together convergence in every direction to a type of uniform convergence. Here we do this under the assumption:

$$\mathbb{P}(t_e \in [a,b]) = 1, \text{ where } 0 < a < b < \infty.$$

This implies

$$T(0,x), g(x) \in [a\|x\|_1, b\|x\|_1] \text{ for } x \in \mathbb{Z}^d.$$

So define the event

$$\Omega' = \{t_e \in [a,b] \text{ for all } e\} \cap \left\{\lim_n \frac{T(0,nx)}{n} = g(x) \text{ for all } x \in \mathbb{Z}^d\right\}.$$

(Exercise.) Show that on the above event we actually have

$$\lim_{\alpha\to\infty} \frac{T(0,\alpha x)}{\alpha} = g(x) \text{ for all } x \in \mathbb{Z}^d.$$

(Here α is real instead of just being an integer.)

Fix $\omega \in \Omega'$, a set which has probability one, for the rest of the argument. We will show the equivalent statement

$$\limsup_{x\to\infty} \frac{|T(0,x)-g(x)|}{\|x\|_1} = 0,$$

so suppose it fails: for some $\epsilon > 0$, there is a sequence (x_n) going to infinity such that

$$|T(0, x_n) - g(x_n)| \geq \epsilon \|x_n\|_1 \text{ for all } n.$$

We may assume by compactness that $x_n/\|x_n\|_1 \to z$ for some z with $\|z\|_1 = 1$. We will show that for some x in a nearby direction to z, one cannot have $T(0, nx)/n \to g(x)$. Fix

$$\delta \in \left(0, \frac{\epsilon}{4b+1}\right)$$

and choose $x \in \mathbb{Z}^d$ such that

$$\left\| \frac{x}{\|x\|_1} - z \right\|_1 < \delta,$$

so that for n large,

$$\left\| \frac{x_n}{\|x_n\|_1} - \frac{x}{\|x\|_1} \right\|_1 < 2\delta.$$

We will compare the passage time from 0 to x_n to the passage time to the "nearby" point $\|x_n\|_1 x/\|x\|_1$. Then

$$|T(0, x_n) - g(x_n)| \leq \left| T(0, x_n) - T\left(0, \|x_n\|_1 \frac{x}{\|x\|_1}\right) \right| + \left| T\left(0, \|x_n\|_1 \frac{x}{\|x\|_1}\right) - g\left(\|x_n\|_1 \frac{x}{\|x\|_1}\right) \right|$$
$$+ \left| g\left(\|x_n\|_1 \frac{x}{\|x\|_1}\right) - g(x_n) \right|.$$

The first and last terms are bounded by

$$b \left\| \|x_n\|_1 \frac{x}{\|x\|_1} - x_n \right\|_1 = b\|x_n\|_1 \left\| \frac{x}{\|x\|_1} - \frac{x_n}{\|x_n\|_1} \right\|_1 < 2b\delta \|x_n\|_1.$$

However since we have radial convergence in direction x, the above exercise gives

$$T\left(0, \|x_n\|_1 \frac{x}{\|x\|_1}\right) = g\left(\|x_n\|_1 \frac{x}{\|x\|_1}\right) + o(\|x_n\|_1),$$

so for n large, the second term is bounded by $\delta\|x_n\|_1$. In total,

$$\epsilon \|x_n\|_1 \leq |T(0, x_n) - g(x_n)| \leq (4b+1)\delta \|x_n\|_1 < \epsilon \|x_n\|_1,$$

a contradiction.

□

Remarks.

1. Boivin [8] showed a shape theorem in the ergodic case (first done for bounded marginals by Derriennic, reported in [24, (9.25)]). He assumed (a) the weight distribution is ergodic under integer translations and (b) $\mathbb{E}t_e^{d+\epsilon} < \infty$ for some $\epsilon > 0$. (Actually this moment condition is slightly weaker in his paper.)

2. The key to extending the above argument to the non-bounded cases (and ergodic cases) is to see that the relevant property used here is that

$$\mathbb{P}\left(\sup_{x\neq 0}\frac{T(0,x)}{\|x\|_1}<\infty\right)>0.$$

Our boundedness condition allowed us to say this supremum is at most b a.s., but this weaker condition is enough. It is implied by $\mathbb{E}\min\{t_1,\ldots,t_{2d}\}^d<\infty$ in the i.i.d. case or $\mathbb{E}t_e^{d+\epsilon}<\infty$ in the ergodic case.

4 Newman and curvature

4.1 Newman's theorems and conjectures

For this section, we will assume:

1. $\mathbb{E}e^{\alpha t_e}<\infty$ for some $\alpha>0$,
2. (t_e) is i.i.d., continuous, so there is a unique geodesic $G(x,y)$ from each x to each y.

Our aim is to show that there are infinite geodesics with asymptotic directions.

Definition 4.1. *We say that γ (with vertices $x_0,x_1,x_2,\ldots$) has asymptotic direction θ (where $\theta\in\mathbb{R}^d$ has Euclidean norm 1) if*

$$\arg x_n:=\frac{x_n}{\|x_n\|}\to\theta.$$

Newman also makes the "uniform curvature" assumption. For illustration, we will give a slightly more transparent condition, which is stronger, but has the same flavor.

Let us assume:

A. $\partial\mathcal{B}$ is differentiable (meaning each $x\in\partial\mathcal{B}$ has a unique supporting hyperplane – a hyperplane H_x such that (a) $x\in H_x$ and $\mathcal{B}$ does not intersect both components of H_x^c) and

B. there are uniform constants $C,\delta>0$ such that for all $x\in\partial\mathcal{B}$,

$$g(x+u)-g(x)\geq C\|u\|^2 \text{ for } u \text{ with } x+u\in H_x \text{ and } \|u\|<\delta.$$

This assumption is a type of curvature of the boundary of $\mathcal{B}$. Generally it is believed that the limit shape has differentiable boundary and is strictly convex (even with positive curvature). However it is not even known that it is not a polygon. For a certain class of distributions in $2d$ (mentioned above in the remark about $\mathcal{K}=\infty$), Auffinger-Damron '14 [4, Cor. 2.1] showed the limit shape is not a polygon.

Open problem. Show that if (t_e) are i.i.d., continuous, and $\mathbb{E}\min\{t_1,\ldots,t_{2d}\}^d<\infty$, then the limit shape cannot be a polygon.

The following is a version of [29, Theorem 2.1].

Theorem 4.2 (∼Newman)**.** *Assuming 1, 2, A, B,*

1. *a.s., for all θ, there is an infinite geodesic starting from 0 with direction θ.*
2. *a.s., all infinite geodesics have asymptotic directions.*

Remarks.

1. This result says in particular that $\mathcal{N} = \infty$.
2. One way to state it is to use the tree of infection $\mathcal{T}$ of the origin. Each infinite path in $\mathcal{T}$ has a direction and there is an infinite path in $\mathcal{T}$ in each direction.

4.2 Proofs

So let's get to the main question: how does curvature help to control geodesics? Actually differentiability is not needed at all for this method. (See Newman's curvature condition in his ICM paper [29].)

Lemma 4.3 (Geodesic wandering bound – Newman-Piza argument)**.** *Assume 1, 2, and B for $x = e_1$ and $H_x \perp e_1$. Then for each $\epsilon > 0$, there are $C_1, C_2 > 0$ such that*

$$\mathbb{P}(D(0, ne_1) \geq n^{\frac{3}{4}+\epsilon}) \leq C_1 e^{-n^{C_2}} \text{ for all } n.$$

Here, $D(0, ne_1)$ is the maximal distance from any point in $G(0, ne_1)$ to the e_1-axis.

Proof. For z of distance $n^{3/4+\epsilon}$ to the e_1-axis, note

$$\mathbb{P}(D(0, ne_1) \geq n^{3/4+\epsilon}) \leq \sum_z \mathbb{P}(T(0, ne_1) = T(0, z) + T(z, ne_1)).$$

We claim that for $C_1, C_2 > 0$,

$$\mathbb{P}(T(0, ne_1) = T(0, z) + T(z, ne_1)) \leq C_1 e^{-n^{C_2}},$$

where $z = (n/2)e_1 + n^{3/4+\epsilon} e_2$.

(Exercise.) A similar statement holds for other z. Prove one and sum over z to get the lemma.

If it were up to the g function, this event would never occur:

$$g(z) + g(ne_1 - z) - g(ne_1) = [g(z) - g((n/2)e_1)] + [g(ne_1 - z) - g((n/2)e_1)],$$

and for n large,

$$\begin{aligned} g(n/2e_1 + n^{3/4+\epsilon} e_2) - g(n/2e_1) &= (n/2)\left(g\left(e_1 + \frac{2}{n^{1/4-\epsilon}} e_2\right) - g(e_1)\right) \\ &\geq C(n/2)\left(\frac{2}{n^{1/4-\epsilon}}\right)^2 \\ &\geq Cn^{1/2+2\epsilon}. \end{aligned}$$

(By symmetry this holds for $g(n - z) - g((n/2)e_1)$ as well.) So we get

$$g(z) + g(ne_1 - z) - g(ne_1) \geq Cn^{1/2+2\epsilon}.$$

In the approximation of T by g, we have some random (and nonrandom) error, which we must control. Namely, on the above event, one has

$$\begin{aligned}0 &= T(0,z) + T(z,ne_1) - T(0,ne_1)\\ &\geq [T(0,z) - g(z)] + [T(z,ne_1) - g(ne_1 - z)] - [T(0,ne_1) - g(ne_1)] + Cn^{1/2+2\epsilon},\end{aligned}$$

meaning at least one of the three bracketed terms is in absolute value bigger than $(C/3)n^{1/2+2\epsilon}$.

The following lemma is a combination of a concentration inequality of Kesten [25, Eq. (1.15)] and nonrandom fluctuation bounds of Alexander [1, Theorem 3.2].

Lemma 4.4. *Assume $\mathbb{E}e^{\alpha t_e} < \infty$ for some $\alpha > 0$. There exist c_1, c_2 such that*

$$\mathbb{P}\left(|T(0,x) - g(x)| \geq \lambda\sqrt{\|x\|}\right) \leq c_1 e^{-c_2\lambda}$$

for $C_1 \log\|x\| \leq \lambda \leq C_1\|x\|$.

(Improvements have been made by Talagrand [33, Prop.8.3], Benaïm-Rossignol [7, Theorem 5.4], Damron-Hanson-Sosoe [12].)

Applying this to $x = ne_1$, we obtain

$$\mathbb{P}(|T(0,ne_1) - g(ne_1)| \geq (C/3)n^{1/2+2\epsilon}) \leq c_1 e^{-c_2' n^{2\epsilon}},$$

and similarly for $|T(0,z) - g(z)|$ and $|T(z,ne_1) - g(ne_1 - z)|$. □

A similar proof gives the following useful (for us) result from Newman. First some definitions.

Definition 4.5. 1. *For a vertex $x \neq 0$ and $\epsilon > 0$, let C_x be the annulus portion:*

$$C_x = \left\{z \in \mathbb{R}^d : \|z\| \in [\|x\|/2, 2\|x\|],\ \|\arg z - \arg x\| \leq \|x\|^{-1/4+\epsilon}\right\}.$$

2. *$\partial' C_x$ is the boundary of C_x minus the "forward" boundary of C_x:*

$$\partial' C_x = \partial C_x \setminus \{z \in \mathbb{R}^d : \|z\| = 2\|x\|\}.$$

3. *$out(x)$ is the set of vertices z such that $T(0,z) = T(0,x) + T(x,z)$ (vertices whose geodesic from 0 goes through x).*
4. *Define G_x as the event that*

$$out(x) \cap \partial' C_x \neq \emptyset,$$

Theorem 4.6. *Assume 1, 2, A, and B. Given $\epsilon > 0$ there exist $C_1, C_2 > 0$ such that*

$$\mathbb{P}(G_x) \leq C_1 e^{-\|x\|^{C_2}} \text{ for all } x.$$

To derive this bound for G_x, one needs uniform curvature for directions in a small interval near the argument of x. So B suffices.

Proof of item 2 in ∼Newman's. Let $\gamma = (0, x_1, x_2, \ldots)$ be an infinite geodesic (self-avoiding) and choose $M > 0$ such that if $\|x\| \geq M$, then G_x occurs. Now pick any k such that $\|x_k\| \geq M$ and define inductively

$$y_1 = x_k \text{ and } y_{i+1} = \text{ first exit of } \gamma \text{ from } C_{y_i} \text{ after hitting } y_i.$$

Then the portion of γ after x_k is contained in the union of the C_{y_i}'s, so any $z \in \gamma$ after x_k is in some C_{y_I} and thus satisfies

$$\begin{aligned} \|\arg z - \arg x_k\| \leq \sum_{i=1}^{I-1} \|\arg y_{i+1} - \arg y_i\| + \|\arg z - \arg y_I\| &\leq \sum_{i=1}^{I} \|y_i\|^{-1/4+\epsilon} \\ &= \sum_{i=1}^{I} (2^{i-1}\|y_1\|)^{-1/4+\epsilon} \\ &\leq C\|x_k\|^{-1/4+\epsilon}. \end{aligned}$$

This suffices to show that $(\arg x_n)$ is a Cauchy sequence. □

(Exercise.) Use this method to show that not only is every infinite geodesic directed, but for each direction, there is an infinite geodesic from 0 with this direction. That is, prove item 1 of ∼Newman's.

5 Weakening assumptions: back to the present day

We aim to improve ∼Newman's in two directions. First, we want to include all translation-ergodic passage times. Next, we want to not assume curvature! It is pretty reasonable not to want to assume curvature because:

Theorem 5.1 (Häggstrom-Meester '95 [19])**.** *Given any compact, convex subset $\mathcal{C}$ of $\mathbb{R}^d$ with nonempty interior and which is symmetric about the axes, there is a translation invariant measure on passage times with $\mathcal{C}$ as its limit shape.*

In particular, there are limit shapes for translation invariant FPP which are polygons! These shapes do not satisfy uniform curvature, so we should not assume it. But in that case, should we really expect geodesics still to have asymptotic directions? Well maybe not in directions that are interiors of "sides" of the limit shape, but perhaps in exposed directions. So let us assume for now:

Assumptions of Hoffman.

1. $\mathbb{P}$ is ergodic under lattice translations and has the symmetries of $\mathbb{Z}^d$,
2. $\mathbb{P}$ has unique passage times: a.s., any two distinct paths have different passage times,
3. $\mathbb{E}t_e^{d+\delta} < \infty$ for some $\delta > 0$,
4. the limit shape for $\mathbb{P}$ is bounded.

(These suffice for Boivin's shape theorem, but can be weakened in the i.i.d. case.) Here we will try to work in the direction of:

Open problem. Show that if (t_e) are i.i.d. and continuous, almost surely there is an infinite geodesic with an asymptotic direction.

5.1 In direction e_1

To improve the Newman results, we continue where Hoffman left off, analyzing Busemann functions. In '08, he considered some Busemann-type limits, and with these, was able to show existence of at least 4 disjoint infinite geodesics. We present here a version of a combination of arguments from that paper and from Damron-Hanson '14 [10], in an ideal scenario (Assumption C below). Put $H_n = \{x \in \mathbb{Z}^d : x \cdot e_1 = n\}$ and

$$B_n(x,y) = T(x,H_n) - T(y,H_n) \text{ for } x,y \in \mathbb{Z}^d.$$

(Exercise.) Show the above conditions imply existence of a unique minimizing path from x to H_n for each x and n. (Actually we only need existence of this geodesic for what follows.)

Assumption C. (Unjustified!!) Almost surely, for $x, y \in \mathbb{Z}^d$,

$$B(x,y) := \lim_n B_n(x,y) \text{ exists a.s.}$$

Why should this assumption be true? Based on existing results, from each vertex x, there should be a limiting geodesic Γ_x for the sequence $G(x,H_n)$ and for x,y, the geodesics Γ_x, Γ_y should coalesce. If their coalescence point is z, then you can verify:

$$B(x,y) = T(x,z) - T(y,z).$$

Let's first find the mean of B.

Proposition 5.2. *One has*

$$\mathbb{E}B(x,y) = (y-x)\cdot\rho \textit{ for all } x,y \in \mathbb{Z}^d,$$

where $\rho = g(e_1)e_1$.

Proof. First, $x \mapsto \mathbb{E}B(0,x)$ is additive:

$$\mathbb{E}B(0,x+y) = \mathbb{E}B(0,x) + \mathbb{E}B(x,x+y) = \mathbb{E}B(0,x) + \mathbb{E}B(0,y).$$

Also $\mathbb{E}B(0,-x) = -\mathbb{E}B(0,x)$, so there is a vector $\rho \in \mathbb{R}^d$ such that

$$\mathbb{E}B(x,y) = \mathbb{E}B(0,y-x) = (y-x)\cdot\rho.$$

The vector ρ is determined by its dot product with elements of the basis $\{e_1,\ldots,e_d\}$.

First take $x = 0, y = e_1$. Then we use an averaging trick that was in Hoffman [22] and Garet-Marchand [15] and goes back to Kingman [26, Eq. (26)]. We use the following fact:

(Exercise.)

$$\lim_n \frac{\mathbb{E}T(0, H_n)}{n} = g(e_1).$$

Now compute

$$\begin{aligned}
\frac{1}{n}\mathbb{E}T(0, H_n) &= \frac{1}{n}\sum_{k=1}^{n} \left(\mathbb{E}T((k-1)e_1, H_n) - \mathbb{E}T(ke_1, H_n)\right) \\
&= \frac{1}{n}\sum_{k=1}^{n} \left(\mathbb{E}T(0, H_{n-k+1}) - \mathbb{E}T(e_1, H_{n-k+1})\right) \\
&= \frac{1}{n}\sum_{k=1}^{n} \mathbb{E}\left(T(0, H_k) - T(e_1, H_k)\right).
\end{aligned}$$

The term inside converges a.s. to $B(0, e_1)$. Since it is bounded by $T(0, e_1)$ in absolute value, DCT implies that it converges to $\mathbb{E}B(0, e_1)$. So

$$\mathbb{E}B(0, e_1) = \lim_n \frac{1}{n}\mathbb{E}T(0, H_n) = g(e_1).$$

If $i > 1$, then

$$0 = \mathbb{E}B(0, e_i) + \mathbb{E}B(e_i, 0) = \mathbb{E}B(0, e_i) + \mathbb{E}B(0, -e_i),$$

and these are equal by reflection symmetry, giving $\mathbb{E}B(0, e_i) = 0$. Last, $\rho = g(e_1)e_1$ is the unique vector with $\rho \cdot e_1 = g(e_1)$ and $\rho \cdot e_i = 0$ for $i > 1$. □

Further remarks:

1. By the ergodic theorem, as before, for $x \in \mathbb{Z}^d$,

$$\lim_n \frac{1}{n}B(0, nx) = \lim_n \frac{1}{n}\sum_{k=1}^{n} B((k-1)x, kx) = \mathbb{E}B(0, x) = x \cdot \rho \text{ a.s.}$$

2. From here we can replicate the proof of the shape theorem, using $|B(x, y)| \leq T(x, y)$, to show:

   Theorem 5.3 (Busemann shape theorem)**.** *For each $\epsilon > 0$,*

$$\mathbb{P}\left(|B(0, x) - x \cdot \rho| \geq \epsilon\|x\| \text{ for infinitely many } x \in \mathbb{Z}^d\right) = 0.$$

   This means uniformly $B(0, x) = x \cdot \rho + o(\|x\|)$ as $x \in \mathbb{R}^d \to \infty$.

3. Note that the set

$$\mathcal{B}_B = \{z \in \mathbb{R}^d : z \cdot \rho \leq 1\}$$

   is a half-space. Thus the Busemann "limit shape" is a half-space.

4. The boundary

$$H_B = \{z \in \mathbb{R}^d : z \cdot \rho = 1\}$$

   is a supporting hyperplane for $\mathcal{B}$ in direction e_1 (at the point $e_1/g(e_1)$).

Proof. First $e_1/g(e_1) \in H_B \cap \partial\mathcal{B}$. It is in H_B since

$$e_1/g(e_1) \cdot \rho = \frac{g(e_1)}{g(e_1)} e_1 \cdot e_1 = 1,$$

and it is on the boundary of the limit shape in direction e_1 since $g(e_1/g(e_1)) = 1$. Since H_B is perpendicular to e_1 and contains the point $e_1/g(e_1)$, on the boundary of the limit shape, it must be a supporting line. □

5. **(Exercise.)** Let $\gamma = (0 = x_0, x_1, x_2, \ldots)$ be a subsequential limit of geodesics from 0 to H_n. Then

$$B(0, x_j) = T(0, x_j) \text{ for all } j.$$

Define the set of points

$$S = H_B \cap \partial\mathcal{B}$$

of contact between the supporting line H_B and the limit shape.

Theorem 5.4. *Assume C and Hoffman's conditions. Any subsequential limit γ of geodesics from 0 to H_n is asymptotically directed in S. This means any limit point of $\{x/g(x) : x \in \gamma\}$ is contained in S.*

Proof. Let $\gamma = (0, x_1, x_2, \ldots)$ be such a subsequential limit and pick a convergent subsequence of $(x_n/g(x_n))$, so that, say,

$$x_{n_k}/g(x_{n_k}) \to z.$$

We must show $z \in S$, but since $z \in \partial\mathcal{B}$, we must show $z \in H_B$, or $z \cdot \rho = 1$.

Since g is a norm, the Busemann shape theorem gives

$$\frac{B(0, x_{n_k}) - x_{n_k} \cdot \rho}{g(x_{n_k})} \to 0,$$

meaning

$$\frac{B(0, x_{n_k})}{g(x_{n_k})} \to \lim_k \frac{x_{n_k}}{g(x_{n_k})} \cdot \rho = z \cdot \rho.$$

But also this equals $\frac{T(0,x_{n_k})}{g(x_{n_k})} = \frac{g(x_{n_k}) + o(\|x_{n_k}\|)}{g(x_{n_k})} = 1$. Thus $z \in H_B$. However $g(z) = 1$, so $z \in \partial\mathcal{B}$. □

Corollary 5.5 (Existence of directed geodesics)**.** *Assume C and Hoffman's conditions. If $e_1 g(e_1)$ is an exposed point of $\mathcal{B}$, then any subsequential limit of geodesics from* 0 *to H_n is asymptotically directed in direction e_1.*

Note that we do not need curvature here, only a local condition on the boundary. However, we need existence of Busemann limits.

Open problem. Prove existence of the limit of $T(0, H_n) - T(x, H_n)$.

5.2 In other directions and what is known about limits

A similar proof to that in the last section gives the following. Suppose that H is some supporting hyper-plane for $\mathcal{B}$ and set $H_n = \{w : w \cdot \rho_H = n\}$, where ρ_H is such that $H = \{w : w \cdot \rho_H = 1\}$. Assume that

$$B_H(x,y) := \lim_n [T(x,H_n) - T(y,H_n)] \text{ exists a.s.}$$

Theorem 5.6. *Under the above assumption (and Hoffman's conditions), one has:*

1. *for any* $\epsilon > 0$,

$$\mathbb{P}\left(|B_H(0,x) - \rho_H \cdot x| \geq \epsilon \|x\| \text{ for infinitely many } x \in \mathbb{Z}^d\right) = 0.$$

2. *a.s., each subsequential limit of* $G(0,H_n)$ *is directed in the set* $H \cap \partial\mathcal{B}$.

What is known about these "Busemann limits?"

1. Under an assumption of uniform curvature and finite exponential moments, Newman [29, Theorem 1.1] showed that in 2d, there is a set $D \subset [0,2\pi)$ of countable complement such that if $\theta \in D$, then a.s., if (x_n) is any sequence with $\arg x_n \to \theta$, then for all x, y, the following limit exists:

$$\lim_n [T(x,x_n) - T(y,x_n)].$$

2. Howard-Newman [23, Theorem 1.13] showed this for all directions in a rotationally-invariant model, called Euclidean FPP, in $2d$. They also predicted [23, p. 589] the value of the mean and fluctuations.
3. Damron-Hanson '15 [11, Theorem 2] showed that this limit exists for **any fixed** θ in $2d$ if the limit shape boundary is differentiable. They also showed existence of limiting geodesics for the sequence $G(0,x_n)$. (And uniqueness statements.)
4. Auffinger-Damron-Hanson '14 [5, Theorem 1.1] showed existence of these limits on a half-plane (going toward the boundary). That is, if we consider FPP in the upper half-plane $\mathbb{H}$, and let x_n be the point ne_1, then the above limit exists surely (!) for $x, y \in \mathbb{H}$. Furthermore limiting geodesics exist. This was actually proved in [5] for general subsets of $\mathbb{Z}^2$ with boundary.

   It is not hard to show this for x, y on the boundary of the half-plane by the "paths crossing" trick of Alm [2] and Alm-Wierman [3]. Indeed, note that if x lies to the left of y on the boundary, then for large n, the geodesics $G(x, ne_1)$ and $G(y,(n+1)e_1)$ must cross at some point w. Thus

$$\begin{aligned} T(x,ne_1) + T(y,(n+1)e_1) &= T(x,w) + T(w,ne_1) + T(y,w) + T(w,(n+1)e_1) \\ &\leq T(x,(n+1)e_1) + T(y,ne_1). \end{aligned}$$

   Rearranging,

$$T(x,ne_1) - T(y,ne_1) \leq T(x,(n+1)e_1) - T(y,(n+1)e_1),$$

   meaning that $B(x,y) := \lim_n [T(x,ne_1) - T(y,ne_1)]$ exists. One can extend this argument to the case that geodesics do not exist, and it holds surely. For x, y in the interior of the half-plane, more arguments are needed.

5. As an aside, the first use of the paths crossing trick appears to have been in Alm's paper [2], where it was shown that in $2D$, $\mathbb{E}T(0,e_1) \leq \mathbb{E}T(0,e_1+e_2)$. The argument, as before, is that the geodesics $G(0,e_1+e_2)$ and $G(e_2,e_1)$ must cross, and so
$$T(0,e_1+e_2)+T(e_2,e_1) \geq T(0,e_1)+T(e_2,e_1+e_2).$$
Taking expectation on both sides gives
$$2\mathbb{E}T(0,e_1+e_2) \geq 2\mathbb{E}T(0,e_1).$$

6 Busemann gradient fields

In this section, we will sketch the proof of the following result from Damron-Hanson [10]. It was proved in 2d there, but the ideas work for all dimensions as outlined below.

The main difficulty with making the results of the previous section unconditional is that we do not know that Busemann limits exist. One possible solution to this is to note that $B_n(x,y)$ is bounded in n (a.s.) and so has a convergent subsequence. How do we select a subsequence? If we were to define $B(x,y)$ to be a subsequential limit, we would like B to have a translation-covariance property (to apply the ergodic theorem as in last section). It is not very clear how to do this, as we would need to choose compatible subsequences for a configuration and all its translates.

This same issue appears in the study of short-range disordered systems by Aizenman-Wehr and Newman-Stein, and was dealt with by introducing "metastates." The approach below is roughly modeled off of that. The main idea is to look one level up, at the level of distributions, and to take a subsequential limit of the joint distribution of all Busemann increments. We will need an averaging procedure so that the resulting distribution is translation invariant. The advantage will be that Busemann limits will always be able to be extracted (in a distributional sense), but we will obtain somewhat weaker results.

6.1 Existence of geodesics directed in sectors

Our general assumptions are:

1. $\mathbb{P}$ is ergodic under lattice translations,
2. $\mathbb{E}t_e^{d+\delta} < \infty$ for some $\delta > 0$,
3. the limit shape for $\mathbb{P}$ is bounded.

Theorem 6.1 (Damron-Hanson)**.** *Let $z_0 \in \partial\mathcal{B}$ be a point with a unique supporting hyperplane H for $\mathcal{B}$. Letting*
$$S = H \cap \partial\mathcal{B},$$
a.s., there is an infinite geodesic starting from 0 that is directed in S. This means that if the vertices of the geodesic are $0, x_1, x_2, \ldots$, then $(x_n/g(x_n))$ has all limit points in S.

Note that if z_0 is exposed and differentiable, then a.s. there is an infinite geodesic from 0 that is asymptotically directed in $\{z_0\}$.

Corollary 6.2. *In 2d, under the above assumptions, and unique geodesics from point to hyperplanes, assume that $\mathcal{B}$ is strictly convex. The following hold a.s.:*

1. *for all z_0, there is an infinite geodesic directed in $\{z_0\}$ and*
2. *all infinite geodesics have directions.*

This should be compared to ∼Newman's. It is the same result except: we do not need curvature or i.i.d.

Remarks.

- In the main theorem, we removed the limit assumption, but now the result only holds in directions of differentiability.
- For even the main theorem (a geodesic in one direction), ∼Newman's needed curvature in an interval around z_0 to obtain existence of a directed geodesic. Here this is replace by exposed differentiability at z_0.
- Note here that we cannot get infinite geodesics with true asymptotic directions, since we cannot take z_0 to be an exposed point necessarily. There are always exposed points (found by shrinking a Euclidean ball to make contact with $\mathcal{B}$), but we do not know that these are directions of differentiability. In only one i.i.d. case do we know that there is an exposed point of differentiability of $\mathcal{B}$, due to Auffinger-Damron '13 [4]. In $2D$, if $\mathbb{P}(t_e < I) = 0$ for some $I > 0$ and $\mathbb{P}(t_e = I) = \vec{p_c}$, the oriented percolation threshold, then the direction $\pi/4$ is exposed and differentiable. (This implies in particular that $\mathcal{B}$ is not a polygon.)
- In 2D, statements were also proved about coalescence of geodesics and nonexistence of certain doubly-infinite geodesics. (This is covered in the next section.)
- A similar theorem has been proved by Georgiou-Rassoul-Agha-Seppäläinen [16] in a related model: directed last-passage percolation in two dimensions. To construct limits of Busemann fields, they appealed to a relation to queueing systems.

6.2 Sketch of proof

Proof sketch. For most of the proof, we take $z_0 \in \partial\mathcal{B}$, and H **any** supporting hyperplane to $\mathcal{B}$ at z_0. (That is, we do not assume differentiability until the end.) Write $\rho \in \mathbb{R}^d$ for the unique vector with $H = \{w : w \cdot \rho = 1\}$.

The trick will be to enlarge the probability space and take distributional limits there. We consider $\widetilde{\Omega} = \Omega_1 \times \Omega_2 \times \Omega_3$, where

$$\Omega_1 = [0,\infty)^{\mathcal{E}^d},$$

$$\Omega_2 = \mathbb{R}^{\mathbb{Z}^d \times \mathbb{Z}^d}$$

$$\Omega_3 = \{0,1\}^{\vec{\mathcal{E}^d}}.$$

Here, $\vec{\mathcal{E}^d}$ is the set of directed edges of $\mathbb{Z}^d$. A typical element of $\widetilde{\Omega}$ is

$$\widetilde{\omega} = ((t_e)_{e\in\mathcal{E}^d}, (B(x,y))_{x,y\in\mathbb{Z}^d}, (\eta(e))_{e\in\vec{\mathcal{E}^d}}) = ((t_e), B, \eta).$$

For an element of our original probability space Ω, we can define elements in each of the Ω_i's. Of course, the element of Ω_1 will be $(t_e(\omega))$. Next, for the hyper-plane H and ρ the unique vector with $H = \{w : w \cdot \rho = 1\}$, set $H_\alpha = \{w : w \cdot \rho = \alpha\}$ and put

$$B_\alpha = B_\alpha(\omega) = (T(x, H_\alpha) - T(y, H_\alpha))_{x,y \in \mathbb{Z}^d}.$$

This is our element of Ω_2. For $e = (x, y) \in \vec{\mathcal{E}}^d$, define

$$\eta_\alpha(e) = \begin{cases} 1 & \text{if } T(x, H_\alpha) = T(y, H_\alpha) + t_e \\ 0 & \text{otherwise} \end{cases}.$$

This means that the directed edge e gets a value of 1 if it is in a geodesic from some point to H_α (and it is traversed in the correct direction). Finally set

$$\eta_\alpha = \eta_\alpha(\omega) = (\eta_\alpha(e))_{e \in \vec{\mathcal{E}}^d}$$

and

$$\Phi_\alpha : \Omega \to \widetilde{\Omega}$$

by

$$\Phi_\alpha(\omega) = ((t_e), B_\alpha, \eta_\alpha), \ \mu_\alpha = \mathbb{P} \circ \Phi_\alpha.$$

This is the joint distribution of the edge weights, Busemann gradients, and geodesic graph configurations. Note that if θ is a translation by the integer vector x, which acts as:

$$\theta((t_e), (B(x, y)), (\eta(e))) = ((t_{\theta^{-1}e}), (B(\theta^{-1}x, \theta^{-1}y)), (\eta(\theta^{-1}e))),$$

then

$$\mu_\alpha \circ \theta^{-1} = \mu_{\alpha + \rho \cdot x}.$$

For instance,

$$\begin{aligned}
(\mu_\alpha \circ \theta^{-1})(B(w, z) \in [a, b]) &= \mu_\alpha(B(\theta^{-1}w, \theta^{-1}z) \in [a, b]) \\
&= \mathbb{P}(T(\theta^{-1}w, H_\alpha) - T(\theta^{-1}z, H_\alpha) \in [a, b]) \\
&= \mathbb{P}(T(w, H_{\alpha + x \cdot \rho}) - T(z, H_{\alpha + x \cdot \rho}) \in [a, b]) \\
&= \mu_{\alpha + x \cdot \rho}(B(w, z) \in [a, b]).
\end{aligned}$$

From the second to third line, we applied the translation θ, used the fact that $\mathbb{P}$ is translation invariant, and the fact that the image of H_α under the map θ is $H_{\alpha + x \cdot \rho}$.

Here are some basic properties of the measures μ_α, which follow from the definitions. Given the configuration η in Ω_3, we can define a directed graph $\mathbb{G} = \mathbb{G}(\eta)$ induced by the edges with value 1. Write $x \to y$ if there is a directed path from x to y.

Lemma 6.3. *Let $\alpha \in \mathbb{R}$. The following properties hold μ_α-a.s. for all $x, y \in \mathbb{Z}^d$.*

1. *B is additive and $|B(x, y)| \leq T(x, y)$,*
2. *from each $x \notin H_\alpha$, there is a directed path to H_α in $\mathbb{G}$,*
3. **(Exercise.)** *each directed path in $\mathbb{G}$ is a geodesic for (t_e). If $x \to y$ then $B(x, y) = T(x, y)$.*

We want to take a subsequential limit of the μ_α's, but we need it to be translation invariant, so we average. Set

$$\mu_n^* = \frac{1}{n}\int_0^n \mu_\alpha \, \mathrm{d}\alpha.$$

(Need to know that for a generating class of events A, $\alpha \mapsto \mu_\alpha(A)$ is Lebesgue measurable.) The way to think of μ_n^* is that we select a random hyperplane H_α uniformly from $\alpha \in [0,n]$ and then sample our variables. Note now that for an integer translation θ, and A an event in $\widetilde{\Omega}$,

$$\begin{aligned}(\mu_n^* \circ \theta^{-1})(A) = \mu_n^*(\theta^{-1}A) = \frac{1}{n}\int_0^n \mu_\alpha(\theta^{-1}A)\,\mathrm{d}\alpha &= \frac{1}{n}\int_0^n \mu_\alpha \circ \theta^{-1}(A)\,\mathrm{d}\alpha \\ &= \frac{1}{n}\int_0^n \mu_{\alpha + x\cdot\rho}(A)\,\mathrm{d}\alpha.\end{aligned}$$

Changing variables, this becomes

$$(\mu_n^* \circ \theta^{-1})(A) = \frac{1}{n}\int_{x\cdot\rho}^{n+x\cdot\rho} \mu_\alpha(A)\,\mathrm{d}\alpha,$$

and so

$$\left|(\mu_n^* \circ \theta^{-1})(A) - \mu_n^*(A)\right| \le \frac{1}{n}\left|\int_0^{x\cdot\rho} \mu_\alpha(A)\,\mathrm{d}\alpha\right| + \frac{1}{n}\left|\int_n^{n+x\cdot\rho} \mu_\alpha(A)\,\mathrm{d}\alpha\right|. \tag{6.1}$$

(Exercise.) Show the following.

1. the sequence (μ_n^*) is tight, so it has a subsequential limit μ. Furthermore, The above lemma also holds for μ (the three properties above), except item 2 would say that from each $x \in \mathbb{Z}^d$, there is an infinite directed path in $\mathbb{G}$.
2. μ is invariant under lattice translations. (Use (6.1) above.)

We now want to try to redo last section.

Proposition 6.4. *The mean of the Busemann function B is given by*

$$\mathbb{E}_\mu B(x,y) = (y-x)\cdot\rho.$$

Proof. The following is an integrated version of the averaging argument (inspired by Gouéré [18, Lemma 2.6]) from last section. For any $x \in \mathbb{Z}^d$,

$$\begin{aligned}\mathbb{E}_{\mu_n^*} B(-x,0) &= \frac{1}{n}\int_0^n \mathbb{E}_{\mu_\alpha} B(-x,0)\,\mathrm{d}\alpha \\ &= \frac{1}{n}\int_0^n \mathbb{E}B_\alpha(-x,0)\,\mathrm{d}\alpha \\ &= \frac{1}{n}\left[\int_0^n \mathbb{E}T(-x,H_\alpha)\,\mathrm{d}\alpha - \int_0^n \mathbb{E}T(0,H_\alpha)\,\mathrm{d}\alpha\right].\end{aligned}$$

By shifting by x, $\mathbb{E}T(-x,H_\alpha)=\mathbb{E}T(0,H_{\alpha+x\cdot\rho})$, and by changing variables, we get

$$\begin{aligned}
&\frac{1}{n}\left[\int_{x\cdot\rho}^{n+x\cdot\rho}\mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha-\int_0^n\mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha\right]\\
&=\frac{1}{n}\left[\int_{n}^{n+x\cdot\rho}\mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha-\int_0^{x\cdot\rho}\mathbb{E}T(0,H_\alpha)\ \mathrm{d}\alpha\right]\\
&\sim\int_n^{n+x\cdot\rho}\frac{\mathbb{E}T(0,H_\alpha)}{n}\ \mathrm{d}\alpha\\
&=\int_0^{x\cdot\rho}\frac{\mathbb{E}T(0,H_{n+\alpha})}{n+\alpha}\cdot\frac{n+\alpha}{n}\ \mathrm{d}\alpha.
\end{aligned}$$

A similar result to an earlier exercise here gives

$$\mathbb{E}T(0,H_\alpha)/\alpha\to 1 \text{ as } \alpha\to\infty.$$

So we obtain as a limit $x\cdot\rho$. Now we simply need to know that $\mathbb{E}_{\mu_n^*}B(-x,0)\to\mathbb{E}_\mu B(-x,0)$. This follows from:

(Exercise.) Show that $\sup_n\mathbb{E}_{\mu_n^*}B(x,y)^{d+\delta}<\infty$ for all x,y, and conclude that the above convergence holds.

We conclude that $\mathbb{E}_\mu B(-x,0)=\rho\cdot x$, and for $x,y\in\mathbb{Z}^d$, by translation invariance,

$$\mathbb{E}_\mu B(x,y)=\mathbb{E}_\mu B(-(y-x),0)=(y-x)\cdot\rho.$$

□

As usual, we can try to upgrade to a shape theorem, but in the current case, we do not necessarily have ergodicity. Nonetheless, we can prove a random shape theorem by breaking μ into ergodic components. We would have ergodicity if the coordinates B and η we simply functions of (t_e), since the t_e's are ergodic. But although this holds μ_α-a.s. for each α, it does not have to be true in the limit (that is, using the measure μ). This is in fact the main issue why we must choose z_0 to be a point of differentiability for the main theorem.

Indeed, in a simpler context, one can find random variables X_n and functions f_n such that the vectors $(X_n,f_n(X_n))$ converge (jointly) in distribution to a vector (X,Y), but Y is not a function of X.

Theorem 6.5 (Busemann shape theorem)**.** *There exists a random vector $\hat\rho$ with $\mathbb{E}_\mu\hat\rho=\rho$ such that for all $\epsilon>0$,*

$$\mathbb{P}_\mu\left(|B(0,x)-\hat\rho\cdot x|>\epsilon\|x\| \text{ for infinitely many } x\in\mathbb{Z}^d\right)=0.$$

Proposition 6.6. *μ-a.s., the hyperplane $\hat H=\{w:\hat\rho\cdot w=1\}$ is a supporting hyperplane for $\mathcal{B}$ at z_0.*

Proof. We will show that (a) for $z\in\mathcal{B}$, one has μ-a.s. that $\hat\rho\cdot z\le 1$ and (b) $\mathbb{E}_\mu\hat\rho\cdot z_0=1$. These two statements imply that $\hat\rho\cdot z_0=1$ a.s. (meaning $z_0\in\hat H\cap\partial\mathcal{B}$) and $\mathcal{B}$ lies on one side of $\hat H$.

For the first, since $|B(x,y)| \le T(x,y)$ for all x,y, if $z \in \mathcal{B}$, μ-almost surely

$$\hat{\rho} \cdot z = \lim_n \frac{B(0,nz)}{n} \le \lim_n \frac{T(0,nz)}{n} = g(z) \le 1.$$

For the second since $z_0 \in H$,

$$\mathbb{E}_\mu \hat{\rho} \cdot z_0 = \rho \cdot z_0 = 1.$$

□

If H is a veritable tangent plane to $\mathcal{B}$, then we can determine $\hat{\rho} = \rho$ almost surely.

Corollary 6.7. *If H is the unique supporting hyperplane for $\mathcal{B}$ at z_0, then μ-a.s. $\hat{H} = H$ and $\hat{\rho} = \rho$.*

From here, we can basically repeat everything from the last section, to obtain the main result:

Theorem 6.8. *Let $\hat{S} = \partial\mathcal{B} \cap \hat{H}$. μ-almost surely, every infinite directed path in $\mathbb{G}$ is directed in $\hat{S}$.*

Proof. Identical to the last section. If $\gamma = 0, x_1, x_2, \ldots$ is an infinite directed path from 0, then $B(0,x_n) = \hat{\rho} \cdot x_n + o(\|x_n\|)$ but also equals $T(0,x_n) = g(x_n) + o(\|x_n\|)$. Thus if $x_{n_k}/g(x_{n_k}) \to z$,

$$\frac{B(0,x_{n_k})}{g(x_{n_k})} = \hat{\rho} \cdot \left(\frac{x_{n_k} + o(\|x_{n_k}\|)}{g(x_{n_k})} \right) \to \hat{\rho} \cdot z$$

and also equals

$$\frac{T(0,x_{n_k})}{g(x_{n_k})} \to 1.$$

This means $z \in \partial\mathcal{B} \cap \hat{H}$.

□

In the differentiable case, let

$$A = \{\text{every infinite directed path in } \mathbb{G} \text{ is directed in } S\}.$$

Then $\mu(A) = 1$ and so

$$\mu\left[\mu(A \mid (t_e)) = 1\right] = 1.$$

(The inside is the regular conditional probability measure.) On the event $\{\mu(A \mid (t_e)) = 1$, there are geodesic graph configurations for (t_e) in which 0 has an infinite directed path in S. This edge-weight event has probability one in μ and thus in our original $\mathbb{P}$, and this completes the proof. □

7 Bigeodesics

We have talked for the whole course about infinite geodesics, but we have not yet mentioned that they can come in two versions.

Definition 7.1. *An infinite geodesic indexed by* $\mathbb{N}$ *is a unigeodesic. One indexed by* $\mathbb{Z}$ *is a bigeodesic.*

A bigeodesic has both ends which are themselves unigeodesics. Although we have seen that there are always unigeodesics (and often many of them), the opposite is expected to be true for bigeodesics (at least in low dimensions). The following question is attributed to Furstenberg by Kesten [24, (9.22)]:

Question. Do bigeodesics exist with positive probability?

It is believed that for $d = 2$ (and suitably low dimension), with probability one, there are no bigeodesics. This is based on nonrigorous physics arguments and heuristic arguments from values of scaling exponents. (Such an argument appears in [6, Sec. 4.5.1].)

The main result we want to sketch in this section is in the recent paper of Damron-Hanson [11]. It is a consequence of uniqueness statements we give later. Here we take i.i.d. continuous weights with finite mean, although their results apply to translation invariant weights as well.

Theorem 7.2 (Nonexistence of bigeodesics in fixed directions.)**.** *Suppose* (t_e) *is i.i.d. with continuous marginals and* $\mathbb{E}t_e < \infty$ *and assume that* $\partial\mathcal{B}$ *is differentiable. In dimension* $d = 2$*, let* $z_0 \in \partial\mathcal{B}$ *and* $S_{z_0} = \partial\mathcal{B} \cap H_{z_0}$*, where* H_{z_0} *is the unique supporting line for* $\mathcal{B}$ *at* z_0*. A.s., there is no bigeodesic with an end directed in* S_{z_0}*.*

Since $z_0 \in S_{z_0}$, the result gives nonexistence of bigeodesics with an end directed in $\{z_0\}$, for any deterministic z_0. This result was proved earlier in a related model, directed last-passage percolation, by Georgiou-Rassoul-Agha-Seppäläinen [16, 17].

7.1 Previous results

Work on bigeodesics started with Wehr, Licea-Newman, and Wehr-Woo in the '90s.

- Let $\hat{\mathcal{N}}$ be the (a.s. constant) number of bigeodesics. Then Wehr [34] showed in '97 that under the above assumptions, $\hat{\mathcal{N}} \in \{0, \infty\}$ (for any dimension).
- Wehr-Woo [35] showed in '98 that a.s. there are no bigeodesics in first-passage percolation on a $2d$ half-plane.
- Licea-Newman [27] showed in '96 that in $2d$ if the (t_e)'s are i.i.d. and continuous, there exists a deterministic set $D \subset [0, 2\pi)$ with D^c of measure zero such that for $\theta_1, \theta_2 \in D$,

  $$\mathbb{P}(\exists \text{ bigeodesic with ends directed in } \theta_1, \theta_2) = 0.$$

  Note that for this result, no curvature assumption is needed. However, as we have seen, to know that infinite geodesics have directions at all, we need to assume curvature. So without curvature, this statement could in principle be vacuously true.

The main difficulty with the Licea-Newman result is that D is not specified. (However, Zerner [30, Theorem 1.5] showed that D^c is at most countable, improving the above result.) For instance, it was not known if $0 \in D$. Theorem 7.2 takes care of this, showing that $D = [0, 2\pi)$. However, it is still far from the bigeodesic conjecture, since we would want to show that there is no bigeodesic with an end directed in $\{z_0\}$ for all z_0 simultaneously, and we cannot conclude this from the theorem, since there are uncountably many z_0's.

The route to studying bigeodesics with deterministic directions goes through uniqueness of infinite geodesics with certain directions. The main result of Licea-Newman is:

Theorem 7.3. *Given $\theta \in D$, a.s. any two infinite geodesics with direction θ coalesce.*

Why does this suffice to prove nonexistence of bigeodesics with fixed directions?

Sketch of Licea-Newman given Theorem 7.3. Given $\theta_1, \theta_2 \in D$, one can strengthen the Wehr result on the number of bigeodesics to ones with these directions. That is, letting $\mathcal{N}'$ be the (a.s. constant) number of bigeodesics with directions θ_1, θ_2, one has $\mathcal{N}' \in \{0, \infty\}$. So supposing it is ∞, we can a.s. find two points $x, y \in \mathbb{Z}^2$ such that x is in a bigeodesic P_x and y is in a bigeodesic P_y, and further $x \notin P_y$, $y \notin P_x$. However since these bigeodesics have directions θ_1, θ_2, the uniqueness theorem above implies that their ends need to coalesce. So we can find points $v, w \in P_x \cap P_y$ such that the portion of P_x from v to w is not equal to the portion of P_y from v to w. But both of these portions are finite geodesics, and this means there are two distinct geodesics from v to w, a contradiction, since the weights (t_e) are continuous. $\square$

7.2 Sketch of proof of Theorem 7.2

By the last section, we need only show:

Theorem 7.4. *Let (t_e) be i.i.d. with continuous marginals, and finite mean and assume $\partial\mathcal{B}$ is differentiable. Given $z_0 \in \partial\mathcal{B}$, a.s. all infinite geodesics that are directed in S_{z_0} coalesce.*

To be honest, a slightly different argument is needed to show that this theorem suffices to prove nonexistence of bigeodesics. The reason is that Theorem 7.2 concerns bigeodesics which have at least one end directed in S_{z_0}, whereas the bigeodesics in Licea-Newman were forced to have both ends directed in deterministic directions in D (and so is more restrictive). We will overlook this detail, though.

For the rest of the notes, we sketch the proof of Theorem 7.4.

Proof. For simplicity, we take $z_0 = e_2/g(e_2)$, the point on the boundary of the limit shape in direction e_2. By our differentiability assumptions, one can show that S_{z_0} does not touch the e_1-axis, and therefore each S_{z_0}-directed geodesic intersects the lower half-plane only finitely often. This allows us to reduce the problem to upper half-plane geodesics. (This is actually the biggest step in the paper but in a sense it is the least interesting, so we omit it.)

So for the remainder, we show that a.s., all infinite upper half-plane geodesics that are directed in S_{z_0} coalesce. Upper half-plane geodesics are defined relative to the first-passage model in the upper half-plane. That is, we set

$$V_H = \{x \in \mathbb{Z}^d : x \cdot e_2 \geq 0\}$$

and for $x, y \in V_H$, $T_H(x, y)$ is the minimal passage time of all paths from x to y that take only vertices in V_H. An infinite geodesic is defined similarly to a full-plane one. We will content ourselves with showing the similar statement that a.s., for each $x, y \in L_0 = \{z : z \cdot e_2 = 0\}$, any infinite geodesics from x and y directed in S_{z_0} coalesce.

Step 1. Definition of left- and right-most geodesics. Note that all infinite geodesics from 0 that are directed in S_{z_0} are ordered from left to right. This is a consequence of uniqueness of geodesics. (That is, if an infinite geodesic moves to the "left" of another one, it cannot move back to the "right.") Thus we can define from each $x \in L_0 = \{z : z \cdot e_1 = 0\}$ a left-most and right-most infinite geodesic directed in S_{z_0}: we simply take the right or left boundary of the union of all infinite geodesics from x directed in S_{z_0}. In the full-plane, this is significantly more complicated, but in both cases, if we write these geodesics as

$$\Gamma_x^L \text{ and } \Gamma_x^R \text{ respectively,}$$

then one can show that for each x, a.s., both of these geodesics are also directed in S_{z_0}. This requires the differentiability assumption, as the endpoints of S_{z_0} are points of differentiability of $\partial\mathcal{B}$, so using the results of the last section, we can find infinitely many infinite geodesics from x directed in sectors near these endpoints. These geodesics trap all the ones directed in S_{z_0} from both sides. One can furthermore show that a.s.,

$$\Gamma_x^L \text{ and } \Gamma_y^L \text{ coalesce for } x, y \in L_0,$$

and similarly for rightmost geodesics. (This uses an argument from Licea-Newman.) The goal now is to show that

$$\Gamma_0^L = \Gamma_0^R \text{ a.s.}$$

By extremality, then, any infinite geodesics from x, y must coalesce with these.

Step 2. Left- and right-most Busemann functions. By coalescence, we can define for $x, y \in L_0$,

$$B_*(x, y) = B_{\Gamma_0^*}(x, y) \text{ for } * = L, R.$$

As before, one can show that

$$\mathbb{E}B_*(0, x) = x \cdot \rho^*$$

for $\rho^* \in \mathbb{R}^2$ such that $\{z : z \cdot \rho^* = 1\}$ is a supporting line for $\mathcal{B}$ at z_0. By differentiability, then, one has $\rho^L = \rho^R$ and so

$$\mathbb{E}B_L(0, x) = \mathbb{E}B_R(0, x) \text{ for all } x.$$

(This step requires us to relate to left- and right-most geodesics in the full-plane.) Therefore

$$\Delta(0, x) = \mathbb{E}B_L(0, x) - \mathbb{E}B_R(0, x) = 0.$$

Step 3. Paths crossing argument. However we claim that

$$B_L(0, e_1) \leq B_R(0, e_1) \text{ a.s.}$$

To see why, note that $\Gamma_{e_1}^L$ must intersect Γ_0^R at some vertex w. Taking (x_n) and (y_n) to be sequences on Γ_0^L and Γ_0^R going to infinity, for n large,

$$T(0, y_n) + T(e_1, x_n) = T(0, w) + T(w, y_n) + T(e_1, w) + T(w, x_n) \geq T(0, x_n) + T(e_1, y_n),$$

or

$$T(0, y_n) - T(e_1, y_n) \geq T(0, x_n) - T(e_1, x_n).$$

Taking $n \to \infty$, gives $B_L(0, e_1) \leq B_R(0, e_1)$. Combining with the previous step gives

$$\Delta(0, e_1) = 0 \text{ a.s.}$$

Step 4. The contradiction. Suppose that with positive probability, $\Gamma_0^L \neq \Gamma_0^R$. Then by coalescence, on this event, there is a k large enough so that Γ_0^L and $\Gamma_{ke_1}^L$ coalesce at a different point than that at which Γ_0^R and $\Gamma_{ke_1}^R$ do. By uniqueness of geodesics (using that the Busemann function is just the difference of passage time to the coalescence point), $\Delta(0, k_1) \neq 0$ with positive probability. But by stationarity and the last step,

$$\Delta(0, ke_1) = \sum_{j=1}^{k} \Delta((j-1)e_1, je_1) = 0,$$

a contradiction.

□

Remarks.

1. As in Licea-Newman, we should ask if infinite geodesics must be directed in sectors under our assumptions. Using the existence results from the last section along with trapping, one can show that if $\partial \mathcal{B}$ is differentiable then a.s.
   (a) for each z_0, there is an infinite geodesic from 0 directed in S_{z_0} and
   (b) each infinite geodesic is directed in S_{z_0} for some z_0.
2. The results from [16, 17] in directed LPP are nearly the same as those presented here. Due to directedness of the model, there is no need to reduce to a half-plane, but in Step 1, one must appeal to results analogous to those in the last section (existence of directed geodesics) in LPP, established in [16].

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