Polynomial Chaos and Scaling Limits of Disordered Systems

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Joint work
with

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Nikos Zygouras (Warwick)

Outline

1. Disordered Systems (Disorder Relevance vs Irrelevance)

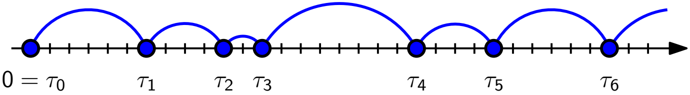
   - Disordered Pinning Model
   - Long-range Directed Polymer Model
   - Random Field Ising Model

2. Disorder Relevance via Continuum and Weak Disorder Limits

   - Polynomial chaos expansions for partition functions
   - Lindeberg Principle for polynomial chaos expansions
   - Convergence of polynomial chaos to Wiener chaos expansions
   - From partition functions to disordered continuum models

3. Some Open Questions

1.1 The Homogeneous Pinning Model



Let $\tau := \{\tau_0 = 0 < \tau_1 < \tau_2 \cdots\} \subset \mathbb{N}_0$ be a recurrent renewal process, with law $\mathbf{P}$, and

$$\mathbf{P}(\tau_1 = n) \sim \frac{C}{n^{1+\alpha}} \qquad \text{for some exponent } \alpha > 0.$$

The Pinning Model is defined by the family of Gibbs measures:

$$\mathbf{P}_{N,h}(\tau) = \frac{1}{Z_{N,h}} e^{h \sum_{n=1}^{N} 1_{\{n \in \tau\}}} \mathbf{P}(\tau) \qquad (\text{expectation } \mathbf{E}_{N,h}[\cdot]),$$

where N is the system size, $h \in \mathbb{R}$ determines the interaction strength, and $Z_{N,h} = \mathbf{E}[e^{h \sum_{n=1}^{N} 1_{\{n \in \tau\}}}]$ is the partition function.

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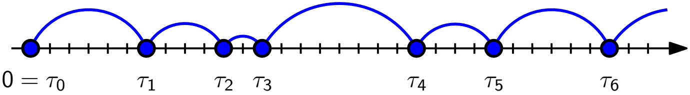

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1.2 Phase Transition for the Pinning Model

As h varies, the pinning model undergoes a localization-delocalization transition. More precisely, there is a critical h_c ($= 0$ in this case) such that

- For $h < h_c$, the limiting contact fraction

$$g(h) := \lim_{N\to\infty} \mathbf{E}_{N,h}\Big[\frac{1}{N}\sum_{n=1}^{N} 1_{\{n\in\tau\}}\Big] = 0;$$

- For $h > h_c$, the limiting contact fraction $g(h) > 0$.

Furthermore, $g(h) = F'(h)$, where the free energy

$$F(h) = \lim_{N\to\infty} \frac{1}{N}\log Z_{N,h} \begin{cases} = \quad 0 & \text{if } h \le h_c, \\ \approx C(h-h_c)^\gamma & \text{as } h \downarrow h_c. \end{cases}$$

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Let $\omega := (\omega_n)_{n\in\mathbb{N}}$ be i.i.d. with $\mathbb{E}[\omega_1] = 0$ and $\mathbb{E}[e^{\lambda\omega_1}] < \infty$ for all λ close to 0.

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For each $\beta > 0$, as h varies, the disordered pinning model also undergoes a localization-delocalization transition.

There exists $\hat{h}_c(\beta) < 0$, s.t. for $\mathbb{P}$-a.e. ω, the contact fraction

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Basic Question: Does disorder modify the qualitative nature of the homogeneous model (without disorder)?

For the pinning model, we say that disorder is

- relevant if the critical exponents $\hat{\gamma}(\beta) \neq \gamma$ for all $\beta > 0$ (no matter how weak is the disorder strength);
- irrelevant if $\hat{\gamma}(\beta) = \gamma$ for $\beta > 0$ sufficiently small.

For the pinning model with renewal exponent α, it has been shown:

- Disorder is relevant for $\alpha > \frac{1}{2}$;
- Disorder is irrelevant for $\alpha < \frac{1}{2}$;
- Disorder is marginally relevant for $\alpha = \frac{1}{2}$.

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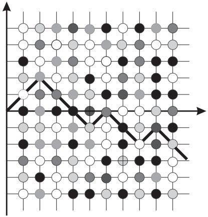
2.1 Directed Polymer Model



Let $X := (X_n)_{n\in\mathbb{N}_0}$ be a mean-zero random walk on $\mathbb{Z}^d$ with law $\mathbf{P}$.

Let $\omega := (\omega(n,x))_{n\in\mathbb{N}_0, x\in\mathbb{Z}^d}$ be i.i.d. with $\mathbb{E}[\omega(0,o)] = 0$, and $\mathbb{E}[e^{\lambda\omega(0,o)}] < \infty$ for all λ close to 0.

Given disorder ω, the Directed Polymer Model on $\mathbb{Z}^{d+1}$ is defined by the family of Gibbs measures

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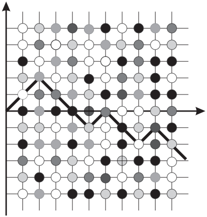

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Assuming X to be diffusive, it has been shown that:

- $\beta_c(d) = 0$ for $d = 1$ and 2, and hence disorder is relevant;
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Assuming that $d = 1$ and X is in the domain of attraction of an α-stable process for some $\alpha \in (0, 2]$, then similarly:

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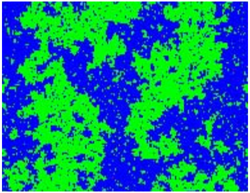

The Ising model on a domain $\Omega \subset \mathbb{Z}^d$ with $+$ boundary condition, at inverse temperature $\beta \geq 0$ and external field $h \in \mathbb{R}$, is given by the following Gibbs measure on spin configurations $(\sigma_x)_{x\in\Omega} \in \{\pm 1\}^\Omega$:

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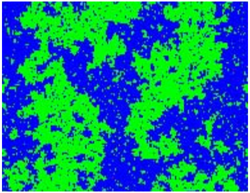

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For $d = 2$, $\beta_c = \frac{1}{2}\log(1+\sqrt{2})$, and as we vary the external field h at $\beta = \beta_c$, Camia-Garban-Newman'12 recently showed that

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3.3 The Two-Dimensional Random Field Ising Model

We now add disorder to the Ising model on $\mathbb{Z}^2$ at $\beta = \beta_c$ in the form of a random external field.

Let $\omega := (\omega_x)_{x \in \mathbb{Z}^2}$ be i.i.d. with $\mathbb{E}[\omega_x] = 0$ and $\mathbb{E}[e^{\lambda \omega_x}] < \infty$ for all λ close to 0.

Given ω, disorder strength $\lambda \geq 0$ and external field $h \in \mathbb{R}$, we define the Random Field version of the critical Ising model on $\Omega \subset \mathbb{Z}^2$ by

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Question: Is disorder relevant in the sense that for arbitrary small disorder strength $\lambda > 0$, the magnetization

$$\hat{m}(\lambda, h) := \lim_{\Omega \uparrow \mathbb{Z}^2} \mathbb{E}\mathbf{E}^{\omega}_{\Omega,\lambda,h}\Big[\frac{1}{|\Omega|}\sum_{x \in \Omega} \sigma_x\Big] \approx C h^{\gamma} \quad \text{as } h \downarrow 0$$

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$$\hat{m}(\lambda, h) := \lim_{\Omega \uparrow \mathbb{Z}^2} \mathbb{E}\mathbf{E}^{\omega}_{\Omega,\lambda,h}\Big[\frac{1}{|\Omega|} \sum_{x \in \Omega} \sigma_x\Big] \approx C h^{\gamma} \quad \text{as } h \downarrow 0$$

for some critical exponent $\gamma \neq \frac{1}{15}$?

3.3 The Two-Dimensional Random Field Ising Model

We now add disorder to the Ising model on $\mathbb{Z}^2$ at $\beta = \beta_c$ in the form of a random external field.

Let $\omega := (\omega_x)_{x \in \mathbb{Z}^2}$ be i.i.d. with $\mathbb{E}[\omega_x] = 0$ and $\mathbb{E}[e^{\lambda \omega_x}] < \infty$ for all λ close to 0.

Given ω, disorder strength $\lambda \geq 0$ and external field $h \in \mathbb{R}$, we define the Random Field version of the critical Ising model on $\Omega \subset \mathbb{Z}^2$ by

$$\mathbf{P}^{\omega}_{\Omega,\lambda,h}(\sigma) = \frac{1}{Z^{\omega}_{\Omega,\lambda,h}} \exp\Big\{\sum_{x\in\Omega} (\lambda\omega_x + h)\sigma_x\Big\} \mathbf{P}_{\Omega,\beta_c,0}(\sigma),$$

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4.1 Disorder Relevance via Scaling Limits (Heuristics)

We propose here a new perspective on disorder relevance/irrelevance, which gives a unified treatment for many disordered systems.

Observation: Disorder relevance means that, fixed disorder strength, however weak, is still too strong since it changes the qualitative features of the homogeneous model in the ∞-volume limit.

To moderate the effect of disorder, it should be possible to tune the disorder strength down to zero as the system size tends to infinity (while rescaling space), so that disorder persists in such a weak disorder and continuum scaling limit.

Disorder relevance thus manifests itself in the existence of a non trivial continuum disordered model in a suitable weak disorder and continuum limit. (Consistent with Harris' Criterion'74 for disorder relevance).

Inspired by Alberts-Khanin-Quastel'12 construction of the Continuum Directed Polymer Model in $\mathbb{Z}^{1+1}$, we cast things in the much more general framework of disorder relevance-irrelevance, give general criteria for convergence to continuum disordered models, and apply them to new models of interest.

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4.2 The General Setting

We first study weak disorder and continuum limits of the partition function in a general setting, which includes all previous models as special cases.

Let $\Omega \subset \mathbb{R}^d$. For $\delta \in (0,1)$, let $\Omega_\delta := \Omega \cap (\delta\mathbb{Z})^d$. Let $(\omega_x)_{x\in\Omega_\delta}$ be i.i.d. with $\mathbb{E}[\omega_x] = 0$ and $\mathbb{E}[e^{\lambda\omega_x}] < \infty$ for all λ close to 0.

Let $\mathbf{P}_{\Omega_\delta}$ be a probability measure on $(\sigma_x)_{x\in\Omega_\delta} \in \{0,1\}^{\Omega_\delta}$ that defines the homogeneous model. Given ω, disorder strength λ and bias h, add disorder in the form of a random field by defining

$$\mathbf{P}^\omega_{\Omega_\delta,\lambda,h}(\sigma) = \frac{1}{Z^\omega_{\Omega_\delta,\lambda,h}} e^{\sum_{x\in\Omega_\delta}(\lambda\omega_x+h)\sigma_x}\mathbf{P}_{\Omega_\delta}(\sigma),$$

where $Z^\omega_{\Omega_\delta,\lambda,h}$ is the partition function.

To identify non-trivial disordered limits of $Z^\omega_{\Omega_\delta,\lambda,h}$ in the continuum and weak disorder limit $\delta \downarrow 0$, $\lambda = \lambda(\delta) \downarrow 0$, $h = h(\delta) \downarrow 0$, we first rewrite $Z^\omega_{\Omega_\delta,\lambda,h}$ in a polynomial chaos expansion.

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4.3 Polynomial Chaos Expansion for Partition Function

Because $\sigma_x \in \{0,1\}$, by cluster expansion,

$$\begin{aligned} Z^{\omega}_{\Omega_\delta,\lambda,h} &= \mathbf{E}_{\Omega_\delta}\Big[\prod_{x\in\Omega_\delta} e^{(\lambda\omega_x+h)\sigma_x}\Big] \\ &= \mathbf{E}_{\Omega_\delta}\Big[\prod_{x\in\Omega_\delta} (1+\xi_x\sigma_x)\Big] \qquad (\xi_x := e^{\lambda\omega_x+h}-1) \\ &= 1+\sum_{k=1}^{\infty}\sum_{\substack{I=\{x_1,\ldots,x_k\}\subset\Omega_\delta \\ |I|=k}} \mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}]\,\xi_{x_1}\cdots\xi_{x_k}, \end{aligned}$$

which is multi-linear in the i.i.d. random variables $(\xi_x)_{x\in\Omega_\delta}$ with

$$\mathbb{E}[\xi_x] \approx h(\delta)+\frac{\lambda^2(\delta)}{2} =: \tilde{h}(\delta), \qquad \mathbb{V}ar(\xi_x) \approx \lambda^2(\delta) \quad \text{as } \delta\downarrow 0.$$

Each ξ_x is associated with a cube Δ_x of side length δ in $(\delta\mathbb{Z})^d$, and we can replace ξ_x by a normal variable with the same mean and variance

$$\xi_x \longrightarrow \int_{\Delta_x} \lambda(\delta)\delta^{-\frac{d}{2}} W(\mathrm{d}u) + \int_{\Delta_x} \tilde{h}(\delta)\delta^{-d}\mathrm{d}u,$$

where $W(\mathrm{d}u)$ is a d-dimensional white noise on $\mathbb{R}^d$. This is justified by a Lindeberg principle, extending Mossel-O'Donnell-Oleszkiewicz'10.

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4.4 Convergence to Wiener Chaos Expansions

We then have

$$Z^{\omega}_{\Omega_\delta,\lambda,h} \overset{\delta\downarrow 0}{\approx} 1+\sum_{k=1}^{\infty}\frac{1}{k!}\int\cdots\int_{\Omega^k} \mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}]\prod_{i=1}^{k}\left(\lambda\delta^{-\frac{d}{2}}W(\mathrm{d}x_i)+\tilde{h}\delta^{-d}\mathrm{d}x_i\right).$$

Key Assumption: There exists $\gamma \geq 0$ such that the rescaled k-point correlation function

$$(\delta^{-\gamma})^k\mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}] \xrightarrow[\delta\downarrow 0]{L^2} \psi_\Omega(x_1,\ldots,x_k)\in L^2(\Omega^k),$$

and let $\lambda(\delta) := \hat{\lambda}\delta^{\frac{d}{2}-\gamma}$, $\quad \tilde{h}(\delta) := \hat{h}\delta^{d-\gamma}$ $\quad$ for some $\hat{\lambda}>0, \hat{h}\in\mathbb{R}$,

then

$$1+\sum_{k=1}^{\infty}\frac{1}{k!}\int\cdots\int_{\Omega^k}\delta^{-k\gamma}\mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}]\prod_{i=1}^{k}\left(\lambda\delta^{\gamma-\frac{d}{2}}W(\mathrm{d}x_i)+\tilde{h}\delta^{\gamma-d}\mathrm{d}x_i\right)$$

$$\underset{\delta\to 0}{\Longrightarrow} \mathcal{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}} := 1+\sum_{k=1}^{\infty}\frac{1}{k!}\int\cdots\int_{\Omega^k}\psi_\Omega(x_1,\ldots,x_k)\prod_{i=1}^{k}\left(\hat{\lambda}W(\mathrm{d}x_i)+\hat{h}\mathrm{d}x_i\right),$$

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4.5 Scaling Limit of the Disordered Pinning Model

$\Omega := [0,1]$, and $\mathbf{P}_{\Omega_\delta}$ is the law of the rescaled renewal process. Then

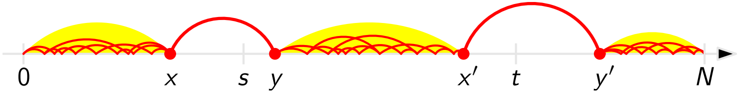
$$(\delta^{\min\{1,\alpha\}-1})^k \mathbf{E}_{\Omega_\delta}[\sigma_{x_1} \cdots \sigma_{x_k}] \xrightarrow[\delta\downarrow 0]{L^2} \psi(x_1, \ldots, x_k),$$

where ψ is the correlation function of the α-stable regenerative set and is in L^2 exactly when $\alpha > \frac{1}{2}$ (disorder relevant regime). Let

$$\lambda(\delta) = \hat{\lambda}\delta^{\min\{1,\alpha\}-\frac{1}{2}}, \qquad h(\delta) = \hat{h}\delta^{\min\{1,\alpha\}} - \lambda^2(\delta)/2.$$

Then the partition function $Z^\omega_{\Omega_\delta,\lambda,h}$ converges weakly to $\mathcal{Z}^W_{\Omega,\hat{\lambda},\hat{h}}$.

The weak convergence can be extended to the family of point-to-point partitions $Z^{\omega,c}_{[a,b]_\delta,\lambda,h}$, indexed by all $[a,b] \subset [0,1]$ with boundary pinning constraints. The limiting family of continuum partition functions $(\mathcal{Z}^W_{[a,b],\hat{\lambda},\hat{h}})_{[a,b]\subset[0,1]}$ can then be used to construct the Continuum Disordered Pinning Model in a white noise random environment.



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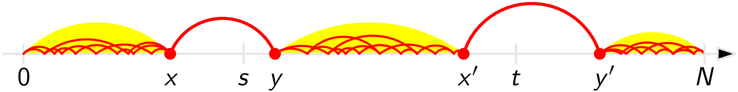

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$$(\delta^{\min\{1,\alpha\}-1})^k \mathbf{E}_{\Omega_\delta}[\sigma_{x_1}\cdots\sigma_{x_k}] \xrightarrow[\delta\downarrow 0]{L^2} \psi(x_1,\ldots,x_k),$$

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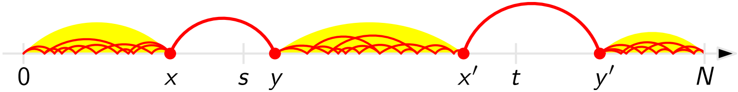

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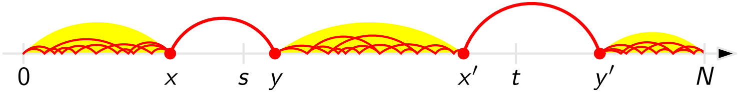

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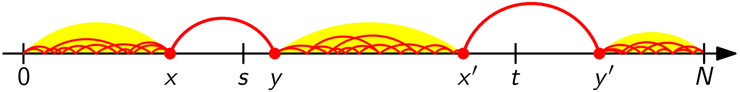

4.6 Scaling Limit of the Long-range Directed Polymer

Let $\Omega := [0,1] \times \mathbb{R}$, and let $\Omega_\delta := \Omega \cap (\delta\mathbb{Z}) \times (\delta^{1/\alpha}\mathbb{Z})$ with $\alpha \in (0,2]$. Let $\mathbf{P}_{\Omega_\delta}$ be the law of a rescaled random walk, which converges in distribution to an α-stable process as $\delta \downarrow 0$. Then

$$(\delta^{-1/\alpha})^k \mathbf{E}_{\Omega_\delta}[\sigma_{(t_1,x_1)} \cdots \sigma_{(t_k,x_k)}] \xrightarrow[\delta\downarrow 0]{L^2} \psi((t_1,x_1),\ldots,(t_k,x_k)),$$

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$$\lambda(\delta) = \hat{\lambda}\delta^{\frac{\alpha-1}{2\alpha}}.$$

Then the random partition function $Z^\omega_{\Omega_\delta,\lambda}$ converges weakly to $\mathcal{Z}^W_{\Omega,\hat{\lambda}}$, generalizing work of Alberts-Khanin-Quastel'12 for the case $\alpha = 2$.

Extending the weak convergence to the family of point-to-point partition functions $(Z^{\omega,c}_\lambda(s,x;t,y))_{0\le s<t\le 1; x,y\in\mathbb{R}}$, we obtain a family of continuum partition functions $(\mathcal{Z}^{W,c}_{\hat{\lambda}}(s,x;t,y))_{0\le s<t\le 1; x,y\in\mathbb{R}}$, which can be used to construct the Continuum Long-range Directed Polymer, extending Alberts-Khanin-Quastel'12.

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4.7 Scaling Limit of the Random Field Ising Model

Let $\Omega \subset \mathbb{R}^2$ be bounded, simply connected, with piecewise smooth boundary. Let $\mathbf{P}_{\Omega_\delta}$ be the law of the critical Ising model on Ω_δ with $+$ boundary condition. Chelkak-Hongler-Izyurov'12 have shown that

$$(\delta^{-\frac{1}{8}})^k \mathbf{E}_{\Omega_\delta}[\sigma_{x_1} \cdots \sigma_{x_k}] \xrightarrow[\delta \downarrow 0]{\text{p.w.}} \psi_\Omega(x_1, \ldots, x_k)$$

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Then the disordered partition function $Z^{\omega}_{\Omega_\delta,\lambda,h}$ converges weakly to $\mathcal{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}}$ with white noise disorder W.

Constructing a Continuum Random Field Ising Model out of $\mathcal{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}}$ seems difficult. Firstly, we need to construct a family of such partition functions indexed by a large enough family of domains Ω with rich enough boundary conditions. Secondly, the continuum model is expected to be a generalized field, as in the case with no disorder ($\lambda = 0$) constructed recently by Camia-Garban-Newman'13.

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5.1 Open Question: Interchanging Limits

Recall that $Z^{\omega}_{\Omega_\delta,\lambda,h}$ is the partition function of the random field perturbation of a homogeneous model with measure $\mathbf{P}_{\Omega_\delta}$. With

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The free energy of the continuum disorder model is given by

$$\mathcal{F}(\hat{\lambda},\hat{h}) = \lim_{\Omega\uparrow\mathbb{R}^d} \frac{1}{|\Omega|}\mathbb{E}[\log \mathcal{Z}^{W}_{\Omega,\hat{\lambda},\hat{h}}] = \lim_{\Omega\uparrow\mathbb{R}^d} \frac{1}{|\Omega|}\lim_{\delta\downarrow 0}\mathbb{E}[\log Z^{\omega}_{\Omega_\delta,\lambda,h}].$$

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For each model, this leads to conjectures on the precise asymptotics for the free energy of the disordered model in the weak disorder limit. For the copolymer model, this interchange of limits has been justified (Bolthausen-den Hollander'97, Caravenna-Giacomin'10).

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5.2 Universality for Long-range Directed Polymer

For each $\alpha \in (1, 2]$, by taking the weak disorder and continuum limit, we can construct a family of disordered point-to-point continuum partition functions $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, x)$.

As a function in $t \geq 0$ and $x \in \mathbb{R}$, $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, x)$ is a mild solution for the stochastic fractional heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta^{\frac{\alpha}{2}} u + \hat{\lambda} W u, \\ u(0, \cdot) = \delta_0(\cdot). \end{cases}$$

For $\alpha = 2$, as $\hat{\lambda} : 0 \uparrow \infty$, the distribution of $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, 0)$ is known to smoothly interpolate between the Gaussian and the Tracy-Widom GUE distribution, which is the universal fluctuation of short-range directed polymers in $\mathbb{Z}^{1+1}$.

Question: For $\alpha \in (1, 2)$, as $\hat{\lambda} \uparrow \infty$, does the law of $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, 0)$ converge to a limit that generalizes Tracy-Widom GUE and governs the universal fluctuation of α-stable directed polymer in $\mathbb{Z}^{1+1}$?

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As a function in $t \geq 0$ and $x \in \mathbb{R}$, $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, x)$ is a mild solution for the stochastic fractional heat equation

$$\begin{cases} \dfrac{\partial u}{\partial t} = \Delta^{\frac{\alpha}{2}} u + \hat{\lambda} W u, \\ u(0, \cdot) = \delta_0(\cdot). \end{cases}$$

For $\alpha = 2$, as $\hat{\lambda} : 0 \uparrow \infty$, the distribution of $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, 0)$ is known to smoothly interpolate between the Gaussian and the Tracy-Widom GUE distribution, which is the universal fluctuation of short-range directed polymers in $\mathbb{Z}^{1+1}$.

Question: For $\alpha \in (1, 2)$, as $\hat{\lambda} \uparrow \infty$, does the law of $\mathcal{Z}^W_{\hat{\lambda}}(0, 0; t, 0)$ converge to a limit that generalizes Tracy-Widom GUE and governs the universal fluctuation of α-stable directed polymer in $\mathbb{Z}^{1+1}$?

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5.3 Open Questions for Random Field Ising

- Go beyond the partition function and construct the Continuum Random Field Ising Model as a generalized random field in a white noise environment (extending Camia-Garban-Newman'13 for the non-disordered case). The law of the disordered field is likely to be singular w.r.t. the non-disordered field.

- Since the partition functions of the random field perturbation of the critical Ising model on $\mathbb{Z}^2$ has non-trivial disordered limits, it is natural to conjecture that disorder is relevant in the sense that:

  Perturbing the critical Ising model on $\mathbb{Z}^2$ by a random field $(\lambda\omega_x + h)_{x\in\mathbb{Z}^2}$ with arbitrarily small $\lambda > 0$, the magnetization

$$\hat{m}(\lambda, h) := \lim_{\Omega\uparrow\mathbb{Z}^2} \mathbb{E}\mathbf{E}^{\omega}_{\Omega,\lambda,h}\Big[\frac{1}{|\Omega|}\sum_{x\in\Omega}\sigma_x\Big] \approx Ch^{\gamma} \quad \text{as } h \downarrow 0$$

  for some critical exponent $\gamma(\lambda)>\gamma(0) = \frac{1}{15}$ (we conjecture that disorder has a smoothing effect on the phase transition in h).

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