The Cut tree of the Brownian Continuum Random Tree and the Reverse Problem

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24 June 2014

Motivation

Introduction to the Brownian CRT

- Let T_n be a uniform tree of n vertices.

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  - Let each edge have length $1/\sqrt{n}$ $\rightsquigarrow$ *metric space*
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  - Denote by $\frac{1}{\sqrt{n}} T_n$ the obtained metric measure space.

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- Aldous ('91):

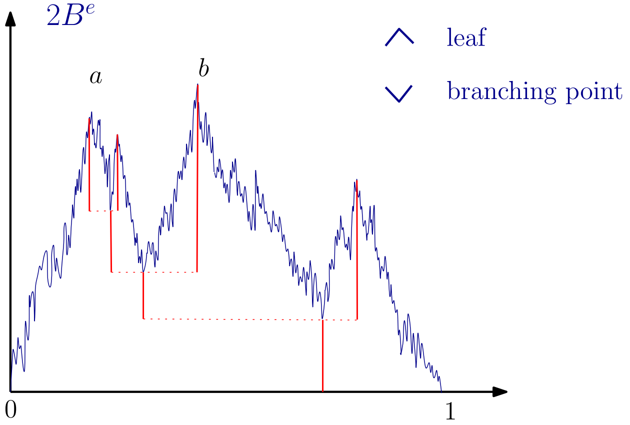
$$\frac{1}{\sqrt{n}} T_n \Longrightarrow \mathcal{T}, \quad n \to \infty,$$

where $\mathcal{T}$ is the Brownian CRT (*Continuum Random Tree*).

Motivation

Brownian CRT seen from Brownian excursion

Let B^e be the normalized Brownian excursion. Then $\mathcal{T}$ is encoded by $2B^e$.



Motivation

Brownian CRT

$\mathcal{T}$ is

- a (random) compact metric space such that $\forall u, v \in \mathcal{T}$, $\exists$ unique geodesic $[\![u, v]\!]$ between u and v;
- equipped with a probability measure μ (mass measure), concentrated on the leaves;
- equipped with a σ-finite measure ℓ (length measure) such that $\ell([\![u, v]\!]) =$ distance between u and v.

Motivation

Aldous–Pitman's fragmentation process

Let $\mathcal{P}$ be a Poisson point process on $[0, \infty) \times \mathcal{T}$ of intensity $dt \otimes \ell(dx)$.

- $\mathcal{P}_t := \{x \in \mathcal{T} : \exists\, s \leq t$ such that $(s, x) \in \mathcal{P}\}$.
- If $v \in \mathcal{T}$, let $\mathcal{T}_v(t)$ be the connected component of $\mathcal{T} \setminus \mathcal{P}_t$ containing v.

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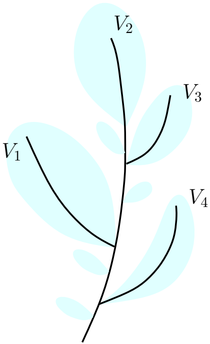
Genealogy of Aldous-Pitman's fragmentation

Let $V_1, V_2, \cdots$ be independent leaves picked from μ.

subtree of $\mathcal{T}$ spanned by $V_1, \cdots, V_k$



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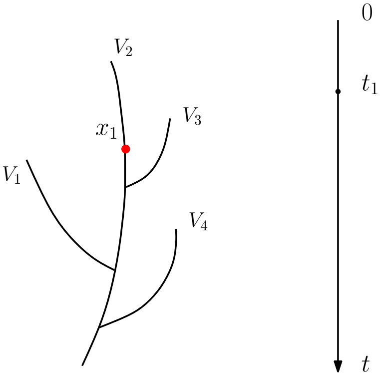

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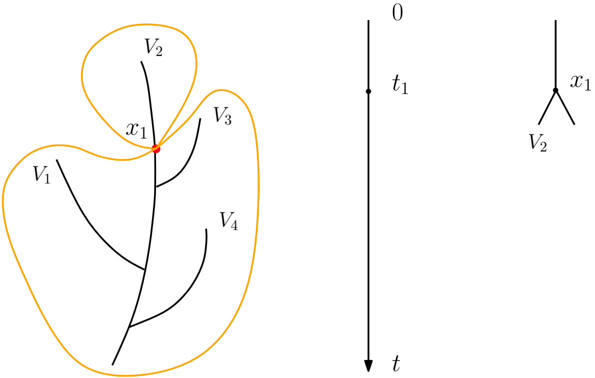

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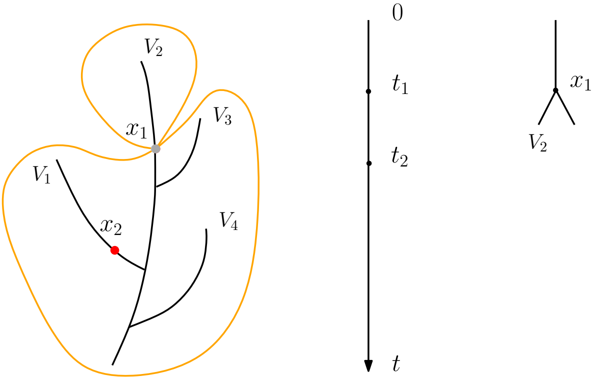

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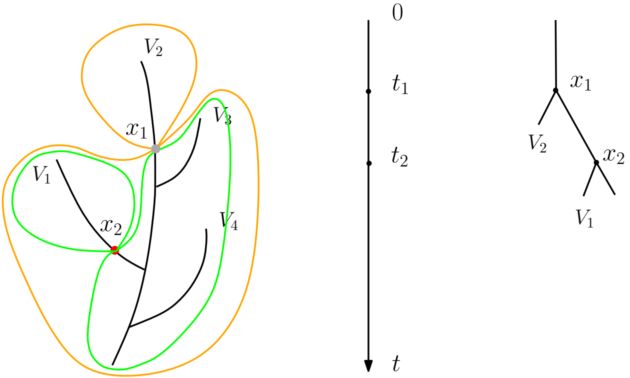

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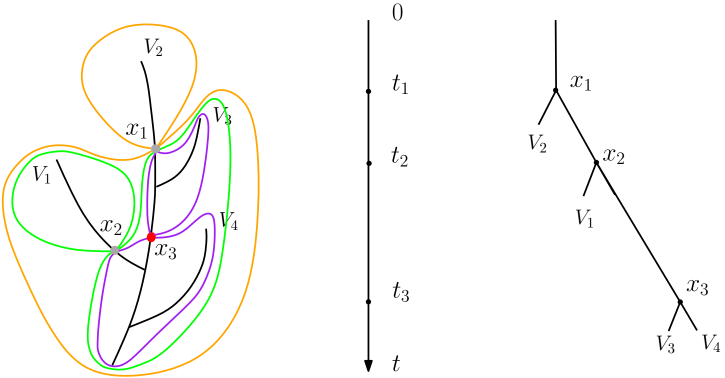

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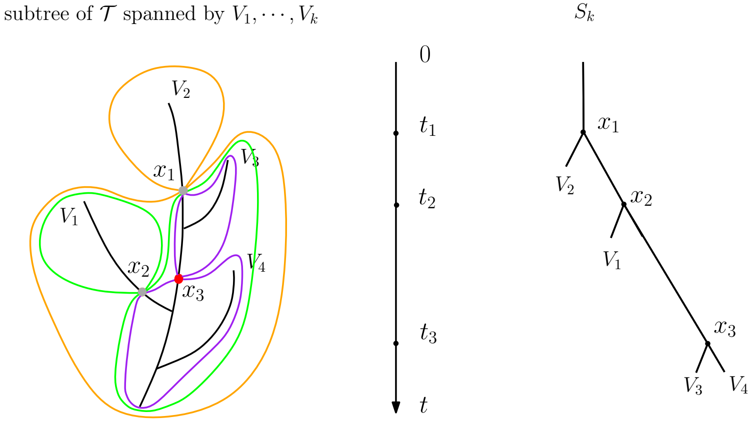

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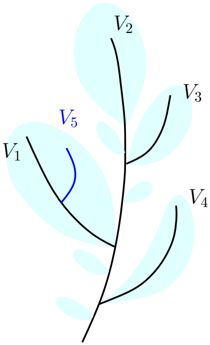

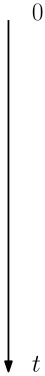

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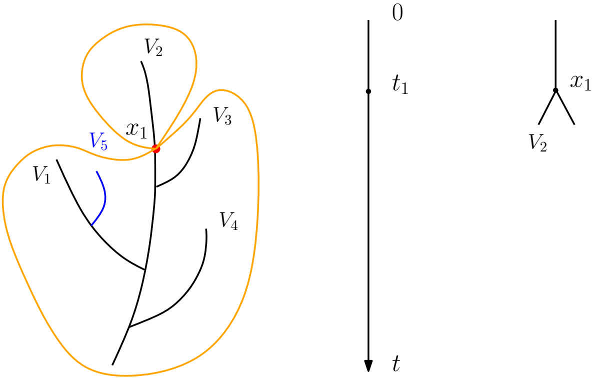

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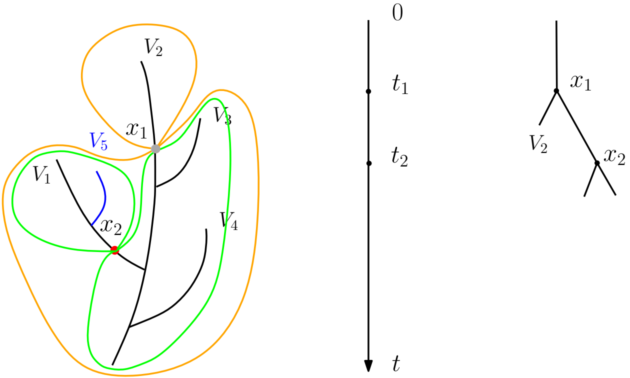

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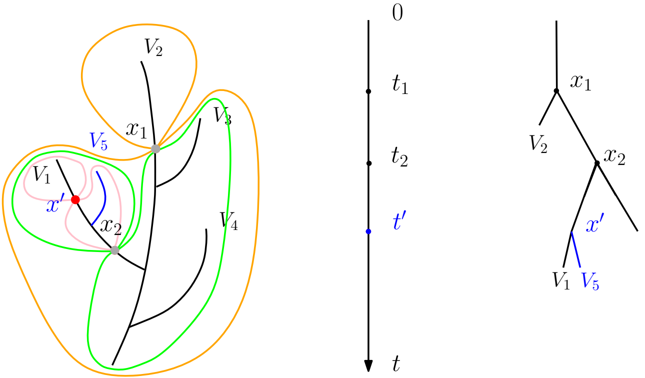

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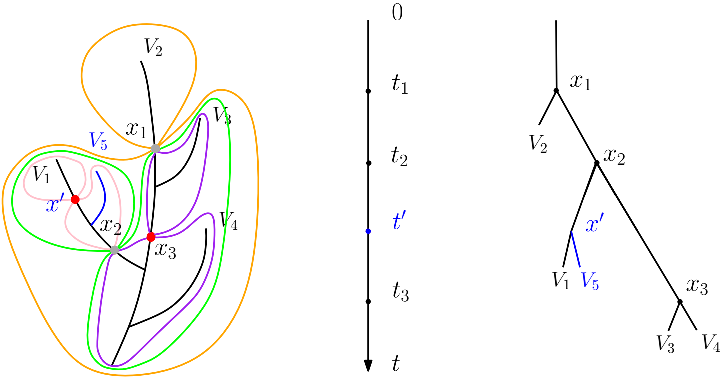

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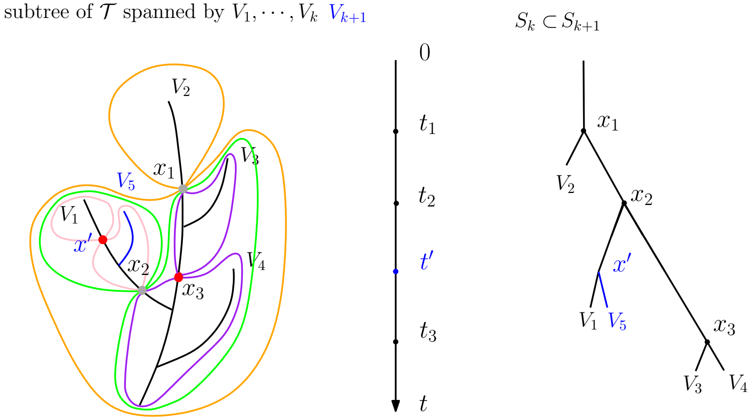


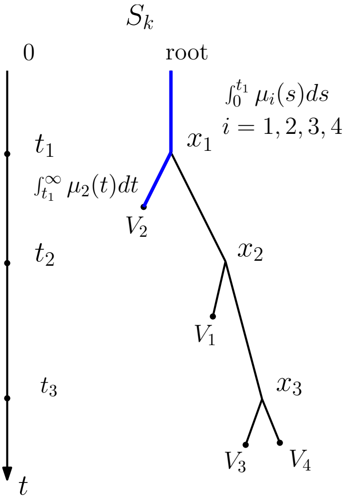
Motivation

Cut tree of the Brownian CRT

Equip S_k with a distance d such that

$$d(\text{root}, V_i) = \int_0^\infty \mu_i(t)dt := L_i,$$

with $\mu_i(t) := \mu(\mathcal{T}_{V_i}(t))$.



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Bertoin & Miermont, 2012

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Question: given $\text{cut}(\mathcal{T})$, can we recover $\mathcal{T}$?

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Theorem (Broutin & W., 2014)

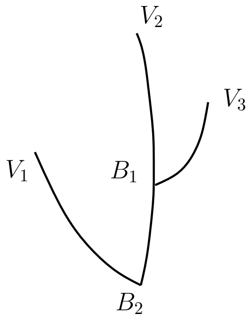
Let $\mathcal{H}$ be the Brownian CRT. Almost surely, there exist $\mathrm{shuff}(\mathcal{H})$ *such that*

$$(\mathrm{shuff}(\mathcal{H}), \mathcal{H}) \stackrel{d}{=} (\mathcal{T}, \mathrm{cut}(\mathcal{T})).$$

Related discrete model

Cutting down uniform tree

A uniform tree T_n



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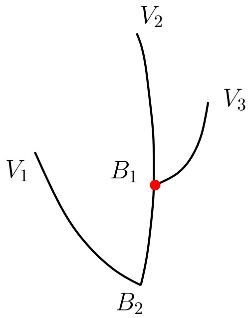

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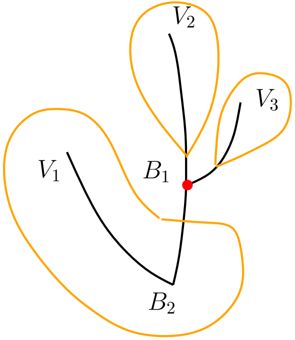

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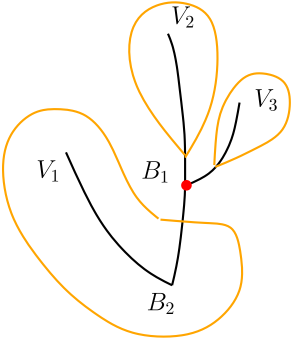

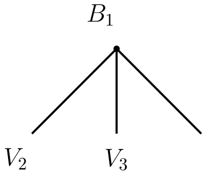

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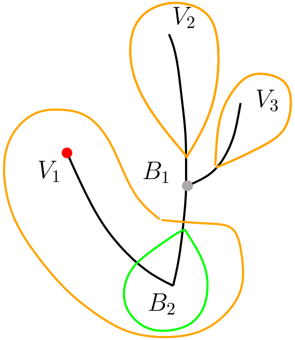

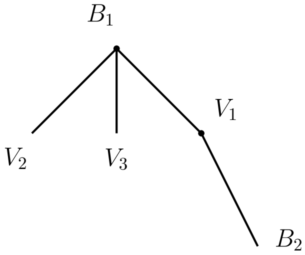

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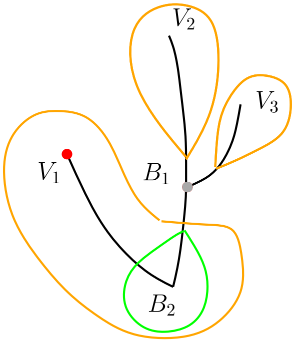
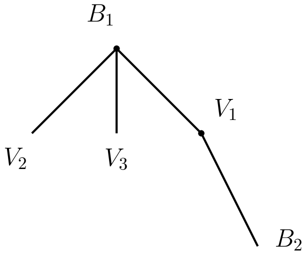




$\mathrm{cut}(T_n)$

Related discrete model

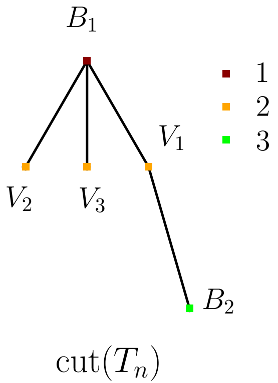
Cut tree of T_n

For $v \in T_n$,

let $L_n(v) :=$ nb. of picks affecting the size of the connected component containing v.

Then, $L_n(v) =$ nb. of vertices between the root and v in $\text{cut}(T_n)$.

$L_n \rightsquigarrow$ distance on T_n



Related discrete model

Convergence of cut trees

- Meir & Moon, Panholzer, etc if V_n is uniform on T_n, then

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$$\Big(\frac{1}{\sqrt{n}} T_n, \frac{1}{\sqrt{n}} \mathrm{cut}(T_n)\Big) \Longrightarrow \big(\mathcal{T}, \mathrm{cut}(\mathcal{T})\big), \quad n \to \infty. \quad \text{(Eq 2)}$$

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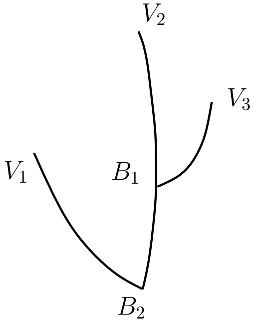
- (Eq 1) and (Eq 2) entail that $\quad \mathrm{cut}(\mathcal{T}) \stackrel{d}{=} \mathcal{T} \quad$ (Eq 3)
- $$\frac{1}{\sqrt{n}} L_n(V_n) \stackrel{\text{(Eq 2)}}{\Longrightarrow} L(V) \stackrel{d}{=} d_{\mathcal{T}}(\text{root}, V), \quad \text{by Eq (3)}$$

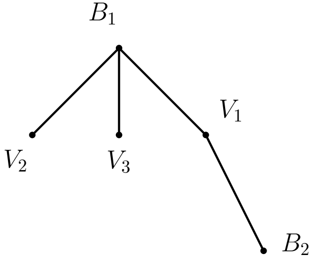
Related discrete model

Reverse transformation

From cut(T_n) to T_n

A uniform tree T_n





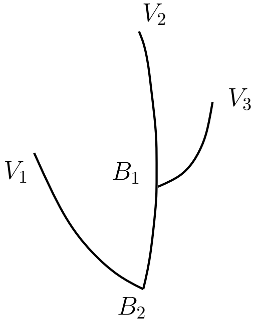
cut(T_n)

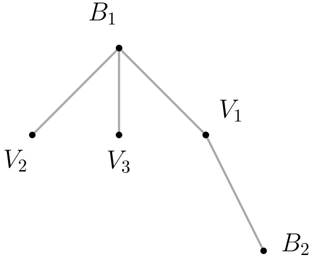
Related discrete model

Reverse transformation

From cut(T_n) to T_n: destroy all the edges in cut(T_n)

A uniform tree T_n





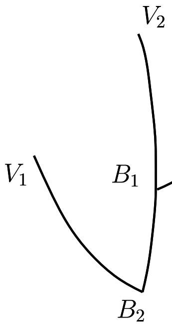
cut(T_n)

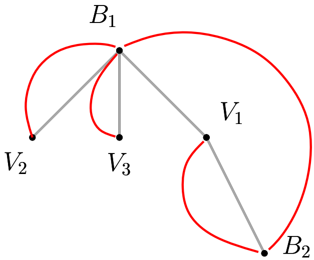
Related discrete model

Reverse transformation

From cut(T_n) to T_n: replace them with the edges in T_n

A uniform tree T_n





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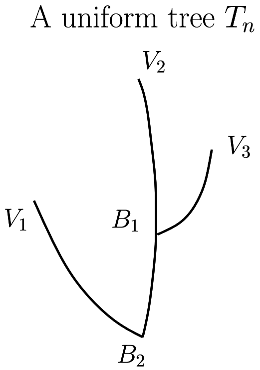
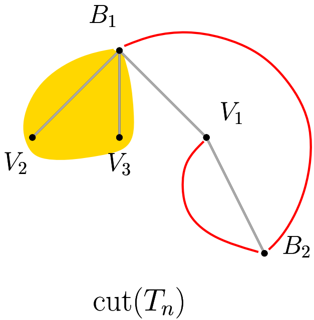
From cut(T_n) to T_n: or equivalently...

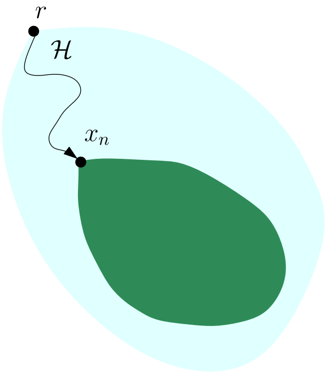




Construction of shuff($\mathcal{H}$)

$\mathrm{Br}(\mathcal{H}) = \{x_1, x_2, x_3, \cdots\}$. For each $n \geq 1$, sample A_n below x_n according to the mass measure μ.

$\mathcal{H}$ = Brownian CRT



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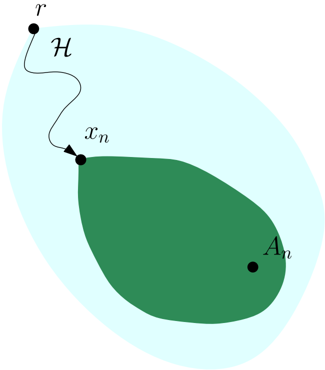

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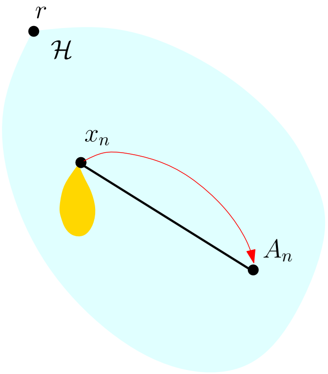

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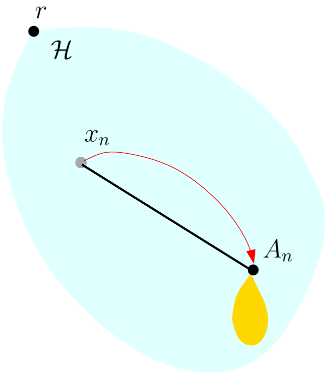

Construction of shuff($\mathcal{H}$)

- Almost surely, the transformation converges as $n \to \infty$. Denote by shuff($\mathcal{H}$) the limit tree.
- shuff($\mathcal{H}$) does not depend on the order of the sequence (x_n)
- It satisfies

$$(\mathrm{shuff}(\mathcal{H}), \mathcal{H}) \overset{d}{=} (\mathcal{T}, \mathrm{cut}(\mathcal{T})).$$

Thank you!

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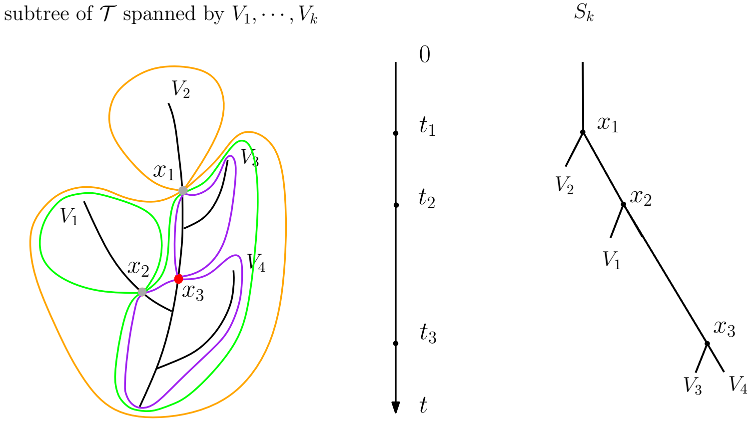

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