Tropical Geometry

Author: Joel Hunter
Supervisor: Alastair Craw

April 15, 2021

Abstract

In this project, we give an introduction to tropical geometry by investigating three key theorems. The first is Kapranov's Theorem, which characterises the tropical hypersurface of a Laurent polynomial in several ways. These hypersurfaces admit a polyhedral structure and this is proved in the second theorem, the Structure Theorem. Finally, we consider a general algebraic set of a Laurent ideal and characterise the tropicalisation of this set in the Fundamental Theorem of Tropical Algebraic Geometry.

Contents

1 Introduction

At its core, *tropical geometry* is the study of polynomials over the *tropical semiring*. This is $\mathbb{R} \cup \{\infty\}$, where the usual addition has been replaced by minimisation, and the usual multiplication has been replaced by addition. This means polynomials in the tropical semiring are piecewise-linear functions. Since systems of piecewise-linear functions are generally easier to work with than systems of polynomials, tropical geometry is becoming a popular field of interest amongst algebraic geometers. As such, some describe tropical geometry as a "piecewise-linear shadow" of classical algebraic geometry [MR18]. However, we study these piecewise-linear objects using techniques from *polyhedral geometry*, and so these objects have a polyhedral structure. So tropical geometry can be thought of as "a marriage between algebraic and polyhedral geometry" [MS15].

For example, consider the polynomial

$$f = 2x^2 + xy + 2y^2 + x + y + 1 \in \mathbb{Q}[x, y].$$

In classical algebraic geometry, we would find the affine hypersurface of f. To turn this hypersurface into something "tropical", we would then apply a *valuation* to each of the points in the hypersurface. This is a function of the form

$$\mathrm{val} : K \longrightarrow \mathbb{R} \cup \{\infty\},$$

where K is a field (so $K = \mathbb{Q}$ in this case). However, this would require finding a closed form for the points in the hypersurface, which isn't always available. Instead, we can *tropicalise* the polynomial to obtain a piecewise-linear function. Denote this by $\mathrm{trop}(f)$. We can then look for points where $\mathrm{trop}(f)$ fails to be linear. Plotting these points in $\mathbb{R}^2$, we obtain an object called the *tropical hypersurface*, $\mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f))$, as shown in Figure 1.

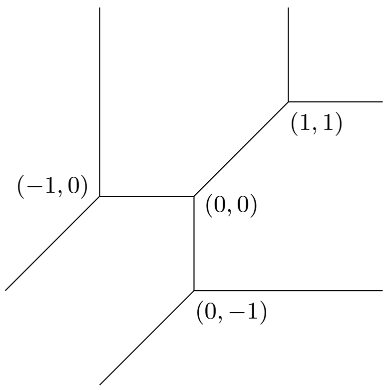

Figure 1: A tropical hypersurface in $\mathbb{R}^2$.

This tropical hypersurface is not just a pretty curve. It is a *polyhedral complex* which subdivides $\mathbb{R}^2$. This is proved in the *Structure Theorem* (Theorem 4.17). The regions it creates represents the parts of $\mathbb{R}^2$ where $\mathrm{trop}(f)$ is linear. If we were to have used our initial approach of finding valuations of points in the hypersurface, then we would obtain the same curve as in Figure 1. The fact that these two methods produce the same hypersurface is the content of *Kapranov's Theorem* (Theorem 3.11), first published in [EKL06].

As in classical algebraic geometry, we can extend our study from one polynomial to ideals in the polynomial ring, and so from hypersurfaces to algebraic sets. If $X = \mathbb{V}(I)$ is an algebraic set, then the corresponding *tropical algebraic set* is given by

$$\mathrm{trop}(X) = \bigcap_{f \in I} \mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f)).$$

Unfortunately, it is not sufficient to take this intersection over the generators of I. However, just like in the hypersurface case, there is another way to compute $\mathrm{trop}(X)$. We can find the points where the *initial ideal* of I is not a monomial (a polynomial with one term). This then forms a polyhedral complex. Another way to consider $\text{trop}(X)$ is the closure in the Euclidean topology on $\mathbb{R}^n$ of the valuation of all points in X. This construction is preferred when studying the theory behind tropical algebraic sets, whereas the polyhedral construction is preferred when computing $\text{trop}(X)$ [Stu20]. The fact that we can view the tropical algebraic set from a polyhedral and an algebraic perspective is due to the *Fundamental Theorem of Tropical Algebraic Geometry* (Theorem 5.19).

This project is a largely self-contained introduction to the study of tropical geometry. It is mainly adapted from *Introduction to Tropical Geometry* by Maclagan-Sturmfels [MS15], with any other references cited when they are used. The only assumed knowledge is from the MA40188 Algebraic Curves course [Cra19].

2 Tropical Seeds

2.1 Tropical Semiring

In this section, we plant the seeds of tropical geometry by defining the tropical semiring and exploring how polynomials behave in this setting.

Definition 2.1. (Tropical Semiring). The *tropical semiring* is the semiring $\overline{\mathbb{R}} := (\mathbb{R} \cup \{\infty\}, \oplus, \odot)$, where

$$x \oplus y := \min\{x, y\} \quad \text{and} \quad x \odot y := x + y,$$

for all $x, y \in \mathbb{R} \cup \{\infty\}$.

So the usual addition is replaced by minimisation, sometimes called the *tropical sum*, and the usual multiplication is replaced by the usual addition, sometimes called the *tropical product.*

Example 2.2. The tropical sum of 5 and 8 is 5, and their tropical product is 13:

$$5 \oplus 8 = \min\{5, 8\} = 5 \quad \text{and} \quad 5 \odot 8 = 5 + 8 = 13.$$

Remark 2.3. The tropical semiring satisfies the usual axioms we'd expect a semiring to uphold, such as additive and multiplicative identities, associativity, distributivity and commutativity (so the tropical semiring is also in fact a commutative semiring). However, it is not a ring since we cannot give a meaningful definition of an additive inverse, or in other words, subtraction doesn't exist in the tropical semiring. For example, we cannot say "8 minus 5" since there is no solution, x, to the equation

$$5 \oplus x = 8 \iff \min\{5, x\} = 8.$$

In classical algebraic geometry, we study the behaviour of spaces defined by systems of multivariate polynomials. To do the same in tropical geometry, we must first define what we mean by a *tropical polynomial.*

Definition 2.4. (Tropical Polynomial). Let $x_1, \dots, x_n$ be variables in the tropical semiring. A *tropical monomial* is defined to be any product of these variables, possibly including repetitions:

$$x_1^{i_1} \dots x_n^{i_n} := \underbrace{x_1 \odot \cdots \odot x_1}_{i_1 \text{times}} \odot \cdots \odot \underbrace{x_n \odot \cdots \odot x_n}_{i_n \text{times}} \quad ,$$

where $i_j \in \mathbb{Z}$, for all $j \in \{1, \dots, n\}$. A *tropical polynomial* is a finite linear combination of tropical monomials,

$$p(x_1, \dots, x_n) = \bigoplus_{i=1}^{N} (c_i \odot x_1^{i_1} \dots x_n^{i_n}) \quad ,$$

where $N \in \mathbb{N}$, $a_i \in \mathbb{R}$, and $i_j \in \mathbb{Z}$, for all $i \in \{1, \dots, N\}$ and all $j \in \{1, \dots, n\}$.

Remarks 2.5. (1) Note that in the definitions of tropical monomials and polynomials, we defined the exponents over the integers; not just the non-negative integers. So in our definition of a tropical monomial, if there exists a $j \in \{1, \dots, n\}$ such that $i_j < 0$, then

$$x^{i_j} = \underbrace{-x \odot \cdots \odot -x}_{-i_j \text{times}}.$$

(2) In classical arithmetic, we can rewrite the general form of a tropical monomial:

$$x_1^{i_1} \dots x_n^{i_n} = i_1 x_1 + \cdots + i_n x_n.$$

Thus the tropical polynomial can be viewed as the minimum of a finite collection of linear functions:

$$p(x_1, \dots, x_n) = \min_{i \in \{1 \dots n\}} \{c_i + i_1 x_1 + \cdots + i_n x_n\}.$$

So, each tropical monomial and polynomial can be regarded as a piecewise-linear function from $\mathbb{R}^n$ to $\mathbb{R}$. We sometimes refer to this function as a *tropical polynomial function.*

Example 2.6. Consider the following cubic polynomial in one variable:

$$f(x) = 1 \odot x^3 \oplus 2 \odot x^2 \oplus 4 \odot x \oplus 7.$$

By Remark 2.5(2), we can regard this as the following piecewise-linear function:

$$f(x) = \min\{3x + 1, 2x + 2, x + 4, 7\}.$$

We can visualise $f(x)$ graphically. Drawing the lines $y = 3x + 1$, $y = 2x + 2$, $y = x + 4$ and $y = 7$ in the (x, y) plane, (Figure 2), we see that the value of $f(x)$ is given by the smallest y value attained by one of the four lines at that value of x. So $f(x)$ is the lower envelope of the four lines, indicated by the bold line.

Note that there are 3 points on $y = f(x)$ where the function fails to be linear; $x = 1, 2, 3$. These points are called the *roots* of $f(x)$.

We can extend this to the general cubic polynomial,

$$p(x) = a \odot x^3 \oplus b \odot x^2 \oplus c \odot x \oplus d,$$

so in classical arithmetic,

$$p(x) = \min\{3x + a, 2x + b, x + c, d\}.$$

Once again, we can sketch this function (Figure 3) and, by examining different cases, obtain the tropical cubic formula:

$$\text{The roots of } p \text{ are: } \begin{cases} b-a,\ c-b,\ d-c, & \text{if } 2b \le a+c,\ d \ge 2c-b, \\ b-a,\ \dfrac{d-b}{2}, & \text{if } 2b \le a+c,\ d \le 2c-b, \\ \dfrac{c-a}{2},\ d-c, & \text{if } 2b \ge a+c,\ d \ge 2c-b, \\ \dfrac{d-a}{3}, & \text{if } 2b \ge a+c,\ d \le 2c-b. \end{cases}$$

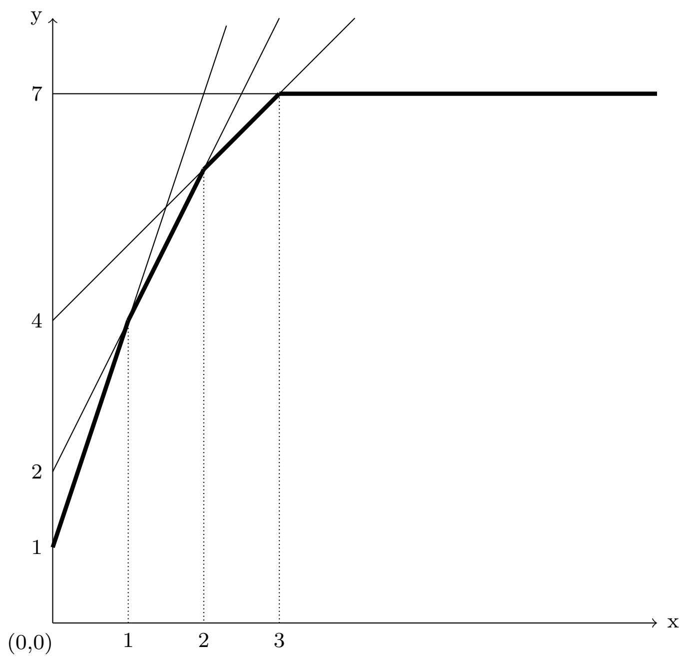

Figure 2: The graph of $y = f(x)$.

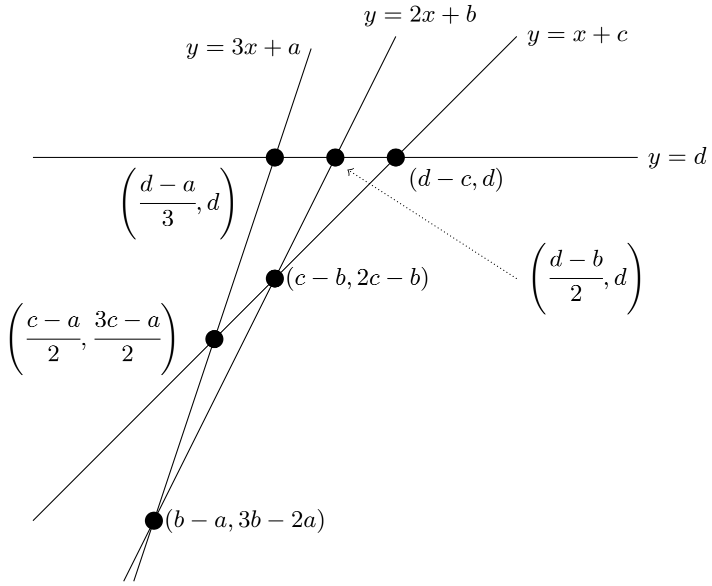

Figure 3: The graph of a $y = p(x)$, a general cubic polynomial

In our earlier example, we had $a = 1, b = 2, c = 4$ and $d = 7$, which corresponds to the first case, giving us the three roots we obtained from Figure 2.

As well as the quadratic formula, this can be extended to obtain a tropical quartic and quintic formula.

2.2 Tropical Hypersurfaces

We extend our discussion on roots in the univariate case to hypersurfaces in the multivariate case. Recall that in classical algebraic geometry, a hypersurface is the set of points which vanish on a given polynomial. We must tweak this definition in order for it to make sense in the context of the tropical semiring.

Definition 2.7. (Tropical Hypersurface). Let $p : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a tropical polynomial function. The *tropical hypersurface*, $\mathbb{V}_{\text{trop}}(p)$, of p is defined as

$$\mathbb{V}_{\text{trop}}(p) := \{\boldsymbol{w} \in \mathbb{R}^n \quad | \quad \text{the minimum in } p(\boldsymbol{w}) \text{ is achieved at least twice}\}.$$

Equivalently, $\boldsymbol{w} \in \mathbb{V}_{\text{trop}}(p)$ if and only if p is not linear at $\boldsymbol{w}$.

Example 2.8. Revisiting our examples from the previous section, we have

$$\mathbb{V}_{\text{trop}}(f) = \{1,\ 2,\ 3\}.$$

For the general cubic polynomial, if $2b \geq a + c$ and $d \geq 2c - b$, then

$$\mathbb{V}_{\text{trop}}(p) = \left\{\frac{c-a}{2},\ d-c\right\}.$$

So, in the univariate case, the elements in the hypersurface are exactly the roots of the polynomial.

We now explore how we can construct these tropical hypersurfaces. Specifically, we'll consider a tropical polynomial in two variables, of the form

$$p(x, y) = \bigoplus_{(i,j)\in\mathbb{Z}^2} c_{ij} \odot x^i \odot y^j,$$

where $c_{ij} \in \overline{\mathbb{R}}$ for all $(i, j) \in \mathbb{Z}^2$. In this case, $\mathbb{V}_{\text{trop}}(p)$ is sometimes referred to as a *plane tropical curve*. The method used in the following example is adapted from [BS14, § 2.1].

Example 2.9. Consider the tropical polynomial

$$f(x,y) = 1 \odot x^2 \oplus 0 \odot xy \oplus 1 \odot y^2 \oplus 0 \odot x \oplus 0 \odot y \oplus 2.$$

So the corresponding tropical polynomial function is given by

$$f(x,y) = \min_{(x,y)\in\mathbb{R}^2} \{2x+1,\ x+y,\ 2y+1,\ x,\ y,\ 2\}.$$

We look for points in $\mathbb{R}^2$ where the minimum in $f(x,y)$ is achieved at least twice. These points form the tropical hypersurface, $\mathbb{V}_{\text{trop}}(f)$. Looking at the six expressions, we see that $(0,0)$ achieves a minimum of 0 three times; in x, y, and $x+y$. So $(0,0) \in \mathbb{V}_{\text{trop}}(f)$. We now look for points which achieve the minimum in two of the three expressions, and thus *may* lie in the hypersurface. We can get three sets of points by setting two expressions equal to each other and less than or equal to the third. For example, we consider the set

$$\{x = y \leq x+y\} = \{(\lambda,\lambda) \mid \lambda \geq 0\}.$$

So we have found a half-ray of points which *may* lie in the hypersurface. The other two half-rays emanating from $(0,0)$ are given by

$$\{x+y = x \leq y\} = \{(\lambda,0) \mid \lambda \leq 0\} \quad \text{and} \quad \{x+y = y \leq x\} = \{(0,\lambda) \mid \lambda \leq 0\}.$$

Figure 4 shows what this looks like in $\mathbb{R}^2$. This is an example of a *tropical line.*

We can repeat this method for the other points which achieve the minimum three times in $f(x,y)$: $(2,2)$ in x, y and 2, $(-1,0)$ in $2x+1, x+y$ and x, and $(0,-1)$ in $2y+1, x+y$ and y. So we obtain the following half-rays:

$$\{(\lambda,\lambda) \mid \lambda \leq 2\},\ \{(2,\lambda) \mid \lambda \geq 2\},\ \{(\lambda,2) \mid \lambda \geq 2\};$$
$$\{(\lambda,0) \mid \lambda \geq -1\},\ \{(-1,\lambda) \mid \lambda \geq 0\},\ \{(\lambda,\lambda+1) \mid \lambda \leq -1\};$$
$$\{(0,\lambda) \mid \lambda \geq -1\},\ \{(\lambda,-1) \mid \lambda \geq 0\},\ \{(\lambda+1,\lambda) \mid \lambda \leq -1\}.$$

Note that some of these half-rays restrict on each other. For example, we found the two half-rays, $\{(\lambda,\lambda) \mid \lambda \geq 0\}$ and $\{(\lambda,\lambda) \mid \lambda \leq 2\}$. So only the intersection of these two half-rays lie in the hypersurface, i.e. $\{(\lambda,\lambda) \mid 0 \leq \lambda \leq 2\} \in \mathbb{V}_{\text{trop}}(f)$. We repeat this for other pairs of half-rays to obtain a full description of $\mathbb{V}_{\text{trop}}(f)$, which is made up of six half-rays and the three lines, $\{(\lambda,\lambda) \mid 0 \leq \lambda \leq 2\}$, $\{(0,\lambda) \mid -1 \leq \lambda \leq 0\}$, and $\{(\lambda,0) \mid -1 \leq \lambda \leq 0\}$. This can be seen in Figure 5.

Tropical hypersurfaces exhibit a very important structure which we explore more thoroughly in Section 4.

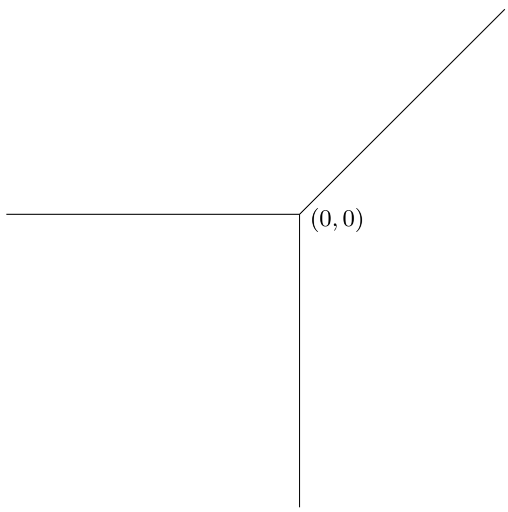

Figure 4: A tropical line in $\mathbb{R}^2$.

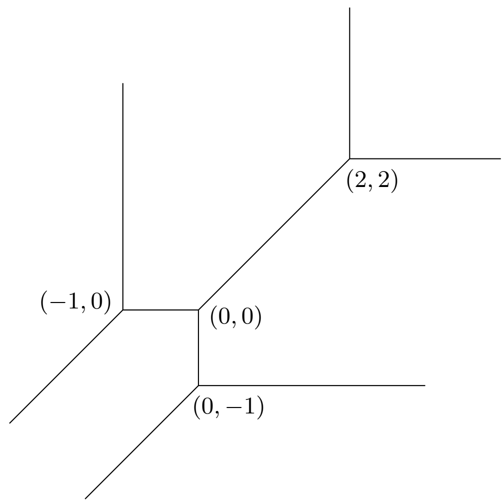

Figure 5: A plane tropical curve in $\mathbb{R}^2$.

2.3 Tropicalisation of polynomials

In the previous sections, we considered a tropical polynomial and looked at it's behaviour when viewed as a function from $\mathbb{R}^n$ to $\mathbb{R}$. In this section, we instead consider a (Laurent) polynomial in a (Laurent) polynomial ring over a field K and *tropicalise* it into a tropical polynomial. This requires equipping K with a function known as a *valuation*.

Definition 2.10. (Valuation). Let K be a field. A *valuation on K* is a function $\text{val} : K \longrightarrow \mathbb{R} \cup \{\infty\}$ such that the following three properties hold:

(1) $\text{val}(a) = \infty$ if and only if $a = 0$;

(2) $\text{val}(ab) = \text{val}(a) + \text{val}(b)$; and

(3) $\text{val}(a + b) \geq \min\{\text{val}(a), \text{val}(b)\}$,

for all $a, b \in K$. We denote the image of val by Γ_{val}, called the *value group* of (K, val). In this case, K is called a *valued field*.

Remarks 2.11. (1) By property (1), we often regard a valuation of K as a function $\text{val} : K^\times \longrightarrow \mathbb{R}$, where $K^\times$ is the set of units of K.

(2) Γ_{val} is an additive subgroup of $\mathbb{R}$.

(3) Let $R := \{c \in K \mid \text{val}(c) \geq 0\}$ be the set of all elements in K with non-negative valuations. Then R is a ring, called the *valuation ring*, and has a unique maximal ideal:

$$m_K := \{c \in K \mid \text{val}(c) > 0\}.$$

Then the quotient ring, $\Bbbk := R/m_K$, is a field called the *residue field* of (K, val).

Examples 2.12. (1) Every field has a trivial valuation, where $\text{val}(c) = 0$ for all $c \in K$.

(2) Let $K = \mathbb{Q}$, $q \in \mathbb{Q}$, and $p \in \mathbb{N}$ which is prime. Then we can uniquely write q in the form

$$q = p^k \frac{a}{b},$$

where $a,\ b,\ k \in \mathbb{Z}$, such that p does not divide either a or b. Then we define the p-adic valuation, $\text{val}_p : \mathbb{Q} \longrightarrow \mathbb{R}$, as

$$\text{val}_p(q) = k.$$

For example, when $p = 2$, we have

$$\text{val}_2\left(\frac{8}{9}\right) = 3, \text{ since } \frac{8}{9} = 2^3 \cdot \frac{1}{9} \text{ and 2 doesn't divide either 1 or 9, and}$$

$$\text{val}_2\left(-\frac{5}{11}\right) = 0, \text{ since } -\frac{5}{11} = 2^0 \cdot -\frac{5}{11} \text{ and 2 doesn't divide either } -5 \text{ or } 11.$$

Note that $\text{val}_p(q) \geq 0$ if and only if $q = \frac{a}{b}$, where p doesn't divide b. So,

$$R = \left\{\frac{a}{b} \in \mathbb{Q} \;\middle|\; a,\ b \in \mathbb{Z},\ p \nmid b\right\} \quad \text{and} \quad m_K = \left\{\frac{a}{b} \in \mathbb{Q} \;\middle|\; a,\ b \in \mathbb{Z},\ p \nmid b,\ p \mid a\right\}$$

Thus, the residue field is the finite field with p elements, i.e. $\Bbbk = \mathbb{Z}/p\mathbb{Z}$.

(3) Let $K = \mathbb{C}\{\{t\}\}$ be the field of Puiseux series with coefficients in $\mathbb{C}$. In this field, elements are of the form

$$c(t) = c_1 t^{a_1} + c_2 t^{a_2} + c_3 t^{a_3} + \ldots,$$

where $c_i \in \mathbb{C}$ for all i, and each a_i is a rational number with common denominator, with $a_1 < a_2 < a_3 < \dots$. Then the natural valuation $\mathrm{val} : \mathbb{C}\{\{t\}\}^\times \longrightarrow \mathbb{R}$ is defined by $\mathrm{val}(c(t)) = a_1$. For example,

$$c(t) = 3t^2 + \frac{5}{4}t^3 + t^5 \Rightarrow \mathrm{val}(c(t)) = 2.$$

Since each exponent, a_i, in an element is a rational number, we have that $\Gamma_{\mathrm{val}} = \mathbb{Q}$. Note that $\mathbb{C}\{\{t\}\}$ is algebraically closed; see [MS15, Theorem 2.1.5] for a proof.

We now prove some useful results about valuations.

Lemma 2.13. *Let (K, val) be a valued field. Then,*

(i) $\mathrm{val}(1) = 0$*;*

(ii) $\mathrm{val}(a) = \mathrm{val}(-a)$*;*

(iii) $\mathrm{val}(\frac{a}{b}) = \mathrm{val}(a) - \mathrm{val}(b)$*;*

(iv) if $\mathrm{val}(a) \neq \mathrm{val}(b)$*, then* $\mathrm{val}(a+b) = \min\{\mathrm{val}(a),\ \mathrm{val}(b)\}$,

for all $a, b \in K$.

Proof. For (i), let $a = b = 1$. Then by property (2) in Definition 2.10, we have $\mathrm{val}(1) = \mathrm{val}(1) + \mathrm{val}(1)$. So $\mathrm{val}(1) = 0$.

For (ii), let $a = b = -1$. Then once again by property (2), we have $\mathrm{val}(1) = \mathrm{val}(-1) + \mathrm{val}(-1)$. So $0 = 2\,\mathrm{val}(-1)$ which implies $\mathrm{val}(-1) = 0$. Now, if we set $b = -1$ and $a \in K$ and arbitrary element in property (2) , then we get $\mathrm{val}(-a) = \mathrm{val}(-1) + \mathrm{val}(a)$, and thus $\mathrm{val}(-a) = \mathrm{val}(a)$.

For (iii), note that property (2) implies $\mathrm{val}(a) + \mathrm{val}(\frac{1}{a}) = \mathrm{val}(\frac{a}{a}) = \mathrm{val}(1) = 0$. So $\mathrm{val}(\frac{1}{a}) = -\mathrm{val}(a)$. Thus, by property (2) again, $\mathrm{val}(\frac{a}{b}) = \mathrm{val}(a) + \mathrm{val}(\frac{1}{b}) = \mathrm{val}(a) - \mathrm{val}(b)$.

For (iv), suppose, without loss of generality, $\mathrm{val}(b) > \mathrm{val}(a)$. We show that $\mathrm{val}(a) = \mathrm{val}(a+b)$. By property (3), $\mathrm{val}(a) \geq \min\{\mathrm{val}(a+b), \mathrm{val}(-b)\} = \min\{\mathrm{val}(a+b), \mathrm{val}(b)\}$. So, by assumption, $\mathrm{val}(a) \geq \mathrm{val}(a+b)$. We also have $\mathrm{val}(a+b) \geq \min\{\mathrm{val}(a), \mathrm{val}(b)\} = \mathrm{val}(a)$. Therefore, we must have $\mathrm{val}(a) = \mathrm{val}(a+b)$. □

We are now in a position to define the *tropicalisation* of a polynomial. Recall that in Remarks 2.5(1) we allowed negative exponents in the definition of a tropical polynomial. Thus we allow negative exponents for the polynomials we wish to tropicalise. In particular, we work over the Laurent polynomial ring in n variables, $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where the general polynomial is of the form $f = \sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}}$. Here, we use the notation $x^{\boldsymbol{u}} := x_1^{u_1} \cdot \ldots \cdot x_n^{u_n}$ for $\boldsymbol{u} = (u_1, ..., u_n)$.

Definition 2.14. (Tropicalisation). Let (K, val) be a valued field and $f = \sum\limits_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial. The *tropicalisation* of f is defined by

$$\mathrm{trop}(f) := \bigoplus_{\boldsymbol{u} \in \mathbb{Z}^n} \mathrm{val}(c_{\boldsymbol{u}}) \odot x^{\boldsymbol{u}} = \min_{\boldsymbol{u} \in \mathbb{Z}^n} \{\mathrm{val}(c_{\boldsymbol{u}}) + \boldsymbol{x} \cdot \boldsymbol{u} \mid \mathrm{val}(c_{\boldsymbol{u}}) \neq 0\},$$

where $\boldsymbol{x} = (x_1, \dots, x_n)$.

Example 2.15. Let $K = \mathbb{Q}$ be equipped with the 2-adic valuation and consider the polynomial in two variables

$$f = 8x^2 + xy + 2y^2 + 5x + y + 1.$$

We tropicalise this polynomial term by term, so consider $8x^2$. The 2-adic valuation of 8 is 3 since $8 = 2^3$. The exponent corresponds to $\boldsymbol{u} = (2,\ 0)$. So $8x^2$ has been “tropicalised” to $3 + 2x$. We repeat this for each term to obtain the tropicalisation of f:

$$\mathrm{trop}(f) = \min\{3 + 2x,\ x + y,\ 1 + 2y,\ x,\ y,\ 0\}.$$

We can now compute the tropical hypersurface, $\mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f))$, by using the same technique as in Example 2.9. Recall this involved finding points which achieved the minimum more than twice in our tropical polynomial and constructing half-rays emanating from these points. However, this example differs slightly as we now have a point, $(0, 0)$, which achieves the minimum in $\mathrm{trop}(f)$ four times; in $x + y$, x, y and 0. So we must adapt our approach slightly. Note that if we set any three of the four expressions equal to each other, we get $x = y = 0$. So we find half-rays by setting two expressions equal to each other and then less than or equal to the other two. These are given by

$$\{(\lambda, 0) \mid \lambda \geq 0\},\ \{(\lambda, 0) \mid \lambda \leq 0\},\ \{(0, \lambda) \mid \lambda \geq 0\} \text{ and } \{(0, \lambda) \mid \lambda \leq 0\}.$$

The other points which achieve the minimum at least twice are $(0, -1)$ and $(-3, 0)$, both 3 times, so our approach is the same as before. Repeating this approach gives us the tropical hypersurface seen in Figure 6.

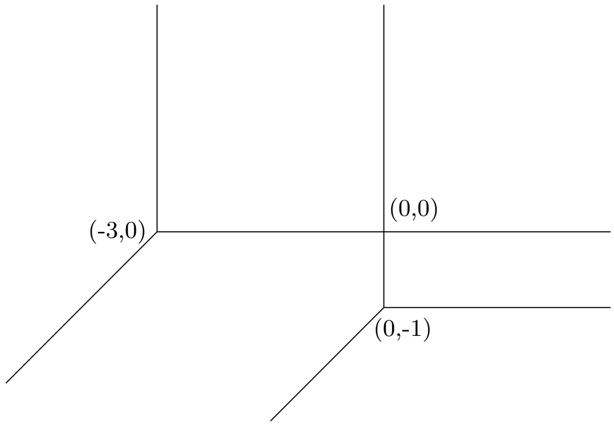

Figure 6: A tropical curve in $\mathbb{R}^2$

3 Kapranov's Theorem

In this section, we state and prove Kapranov's Theorem. This theorem gives us a link between hypersurfaces over an algebraic set and tropical hypersurfaces in $\mathbb{R}^n$, thus establishing a link between classical algebraic geometry and tropical geometry.

3.1 Initial Forms

Before stating Kapranov's Theorem, we need to state some important definitions and results which later become crucial. We first state the definition of a structure that is important in tropical geometry.

Definition 3.1 (**Algebraic torus**). Let K be an algebraically closed field with non-trivial valuation. The *algebraic torus*, T^n, is defined by

$$T^n = \{(a_1, \dots, a_n) \mid a_i \in K^\times \text{ for all } i \in \{1, \dots, n\}\}.$$

An *algebraic set in* T^n, X, is of the form

$$X = \mathbb{V}_t(I) := \{\boldsymbol{x} \in T^n \mid f(\boldsymbol{x}) = 0 \text{ for all } f \in I\},$$

for some ideal $I \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$.

Remarks 3.2. (1) The algebraic torus is an algebraic set, with coordinate ring $K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, the Laurent polynomial ring. Indeed, that T^n is an algebraic set can be seen by noting that $\mathbb{V}_a(I) \cong (K^\times)^n = T^n$, where $I = \langle x_1 y_1 - 1, \dots, x_n y_n - 1\rangle$ and $K[x_1, \dots, x_n, y_1, \dots, y_n]$ is the coordinate ring of $\mathbb{A}^{2n}$. Then the coordinate ring of T^n is isomorphic to

$$K[x_1, \dots, x_n, y_1, \dots, y_n]/I \cong K[x_1^{\pm 1}, \dots, x_n^{\pm 1}],$$

by [Cra19, Proposition 3.3].

(2) Define the quotient ring homomorphism by

$$q : K[x_1, \dots, x_n, y_1, \dots, y_n] \longrightarrow K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]; \quad f \mapsto f + I.$$

This induces an inclusion-preserving bijection between ideals in $K[x_1, \dots, x_n, y_1, \dots, y_n]$ containing I and ideals in the Laurent polynomial ring. Also, algebraic sets in T^n correspond to algebraic sets in $\mathbb{A}^{2n}$ contained in $\mathbb{V}_a(I)$. So, if $\mathbb{V}_t(J)$ is an algebraic set in T^n for some Laurent ideal $J \subseteq K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, then this corresponds to the algebraic set $\mathbb{V}_a(q^{-1}(J))$ in $\mathbb{A}^{2n}$. So we can use statements about affine algebraic sets and ideals in the polynomial ring, and apply them to algebraic sets in T^n and Laurent ideals.

We now outline an important property that we assume the field K to satisfy from now on.

Definition 3.3. (Splitting). A valued field (K, val) is said to have a *splitting* if there exists a homomorphism,

$$\phi : (\Gamma_{\mathrm{val}}, +) \longrightarrow (K^\times, \cdot)\ ;\ w \longmapsto t^w := \phi(w),$$

such that $\mathrm{val}(\phi(w)) = \mathrm{val}(t^w) = w$.

Remarks 3.4. (1) We use the notation t^w to denote $\phi(w)$ in order to be consistent with the Puiseux series field $\mathbb{C}\{\{t\}\}$, where the elements t^w are simply the powers of t.

(2) Suppose $a \in K$ and $\mathrm{val}(a) \geq 0$, so $a \in R$, recalling that R is the valuation ring. Then we denote the image of a in the residue field, $\Bbbk$, by $\overline{a}$. Splitting allows us to always obtain a non-zero element in $\Bbbk$ since $\overline{t^{-\mathrm{val}(a)} a} \in \Bbbk^\times$ for all $a \in K^\times$. Indeed, by property (2) in Definition 2.10 and Definition 3.3,

$$\mathrm{val}(t^{-\mathrm{val}(a)} a) = \mathrm{val}(t^{-\mathrm{val}(a)}) + \mathrm{val}(a) = -\mathrm{val}(a) + \mathrm{val}(a) = 0,$$

meaning $\overline{t^{-\mathrm{val}(a)} a}$ is a non-zero element in the residue field, $\Bbbk$.

The following is a result which is important when proving Kapranov's Theorem.

Lemma 3.5. *Let* (K, val) *be a valued field with a splitting as defined above. Let* $\alpha_1, \dots, \alpha_n \in \Bbbk^\times$, $w_1, \dots, w_n \in \Gamma_{\mathrm{val}}$ *and consider the set*

$$\mathcal{Y}^{(n)} = \{\boldsymbol{y} = (y_1, \dots, y_n) \in T^n \mid \mathrm{val}(y_i) = w_i,\ \overline{t^{-w_i} y_i} = \alpha_i\ \forall i \in \{1, \dots, n\}\}.$$

Then this set is Zariski dense in T^n.

Proof. We wish to show that the Zariski closure of $\mathcal{Y}^{(n)}$ is T^n. In other words, we wish to show that for all non-zero Laurent polynomials $h \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, there exists an element $\boldsymbol{y} \in \mathcal{Y}^{(n)}$ such that $h(\boldsymbol{y}) \neq 0$.

For each $i \in \{1, \dots, n\}$, fix an element z_i in the valuation ring R such that $\overline{z_i} = \alpha_i \in \Bbbk^\times$, so $\mathrm{val}(z_i) = 0$. Then, setting $y_i = t^{w_i} z_i$, we get $\mathrm{val}(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$. Indeed, since $\mathrm{val}(z_i) = 0$, we have

$$\mathrm{val}(y_i) = \mathrm{val}(t^{w_i} z_i) = \mathrm{val}(t^{w_i}) + \mathrm{val}(z_i) = w_i,$$

and, since $\overline{z_i} = \alpha_i$, we have

$$t^{-w_i} y_i = t^{-w_i} t^{w_i} z_i = z_i \implies \overline{t^{-w_i} y_i} = \alpha_i.$$

In fact, we obtain an infinite number of choices for y_i, namely $y_i = t^{w_i} z_i + t^{w_i + j} = t^{w_i}(z_i + t^j)$, where $j > 0$. Then $(y_1, \dots, y_n) \in \mathcal{Y}^{(n)}$ by noting that $\mathrm{val}(t^j) = j \neq 0$ and so, by Lemma 2.13(*iv*), $\mathrm{val}(z_i + t^j) = 0$.

We now proceed by using induction on n. For the base case, $n = 1$, choose y_1 from one of the infinite number of choices listed above such that $\mathrm{val}(y_1) = w_1$ and $\overline{t^{-w_1} y_1} = \alpha_1$. Since any $h \in K[x^{\pm 1}]$ has finitely many roots, we can always choose a y_1 such that $h(y_1) \neq 0$. Now fix $n > 1$ and let $h = \sum_j h_j x_n^j \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, where $h_j \in K[x_1^{\pm 1}, \dots, x_{n-1}^{\pm 1}]$ for all $j \in J$, where $J \subset \mathbb{Z}$ is an indexing set. Assume there exists $\boldsymbol{y}' = (y_1, \dots, y_{n-1}) \in \mathcal{Y}^{(n-1)}$ such that $h_j(\boldsymbol{y}') \neq 0$ for all $j \in J$. Now choose y_n from the infinite choices such that $\mathrm{val}(y_n) = w_n$, $\overline{t^{-w_n} y_n} = \alpha_n$ and all of the finite number of roots of $h(\boldsymbol{y}', y_n) \in K[x_n^{\pm 1}]$ are avoided.

Therefore, $\boldsymbol{y} = (\boldsymbol{y}', y_n) \in \mathcal{Y}^{(n)}$ and since h is arbitrary, we have $\mathcal{Y}^{(n)}$ is Zariski dense in T^n. ◻

We now introduce a key player in Kapranov's Theorem: *initial forms*. Note that for the rest of this section (and section 5), we identify the vector space $\mathbb{R}^n$ and its dual $(\mathbb{R}^n)^*$ via the usual dot product.

Definition 3.6. (Initial Forms). Fix a weight vector $\boldsymbol{w} \in \mathbb{R}^n$ and let $f \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent polynomial of the form $f = \sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}}$. The *initial form* of f with respect to $\boldsymbol{w}$ is defined to be

$$\mathrm{in}_{\boldsymbol{w}}(f) = \sum_{\boldsymbol{u} \,\mid\, \mathrm{val}(c_{\boldsymbol{u}}) + \boldsymbol{w} \cdot \boldsymbol{u} = \mathrm{trop}(f)(\boldsymbol{w})} \overline{t^{-\mathrm{val}(c_{\boldsymbol{u}})} c_{\boldsymbol{u}}} x^{\boldsymbol{u}} \in \Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}].$$

Example 3.7. Define a Laurent polynomial over the field of Puiseux series,

$$f = 2t^2 x^2 + (1+t) y^2 + 3xy + t^3 x + (2t + t^2) y + x^{-1} \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}],$$

where $\mathbb{C}\{\{t\}\}$ has the natural valuation defined in Example 2.12 (3). To calculate the initial form of f with respect to some weight vector $\boldsymbol{w} = (w_1, w_2) \in \mathbb{R}^2$, we first compute its tropicalisation with respect to $\boldsymbol{w}$:

$$\mathrm{trop}(f)(\boldsymbol{w}) = \min\{2 + 2w_1,\ 2w_2,\ w_1 + w_2,\ 3 + w_1,\ 1 + w_2,\ -w_1\}.$$

Note that, for example, $\mathrm{val}(1) = 0$ and $\mathrm{val}(t) = 1$ so $\mathrm{val}(1+t) = 0$ by Lemma 2.13 (*iv*). For $\boldsymbol{w} = (0, 0)$, we have

$$\mathrm{trop}(f)((0,0)) = \min\{2,\ 0,\ 0,\ 3,\ 1,\ 0\} = 0.$$

Since the minimum in $\operatorname{trop}(f)((0,0))$ is achieved three times, the initial form of f with respect to $\boldsymbol{w} = (0,0)$ has three terms. So,

$$\operatorname{in}_{(0,0)}(f) = \overline{t^{-0}(1+t)}y^2 + \overline{t^{-0} \cdot 3}xy + \overline{t^{-0} \cdot 1}x^{-1}.$$

The quotient map $R \longrightarrow \Bbbk$ is a ring homomorphism so $\overline{1+t} = \overline{1} + \overline{t} = 1 + 0 = 1$. Thus,

$$\operatorname{in}_{(0,0)}(f) = y^2 + 3xy + x^{-1}.$$

For $\boldsymbol{w} = (-4, 2)$, we have

$$\operatorname{trop}(f)((-4,2)) = \min\{-6,\ 4,\ -2,\ -1,\ 3,\ 4\} = -6.$$

So the initial form of f with respect to $(-4, 2)$ has one term given by

$$\operatorname{in}_{(-4,2)}(f) = \overline{2t^{-2}t^2}x^2 = 2x^2.$$

We now state, and later prove, a result which describes the initial form of a product of polynomials.

Lemma 3.8. *Let (K, val) be a valued field. Fix a weight vector $\boldsymbol{w} \in \mathbb{R}^n$ and let $f, g \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be Laurent polynomials. Then, $\operatorname{in}_{\boldsymbol{w}}(fg) = \operatorname{in}_{\boldsymbol{w}}(f)\operatorname{in}_{\boldsymbol{w}}(g)$.*

Before we prove this result, we illustrate it with an example.

Example 3.9. Consider the polynomial f from Example 3.7 and let $g = x + t^{-2}y^{-1} \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}]$. The product of these two polynomials is given by

$$fg = 2t^2x^3 + (1+t)xy^2 + 3x^2y + t^3x^2 + (2t+t^2)xy + 2x^2y^{-1} + (t^{-2}+t^{-1})y + 3t^{-2}x + txy^{-1} + t^{-2}x^{-1}y^{-1} + (2t^{-1}+2).$$

Then the tropicalisation of fg with respect to $\boldsymbol{w} = (-4, 2)$ is given by

$$\operatorname{trop}(fg)(\boldsymbol{w}) = \min\{-10, 0, -6, -5, -1, -10, 0, -6, -5, 0, 0\} = -10.$$

So the initial form of fg with respect to $\boldsymbol{w}$ is given by

$$\operatorname{in}_{\boldsymbol{w}}(fg) = \overline{2t^2t^{-2}}x^3 + \overline{2}x^2y^{-1} = 2x^3 + 2x^2y^{-1}.$$

To calculate the initial form of g with respect to $\boldsymbol{w}$, note that its tropicalisation is given by

$$\operatorname{trop}(g)(\boldsymbol{w}) = \min\{-4, -4\} = -4,$$

and so,

$$\operatorname{in}_{\boldsymbol{w}}(g) = x + y^{-1}.$$

From Example 3.7, we saw $\operatorname{in}_{\boldsymbol{w}}(f) = 2x^2$. Thus,

$$\operatorname{in}_{\boldsymbol{w}}(f)\operatorname{in}_{\boldsymbol{w}}(g) = 2x^2(x + y^{-1}) = 2x^3 + 2x^2y^{-1} = \operatorname{in}_{\boldsymbol{w}}(fg).$$

So the product of the initial forms is the initial form of the product. Not only this, but recalling from Example 3.7 that $\operatorname{trop}(f)(\boldsymbol{w}) = -6$, we have

$$\operatorname{trop}(f)(\boldsymbol{w}) + \operatorname{trop}(g)(\boldsymbol{w}) = -6 - 4 = -10 = \operatorname{trop}(fg)(\boldsymbol{w}),$$

so the sum of the tropicalisation (with respect to a given weight vector) is the tropicalisation of the product. This is always true, as shown by the next Lemma.

Lemma 3.10. *Let $f, g \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, where (K, val) is a valued field. Then for any weight vector $\boldsymbol{w} \in \mathbb{R}^n$,*

$$\operatorname{trop}(f)(\boldsymbol{w}) + \operatorname{trop}(g)(\boldsymbol{w}) = \operatorname{trop}(fg)(\boldsymbol{w}).$$

Proof. We only prove this for when $K = \mathbb{C}\{\{t\}\}$ and val is the natural valuation as defined in Example 2.12 (3). We write the two polynomials in general form, so $f = \sum c_{\boldsymbol{u}}x^{\boldsymbol{u}}$ and $g = \sum d_{\boldsymbol{u}'}x^{\boldsymbol{u}'}$, where each

$c_{\boldsymbol{u}}, d_{\boldsymbol{u}'} \in K$. Then $fg = \sum e_{\boldsymbol{v}} x^{\boldsymbol{v}}$ where $e_{\boldsymbol{v}} = \sum_{\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{u}'} c_{\boldsymbol{u}} d_{\boldsymbol{u}'}$. We now explicitly work in the tropical semiring. By Definition 2.14, we have

$$\begin{aligned}
\operatorname{trop}(f) \odot \operatorname{trop}(g) &= \left(\bigoplus_{\boldsymbol{u} \in \mathbb{Z}^n} \operatorname{val}(c_{\boldsymbol{u}}) \odot x^{\boldsymbol{u}} \right) \odot \left(\bigoplus_{\boldsymbol{u}' \in \mathbb{Z}^n} \operatorname{val}(d_{\boldsymbol{u}'}) \odot x^{\boldsymbol{u}'} \right) \\
&= \bigoplus_{\boldsymbol{u}+\boldsymbol{u}'=\boldsymbol{v}} \operatorname{val}(c_{\boldsymbol{u}}) \odot \operatorname{val}(d_{\boldsymbol{u}'}) \odot x^{\boldsymbol{v}} \\
&= \bigoplus_{\boldsymbol{v} \in \mathbb{Z}^n} \left(\bigoplus_{\boldsymbol{u}+\boldsymbol{u}'=\boldsymbol{v}} \operatorname{val}(c_{\boldsymbol{u}}) \odot \operatorname{val}(d_{\boldsymbol{u}'}) \right) \odot x^{\boldsymbol{v}}.
\end{aligned}$$

Without loss of generality, let $c_{\boldsymbol{u}} = t^{\alpha_{\boldsymbol{u}}}$ and $d_{\boldsymbol{u}'} = t^{\beta_{\boldsymbol{u}'}}$, where $\alpha_{\boldsymbol{u}}, \beta_{\boldsymbol{u}'} \in \mathbb{Z}$ are the minimum exponents of t in $c_{\boldsymbol{u}}$ and $d_{\boldsymbol{u}'}$ respectively, for all $\boldsymbol{u}$, $\boldsymbol{u}' \in \mathbb{Z}^n$. Making this simplification will not affect any of our arguments since $\operatorname{val}(c_{\boldsymbol{u}}) = \alpha_{\boldsymbol{u}}$ and $\operatorname{val}(d_{\boldsymbol{u}'}) = \beta_{\boldsymbol{u}'}$ for all $\boldsymbol{u}$, $\boldsymbol{u}' \in \mathbb{Z}^n$, by definition of the natural valuation. So we can now rewrite $e_{\boldsymbol{v}}$ as $e_{\boldsymbol{v}} = \sum_{\boldsymbol{u}+\boldsymbol{u}'=\boldsymbol{v}} t^{\alpha_{\boldsymbol{u}}+\beta_{\boldsymbol{u}'}}$. So by definition of the natural valuation,

$$\begin{aligned}
\operatorname{val}(e_{\boldsymbol{v}}) &= \min_{\boldsymbol{u}+\boldsymbol{u}'=\boldsymbol{v}} \{\alpha_{\boldsymbol{u}} + \beta_{\boldsymbol{u}'}\} \\
&= \min_{\boldsymbol{u}+\boldsymbol{u}'=\boldsymbol{v}} \{\operatorname{val}(c_{\boldsymbol{u}}) + \operatorname{val}(d_{\boldsymbol{u}'})\}.
\end{aligned}$$

So in the tropical semiring, $\operatorname{val}(e_{\boldsymbol{v}}) = \bigoplus_{\boldsymbol{u}+\boldsymbol{u}'=\boldsymbol{v}} \operatorname{val}(c_{\boldsymbol{u}}) \odot \operatorname{val}(d_{\boldsymbol{u}'})$, and thus,

$$\begin{aligned}
\operatorname{trop}(f) \odot \operatorname{trop}(g) &= \bigoplus_{\boldsymbol{v} \in \mathbb{Z}^n} \operatorname{val}(e_{\boldsymbol{v}}) \odot x^{\boldsymbol{v}} \\
&= \operatorname{trop}(fg).
\end{aligned}$$

Therefore, for any weight vector $\boldsymbol{w} \in \mathbb{R}^n$, it follows that $\operatorname{trop}(f)(\boldsymbol{w}) + \operatorname{trop}(g)(\boldsymbol{w}) = \operatorname{trop}(fg)(\boldsymbol{w})$. □

We can now prove Lemma 3.8, using the same notation as in the proof of Lemma 3.10.

Proof of Lemma 3.6. Using Lemma 3.10 and the fact that the quotient map $R \longrightarrow \Bbbk$ is a ring homomorphism, so in particular,

$$\overline{e_{\boldsymbol{v}}} = \sum_{\boldsymbol{v}=\boldsymbol{u}+\boldsymbol{u}'} \overline{c_{\boldsymbol{u}} d_{\boldsymbol{u}'}},$$

the result can be seen by applying the definition of initial forms to fg. □

3.2 Kapranov's Theorem

We now have everything we need to state the main theorem of this section, Kapranov's Theorem. This theorem was first published and proved in [EKL06]. As previously stated, this theorem allows us to view tropical hypersurfaces in three natural ways: combinatorially, algebraically and geometrically.

Theorem 3.11 (**Kapranov's Theorem**). *Let (K, val) be an algebraically closed valued field, and fix $f = \sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then the following three sets are equal:*

(i) the tropical hypersurface $\mathbb{V}_{\text{trop}}(\operatorname{trop}(f))$ in $\mathbb{R}^n$;

(ii) the set $\{\boldsymbol{w} \in \mathbb{R}^n \mid \operatorname{in}_{\boldsymbol{w}}(f)$ is not a monomial$\}$;

(iii) the closure of $\{(\operatorname{val}(y_1), \ldots, \operatorname{val}(y_n)) \mid (y_1, \ldots, y_n) \in \mathbb{V}_t(f)\}$ in the Euclidean topology on $\mathbb{R}^n$.

Remark 3.12. If K is not algebraically closed and has a non-trivial valuation, then we can pass over to its algebraic closure, $\overline{K}$, and extend the valuation, in which case the statement of Kapranov's Theorem holds over $\overline{K}$. This is demonstrated in the following example.

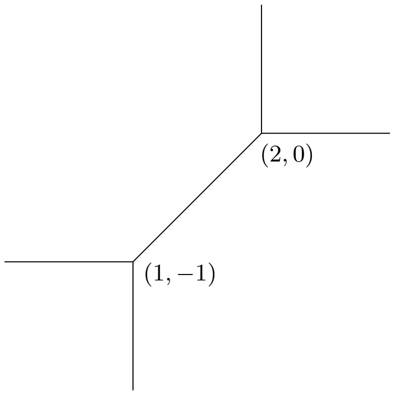

Figure 7: The tropical hypersurface of f, $\mathbb{V}_{\text{trop}}(\text{trop}(f))$, in $\mathbb{R}^2$.

Example 3.13. Let $K = \mathbb{Q}$ and val_2 be the 2-adic valuation. We pass over to $\overline{\mathbb{Q}}$, the field of algebraic numbers. Let $f = 2xy + x - 4y - 4$. Then $\text{trop}(f) = \min\{x + y + 1,\ x,\ y + 2,\ 2\}$ and using the same method as in Example 2.9, we obtain the tropical hypersurface seen in Figure 7.

The elements $\boldsymbol{w} \in \mathbb{R}^2$ such that $\text{in}_{\boldsymbol{w}}(f)$ is not a monomial are $\{(2,0),(1,-1)\}$, where $\text{in}_{\boldsymbol{w}}(f)$ has three terms, and $\{\{(2\lambda, 0)\},\ \{(2, \lambda)\},\ \{(1-\lambda, -1)\},\ \{(1, -1-\lambda)\},\ \{(1+\lambda, -1+\lambda)\}\}$, where $\text{in}_{\boldsymbol{w}}(f)$ has two terms, and in each subset $\lambda > 0$. In the former case, $\text{in}_{\boldsymbol{w}}(f)$ is either $x + y + 1$ or $xy + x + y$, respectively, and in the latter case $\text{in}_{\boldsymbol{w}}(f)$ is either $y + 1$, $x + 1$, $xy + x$, $xy + y$ or $x + y$, respectively. So the set,

$$\big\{\{(2\lambda, 0)\},\ \{(2, \lambda)\},\ \{(1-\lambda, -1)\},\ \{(1, -1-\lambda)\},\ \{(1+\lambda, -1+\lambda)\} \mid \lambda \geq 0\big\},$$

from (ii) in the statement of Kapranov's Theorem coincides with the set from (i), as pictured in Figure 7. Note that we can also write $f = (x-2)(2y+1) - 2$ so the hypersurface of f is given by $\mathbb{V}_t(f) = \{(z,\ \frac{1}{z-2} - \frac{1}{2}) \mid z \in \overline{\mathbb{Q}} \setminus \{0, 2, 4\}\}$. We find the valuations of the points in $\mathbb{V}_t(f)$ by writing $\text{val}(\frac{1}{z-2} - \frac{1}{2})$ in terms of $\text{val}(z)$ and varying $\text{val}(z)$. We have,

$$\text{val}\left(\frac{1}{z-2} - \frac{1}{2}\right) = \text{val}\left(\frac{4-z}{2(z-2)}\right) = \text{val}(4-z) - \text{val}(z-2) - 1,$$

by Lemma 2.13 (iii). Also,

$$\text{val}(4-z) \geq \min\{2,\ \text{val}(z)\} \text{ and } \text{val}(z-2) \geq \min\{1,\ \text{val}(z)\}.$$

So we have the following cases:

$$\left(\text{val}(z), \text{val}\left(\frac{1}{z-2} - \frac{1}{2}\right)\right) = \begin{cases} (\text{val}(z), -1) & \text{if } \text{val}(z) < 1, \\ \left(1, \text{val}\left(\frac{1}{z-2} - \frac{1}{2}\right)\right) & \text{if } \text{val}(z) = 1,\ \text{val}\left(\frac{1}{z-2} - \frac{1}{2}\right) \leq -1, \\ (\text{val}(z), \text{val}(z) - 2) & \text{if } 1 < \text{val}(z) < 2, \\ \left(2, \text{val}\left(\frac{1}{z-2} - \frac{1}{2}\right)\right) & \text{if } \text{val}(z) = 2,\ \text{val}\left(\frac{1}{z-2} - \frac{1}{2}\right) \geq 0, \\ (\text{val}(z), 0) & \text{if } \text{val}(z) > 2. \end{cases}$$

So the closure of these points is the set from (iii) in the statement of Kapranov's Theorem. Also, as z varies over $\overline{\mathbb{Q}} \setminus \{0, 2, 4\}$, the above cases describe all points in $\Gamma^2_{\text{val}_2}$ that lie in the tropical hypersurface, $\mathbb{V}_{\text{trop}}(\text{trop}(f))$. The value group is dense in $\mathbb{R}$ when the field is algebraically closed so the closure of these points in $\Gamma^2_{\text{val}_2}$ is exactly the tropical hypersurface in $\mathbb{R}^2$. Thus we see that the sets from (i) and (iii) coincide, and that Kapranov's Theorem is verified.

Proof of Kapranov's Theorem. We denote the set from (i) in the statement of Kapranov's Theorem as set (i), and similarly for (ii) and (iii). We prove Kapranov's theorem by showing that set (i) is equal to set (ii) and set (i) is equal to set (iii).

For $(i) \subseteq (ii)$, let $\boldsymbol{w} \in \mathbb{V}_{\text{trop}}(\text{trop}(f))$. By Definition 2.7, the minimum in $\text{trop}(f)(\boldsymbol{w})$ is achieved at least twice. When calculating $\text{in}_{\boldsymbol{w}}(f)$, we sum over elements of $\mathbb{Z}^n$ which achieve the minimum in $\text{trop}(f)(\boldsymbol{w})$, so we must sum over at least two elements. Thus, $\text{in}_{\boldsymbol{w}}(f)$ is not a monomial and we have $\boldsymbol{w} \in (ii)$. For the opposite inclusion, let $\boldsymbol{w} \in (ii)$ so $\text{in}_{\boldsymbol{w}}(f)$ is not a monomial. By Definition 3.6, there exists at least two elements $\boldsymbol{u} \in \mathbb{Z}^n$ such that $\text{val}(c_{\boldsymbol{u}}) + \boldsymbol{w} \cdot \boldsymbol{u} = \text{trop}(f)(\boldsymbol{w})$. This means there are at least two elements $\boldsymbol{u} \in \mathbb{Z}^n$ which achieve the minimum in $\text{trop}(f)(\boldsymbol{w})$. Therefore, $\boldsymbol{w} \in \mathbb{V}_{\text{trop}}(\text{trop}(f))$.

For $(iii) \subseteq (i)$, we only consider elements of the set $\{(\text{val}(y_1), \ldots, \text{val}(y_n)) \mid (y_1, \ldots, y_n) \in \mathbb{V}_t(f)\}$ since $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is closed in the Euclidean topology on $\mathbb{R}^n$. Let $\boldsymbol{y} = (y_1, \ldots, y_n) \in \mathbb{V}_t(f)$ so $f(\boldsymbol{y}) = \sum\limits_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} \boldsymbol{y}^{\boldsymbol{u}} = 0$. Taking valuations, we have

$$\begin{aligned}\text{val}(f(\boldsymbol{y})) = \text{val}\left(\sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} \boldsymbol{y}^{\boldsymbol{u}}\right) = \text{val}(0) = \infty > \text{val}(c_{\boldsymbol{u}'} \boldsymbol{y}^{\boldsymbol{u}'}) &\text{ for all } \boldsymbol{u}' \in \mathbb{Z}^n \text{ with } c_{\boldsymbol{u}'} \neq 0 \\ = \text{val}(c_{\boldsymbol{u}'}) + \boldsymbol{u}' \cdot \text{val}(\boldsymbol{y}),&\end{aligned}$$

where $\text{val}(\boldsymbol{y}) := (\text{val}(y_1), \ldots, \text{val}(y_n))$. Assume for contradiction that the minimum in $\text{trop}(f)(\text{val}(\boldsymbol{y}))$ is achieved once. So there exists some $\boldsymbol{v} \in \mathbb{Z}^n$ such that $\text{val}(c_{\boldsymbol{v}} \boldsymbol{y}^{\boldsymbol{v}}) < \text{val}(c_{\boldsymbol{u}'} \boldsymbol{y}^{\boldsymbol{u}'})$ for all $\boldsymbol{u}' \in \mathbb{Z}^n$ with $c_{\boldsymbol{u}'} \neq 0$. Now by property (3) of Definition 2.10, it follows that

$$\text{val}\left(\sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} \boldsymbol{y}^{\boldsymbol{u}}\right) \geq \min_{\boldsymbol{u}' \in \mathbb{Z}^n \mid c_{\boldsymbol{u}'} \neq 0} \{\text{val}(c_{\boldsymbol{u}'} \boldsymbol{y}^{\boldsymbol{u}'})\}.$$

But this minimum is achieved uniquely at $\boldsymbol{v}$ so Lemma 2.13 (iv) implies

$$\text{val}(c_{\boldsymbol{v}} \boldsymbol{y}^{\boldsymbol{v}}) = \text{val}\left(\sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} \boldsymbol{y}^{\boldsymbol{u}}\right) = \infty,$$

which is a contradiction. So we conclude that the minimum in $\text{trop}(f)(\text{val}(\boldsymbol{y}))$ is achieved at least twice and thus, $\text{val}(\boldsymbol{y}) \in \mathbb{V}_{\text{trop}}(\text{trop}(f))$.

The opposite inclusion is more involved and is restated and proved in its own proposition. □

The following result finishes our proof of Kapranov's Theorem. Maclagan and Sturmfels describe this result in [MS15] as "every zero of an initial form lifts to a zero of the given polynomial".

Proposition 3.14. *Fix $f = \sum\limits_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ and a weight vector $\boldsymbol{w} \in \Gamma_{\text{val}}^n$. Suppose $\text{in}_{\boldsymbol{w}}(f)$ is not a monomial (so $\boldsymbol{w} \in \mathbb{V}_{\text{trop}}(\text{trop}(f))$) and there exists some $\boldsymbol{\alpha} \in (\Bbbk^\times)^n$ such that $\text{in}_{\boldsymbol{w}}(f)(\boldsymbol{\alpha}) = 0$. Then there exists some $\boldsymbol{y} \in T^n$ such that $f(\boldsymbol{y}) = 0$, $\text{val}(\boldsymbol{y}) = \boldsymbol{w}$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for all $i \in \{1, \ldots, n\}$.*

Proof. We use induction on n. For the base case, let $n = 1$. Multiply f by x^m where $-m \in \mathbb{Z}$ is the minimal exponent of x which appears in f. So we can rewrite f as $f = \sum_{i=0}^N c_i x^i$, where $N \in \mathbb{N}$, $c_i \in K$ and c_0, $c_N \neq 0$. So for some a_j, $b_j \in K$, we have $f = \prod_{j=1}^N (a_j x - b_j)$. By Lemma 3.8, we have $\text{in}_w(f) = \prod_{j=1}^N \text{in}_w(a_j x - b_j)$. By assumption, there exists some $\alpha \in \Bbbk^\times$ such that $\text{in}_w(f)(\alpha) = 0$, so there exists some $k \in \{1, \ldots, N\}$ such that $\text{in}_w(a_k x - b_k)(\alpha) = 0$. Since α is a unit, it follows that $\text{in}_w(a_k x - b_k)$ is not a monomial. So we have,

$$\text{in}_w(a_k x - b_k) = \overline{t^{-\text{val}(a_k)} a_k} x + \overline{t^{-\text{val}(b_k)} b_k}.$$

So by definition of initial forms, $\operatorname{val}(a_k) + w = \operatorname{trop}(f)(w) = \operatorname{val}(b_k)$. Thus,

$$\begin{aligned}
\operatorname{in}_w(a_k x - b_k)(\alpha) = 0 &\implies \overline{t^{-\operatorname{val}(a_k)} a_k}\alpha + \overline{t^{-\operatorname{val}(b_k)} b_k} = 0 \\
&\implies \alpha = \overline{t^{-w}\frac{b_k}{a_k}},
\end{aligned}$$

where the last implication follows from Lemma 2.13 (*ii*). Take $y = \dfrac{b_k}{a_k}$. Then,

$$\begin{aligned}
f(y) &= \prod_{j=1}^{N}(a_j y - b_j) \\
&= (a_1 y - b_1)\cdots\left(a_k \cdot \frac{b_k}{a_k} - b_k\right)\cdots(a_N y - b_N) \\
&= 0.
\end{aligned}$$

Also, by Lemma 2.13 (*iii*), $\operatorname{val}(y) = \operatorname{val}(b_k) - \operatorname{val}(a_k) = w$, and $\alpha = \overline{t^{-w}y}$. So we've proved the base case.

For the inductive step, assume $n > 1$ and that the proposition holds for all smaller dimensions. We reduce to the case where no two terms in f are divisible by the same power of x_n. Formally, we consider an automorphism with the resulting polynomial satisfying this property. We then prove the proposition for this reduced case, which then implies the general case. More details of this construction can be found in [MS15, Proposition 3.1.5].

In this case, we can regard f as a polynomial in $K[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}][x_n]$, with monomial coefficients. So $f = \sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}}$, where each value of u_n appears only once. Recall, from Lemma 3.5, the set

$$\mathcal{Y}^{(n-1)} = \{\boldsymbol{y} = (y_1, \ldots, y_{n-1}) \in T^{n-1} \mid \operatorname{val}(y_i) = w_i,\ \overline{t^{-w_i} y_i} = \alpha_i\ \forall i \in \{1, \ldots, n-1\}\},$$

which we proved to be Zariski dense in T^{n-1}. So for all $\boldsymbol{y} \in \mathcal{Y}^{(n-1)}$, the polynomial $g(x_n) := f(\boldsymbol{y}, x_n)$ is not the zero polynomial. For the rest of the proof, denote the projection of $\boldsymbol{u} \in \mathbb{Z}^n$ onto the first $n-1$ coordinates by $\boldsymbol{u}'$. Let $g = \sum_{i \in \mathbb{Z}} d_i x_n^i$, where $d_i = c_{\boldsymbol{u}} \boldsymbol{y}^{\boldsymbol{u}'}$ for a unique $\boldsymbol{u} \in \mathbb{Z}^n$ such that $u_n = i$. Taking valuations, we see that

$$\operatorname{val}(d_i) + w_n i = \operatorname{val}(c_{\boldsymbol{u}}) + \operatorname{val}(\boldsymbol{y}^{\boldsymbol{u}'}) + w_n i = \operatorname{val}(c_{\boldsymbol{u}}) + \boldsymbol{w}' \cdot \boldsymbol{u}' + w_n u_n = \operatorname{val}(c_{\boldsymbol{u}}) + \boldsymbol{w} \cdot \boldsymbol{u}.$$

This means that $\operatorname{trop}(g)(w_n) = \operatorname{trop}(f)(\boldsymbol{w})$. It then follows from the definition of initial forms and the fact that $\overline{t^{-w_i} y_i} = \alpha_i$ for all $i \in \{1, \ldots, n-1\}$ that

$$\operatorname{in}_{w_n}(g)(x_n) = \operatorname{in}_{\boldsymbol{w}}(f)(\alpha_1, \ldots, \alpha_{n-1}, x_n).$$

But we know $\operatorname{in}_{\boldsymbol{w}}(f)(\boldsymbol{\alpha}) = 0$ so we must have $\operatorname{in}_{w_n}(g)(\alpha_n) = 0$. So, by the base case, there exists some $y_n \in K^\times$ such that $\operatorname{val}(y_n) = w_n$, $\overline{t^{-w_n} y_n} = \alpha_n$ and $g(y_n) = 0$. Thus, we have a point $(y_1, \ldots, y_n) \in \mathbb{V}_t(f)$. Therefore, for all $\boldsymbol{w} \in \mathbb{V}_{\text{trop}}(\operatorname{trop}(f))$, there exists a point $(y_1, \ldots, y_n) \in \mathbb{V}_t(f)$ such that

$$\operatorname{val}((y_1, \ldots, y_n)) = (w_1, \ldots, w_n) = \boldsymbol{w}.$$

In particular, set (*i*) is contained in set (*iii*) and this completes the proof of Kapranov's Theorem.

□

4 The Structure Theorem

In this section, we explore the structure of the tropical hypersurface when viewing it as a *polyhedral complex*. However, we first need to give an introduction to polyhedral geometry.

4.1 Polyhedral Geometry

This subsection is heavy on definitions. To balance this, we explore many examples.

Definition 4.1. (Polytope). Let $U \subset \mathbb{R}^n$. The *convex hull* of U is the smallest convex set containing U. We denote this set by $\mathrm{conv}(U)$. If U is a finite collection of vectors in $\mathbb{R}^n$, say $U = \{\boldsymbol{u}_1, \dots, \boldsymbol{u}_m\}$, then

$$\mathrm{conv}(U) = \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{u}_i \mid 0 \leq \lambda_i \leq 1, \ \sum_{i=1}^{m} \lambda_i = 1 \right\}$$

is called a *polytope*.

Example 4.2 (Newton polytope). One of the main examples of a polytope we come across is the *Newton polytope*. Let $f = \sum_{\boldsymbol{u} \in \mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. Then the Newton polytope of f is defined by

$$\mathrm{Newt}(f) = \mathrm{conv}\{\boldsymbol{u} \mid c_{\boldsymbol{u}} \neq 0\} \subset \mathbb{R}^n.$$

Consider,

$$f = 2t^2x^2 + (1+t)y^2 + 3xy + t^3x + (2t+t^2)y + x^{-1} \in \mathbb{C}\{\{t\}\}[x^{\pm 1}, y^{\pm 1}],$$

from Example 3.7. Then,

$$\mathrm{Newt}(f) = \mathrm{conv}\{(2,0), (0,2), (1,1), (1,0), (0,1), (-1,0)\}.$$

This can be seen in Figure 8, where we've divided $\mathbb{R}^2$ into lattice points.

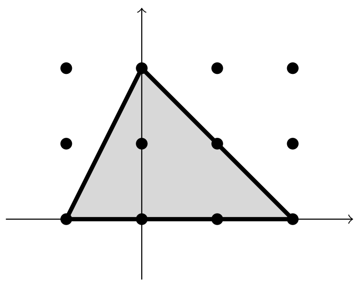

Figure 8: The Newton polytope of f in $\mathbb{R}^2$.

We explore this construction further later on.

Definition 4.3. (Cones and fans). A *polyhedral cone*, C, is the positive hull of a finite set of vectors $\boldsymbol{v}_1, \dots, \boldsymbol{v}_m \in \mathbb{R}^n$, which is defined by

$$C = \mathrm{pos}\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\} := \left\{ \sum_{i=1}^{m} \lambda_i \boldsymbol{v}_i \;\middle|\; \lambda_i \geq 0 \text{ for all } i \in \{1, \dots, m\} \right\}.$$

A cone is said to be *simplicial* if $\{\boldsymbol{v}_1, \dots, \boldsymbol{v}_m\}$ is a set of linearly independent vectors. The *face of a cone* induced by a linear form, $\boldsymbol{w}$, in the dual space, $(\mathbb{R}^n)^*$, is the set

$$\mathrm{face}_{\boldsymbol{w}}(C) := \{\boldsymbol{x} \in C \mid \boldsymbol{w} \cdot \boldsymbol{x} \leq \boldsymbol{w} \cdot \boldsymbol{y} \text{ for all } \boldsymbol{y} \in C\},$$

A *polyhedral fan*, $\mathcal{F} = \{C_1, \dots, C_r\}$, is a collection of polyhedral cones satisfying the following two properties:

- Every face of a cone in $\mathcal{F}$ is also a cone in $\mathcal{F}$.
- The intersection of any two cones in $\mathcal{F}$ is a face of both.

A fan is *simplicial* if every cone it contains is simplicial.

Example 4.4. Let $\boldsymbol{v}_1 = (1, 0)$, $\boldsymbol{v}_2 = (0, 1)$ and $\boldsymbol{v}_3 = (-1, -1)$. Define the two cones $C_1 = \text{pos}\{\boldsymbol{v}_1,\ \boldsymbol{v}_2\}$ and $C_2 = \text{pos}\{\boldsymbol{v}_2,\ \boldsymbol{v}_3\}$. These are pictured in Figures 9 and 10, respectively.

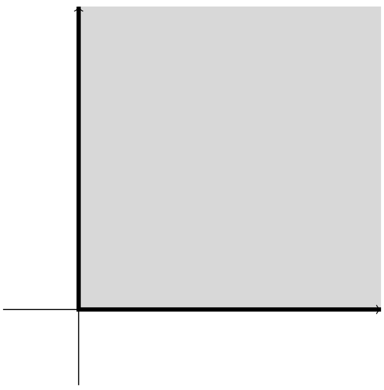

Figure 9: The cone C_1 in $\mathbb{R}^2$.

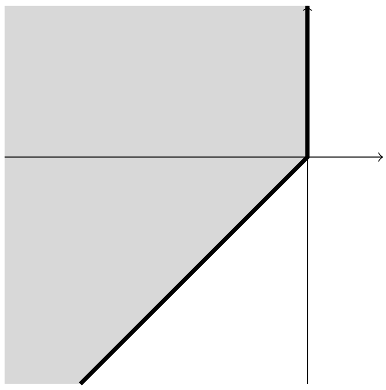

Figure 10: The cone C_2 in $\mathbb{R}^2$.

Each cone has one face of dimension 0, a *vertex*, two faces of dimension 1, an *edge*, and one face of dimension 2. We formally define dimension in Definition 4.8. For C_1, the vertex is $(0, 0)$, induced by the linear form $(\lambda, \mu) \in (\mathbb{R}^2)^*$ for any $\lambda, \mu > 0$. The two edges are the positive x-axis and the positive y-axis, induced by any positive multiple of $(0, 1)$ and $(1, 0)$, respectively. The two-dimensional face is C_1 itself and is induced by $(0, 0)$.

Both C_1 and C_2 are simplicial since no two vectors are multiples of each other. However, if we were to define a third cone, say $C_3 = \text{pos}\{\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3\}$, then C_3 is not simplicial since $\boldsymbol{v}_1 + \boldsymbol{v}_2 + \boldsymbol{v}_3 = 0$. Note that C_3 is the entire $\mathbb{R}^2$ plane.

Let $\mathcal{F}_1 = \{C_1, C_2, \text{pos}\{\boldsymbol{v}_1\}, \text{pos}\{\boldsymbol{v}_2\}, \text{pos}\{\boldsymbol{v}_3\}, \{(0, 0)\}\}$. The intersection of C_1 and C_2 is the positive y-axis which is a face of both C_1 and C_2. Also, every face of C_1 and C_2 is also a cone in $\mathcal{F}_1$. So $\mathcal{F}_1$ is a polyhedral fan and, since both C_1 and C_2 are simplicial, $\mathcal{F}_1$ is simplicial. Now define the cone $C_4 = \text{pos}\{\boldsymbol{v}_1, -\boldsymbol{v}_1\}$, which is the entire x-axis, and its only face is itself. Then,

$$\mathcal{F}_2 = \{C_1, C_4, \text{pos}\{\boldsymbol{v}_1\}, \text{pos}\{\boldsymbol{v}_2\}, \{(0, 0)\}\}$$

is not a polyhedral fan since the intersection of C_1 and C_4 is the positive x-axis, which is not a face in C_4. These examples are pictured in Figures 11 and 12, respectively.

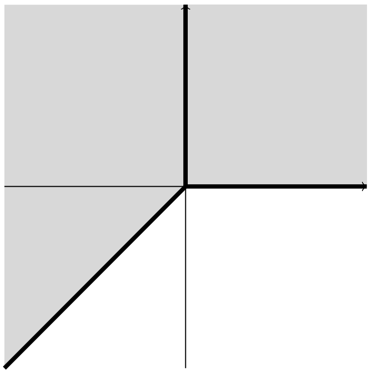

Figure 11: $\mathcal{F}_1$, a polyhedral fan.

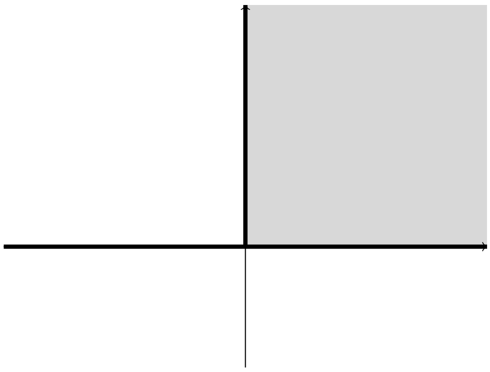

Figure 12: $\mathcal{F}_2$, not a polyhedral fan.

We generalise the concepts of cones and fans to that of *polyhedra* and *polyhedral complexes*, respectively.

Definition 4.5. (Polyhedra and polyhedral complexes). A *polyhedron*, $P \subset \mathbb{R}^n$, is an intersection of finitely many closed affine half-spaces. So for a given $A \in \mathbb{R}^{d\times n}$ and $\boldsymbol{b} \in \mathbb{R}^d$, where d is finite, we have

$$P = \{\boldsymbol{x} \in \mathbb{R}^n \mid A\boldsymbol{x} \leq \boldsymbol{b}\}.$$

We define a *face of a polyhedron* similar to how we defined a face of a cone in Definition 4.3. A face of a polyhedron which is not contained in any larger proper face (a face which is not the polyhedron itself) is called a *facet*.

A *polyhedral complex*, Σ, is a collection of polyhedra satisfying the following two properties:

- If $P \in \Sigma$, then all the faces of P are also in Σ.
- If $P, Q \in \Sigma$, then $P \cap Q$ is either empty or a face of both P and Q.

Polyhedra in a polyhedral complex, Σ, are called the *cells of* Σ. The cells of Σ that are not faces of any larger cell are called *facets* of Σ, and the facets of these cells are called *ridges*. The *support* of Σ is the set

$$|\Sigma| := \{\boldsymbol{x} \in \mathbb{R}^n \mid \boldsymbol{x} \in P \text{ for some } P \in \Sigma\}.$$

Remarks 4.6. (1) By [Zie95, §1.1], polytopes are bounded polyhedra.

(2) In the definition of a polyhedron, if we set $\boldsymbol{b} = \boldsymbol{0}$, then this definition is equivalent to the definition of a cone. See [Zie95, Theorem 1.3] for more details.

(3) Now suppose $\boldsymbol{b} \in \Gamma_{\text{val}}^d \leqslant \mathbb{R}^d$ and $A \in \mathbb{Q}^{d\times n}$, where we recall that Γ_{val} is the value group with respect to some valuation val. Then we say that the polyhedron is Γ_{val}-*rational*. Moreover, a polyhedral complex is said to be Γ_{val}-*rational* if every cell in the complex is Γ_{val}-rational. Note that if $\Gamma_{\text{val}} = \mathbb{Q}$, then we say that the polyhedron/polyhedral complex is *rational*.

Example 4.7. Consider the following polyhedron,

$$P = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \right\},$$

which is the triangle in $\mathbb{R}^2$ with vertices $(0,0)$, $(1,0)$ and $(0,1)$. The facets of P are the "edges" of the triangle, $\{(\lambda, 0)\}$, $\{(0,\lambda)\}$ and $\{(\lambda, 1-\lambda)\}$, where in each case $0 \leq \lambda \leq 1$. Now consider the polyhedron Q defined by

$$Q = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} -1 & 0 \\ 0 & -1 \\ -1 & -1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} 0 \\ 0 \\ -1 \end{pmatrix} \right\},$$

which has facets $\{(1+\lambda, 0) \mid \lambda \geq 0\}$, $\{(0, 1+\lambda) \mid \lambda \geq 0\}$ and $\{(\lambda, 1-\lambda) \mid 0 \leq \lambda \leq 1\}$. Then $\Sigma = \{P, Q\}$ is the polyhedral complex pictured in Figure 13. Indeed, the intersection of P and Q is $\{(\lambda, 1-\lambda) \mid 0 \leq \lambda \leq 1\}$, which is a face of both P and Q. The facets of Σ are P and Q, and the ridges of Σ are the previously described facets of P and Q. The support of Σ is $|\Sigma| = \mathbb{R}^2_{\geq 0}$.

We now define some properties of polyhedra.

Definition 4.8. Let $P \in \mathbb{R}^n$ be a polyhedron. The *lineality space* of P is the largest linear subspace, $V \subset \mathbb{R}^n$, such that for all $\boldsymbol{x} \in P$ and $\boldsymbol{v} \in V$, then $\boldsymbol{x} + \boldsymbol{v} \in P$. The lineality space of a polyhedral complex is the intersection of all the lineality spaces of cells in the complex. The smallest affine subspace containing P is called the *affine span of* P. This is the translate of a linear subspace in $\mathbb{R}^n$, called the *linear space parallel to* P. The *dimension* of P is the dimension of the linear space parallel to P. A polyhedral complex, Σ, is said to be *pure* of dimension d if every facet of Σ has dimension d.

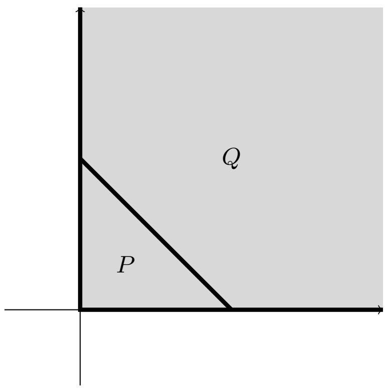

Figure 13: The polyhedral complex, Σ, in $\mathbb{R}^2$.

Example 4.9. Consider the polyhedral complex $\Sigma = \{P, Q\}$ as defined in Example 4.7. The lineality space of both P and Q is $\{(0,0)\}$, and thus the lineality space of Σ is $\{(0,0)\}$. An example of a non-trivial lineality space is that of the polyhedron defined by

$$R = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 \;\middle|\; \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \leq \begin{pmatrix} -1 \\ 1 \end{pmatrix} \right\},$$

which is the line $y = 1$. The lineality space of R is then the x-axis. Note that this is also the linear space parallel to R, and so the dimension of R is 1. The linear space parallel to P is $\mathbb{R}^2$, as is the linear space parallel to Q. In fact, this is true for all 2-dimensional polyhedra in $\mathbb{R}^2$. So, since the dimension of both P and Q is 2, we have that Σ is pure of dimension 2.

We now establish a link between polyhedra and polyhedral fans.

Definition 4.10. (Normal fan). Let $P \subset \mathbb{R}^n$ be a polyhedron. The *(inner) normal fan* of P is the polyhedral fan, $\mathcal{N}_P$, consisting of the cones

$$\mathcal{N}_P(F) := \overline{\{\boldsymbol{w} \in (\mathbb{R}^n)^* \mid \text{face}_{\boldsymbol{w}}(P) = F\}},$$

as F varies over the faces of P. Here, $\overline{\{\cdot\}}$ denotes the closure in the Euclidean topology (on $(\mathbb{R}^n)^*$).

Example 4.11. Consider P from Example 4.7. Recall P has three vertices at $(0,0)$, $(1,0)$ and $(0,1)$. The vertex $(0,0)$ is induced by any $(w_1, w_2) \in (\mathbb{R}^2)^*$ such that $w_1, w_2 > 0$. Denote this region (and vertex) by (1). The vertex $(1,0)$ is induced by any $(w_1, w_2) \in (\mathbb{R}^2)^*$ such that $w_1 < 0$ and $w_2 > w_1$, and the vertex $(0,1)$ is induced by any $(w_1, w_2) \in (\mathbb{R}^2)^*$ such that $w_2 < 0$ and $w_1 > w_2$. Denote these regions by (2) and (3), respectively.

The one-dimensional faces of P are the edges of the triangle. The edge between $(0,0)$ and $(1,0)$ is induced by any positive multiple of $(0,1)$, the edge between $(0,0)$ and $(0,1)$ is induced by any positive multiple of $(1,0)$ and the edge between $(1,0)$ and $(0,1)$ is induced by any positive multiple of $(-1,-1)$. The two dimensional face of P is simply P itself and is induced by $(0,0)$. So we have obtained the normal fan of P which is shown in Figure 14. This fan contains seven cones; three of dimension 2, three of dimension 1 and one of dimension 0.

Definition 4.12. (Star of a cell). Let Σ be a polyhedral complex in $\mathbb{R}^n$ and σ a cell in Σ. The *star of* σ in Σ is a fan in $\mathbb{R}^n$, denoted by $\text{star}_\Sigma(\sigma)$, with cones indexed by the cells $\tau \in \Sigma$ that contain σ as a face. The cone of $\text{star}_\Sigma(\sigma)$ that is indexed by τ is

$$\overline{\tau} = \{\lambda(\boldsymbol{x} - \boldsymbol{y}) \mid \lambda \geq 0,\ \boldsymbol{x} \in \tau,\ \boldsymbol{y} \in \sigma\}.$$

Examples 4.13. (1) Consider the polyhedral complex $\Sigma = \{P, Q\}$ as defined in Example 4.7. We wish to find the star of $\sigma := (1,0)$. The cells of Σ that contain σ as a face are P, Q, $P \cap Q$ and the

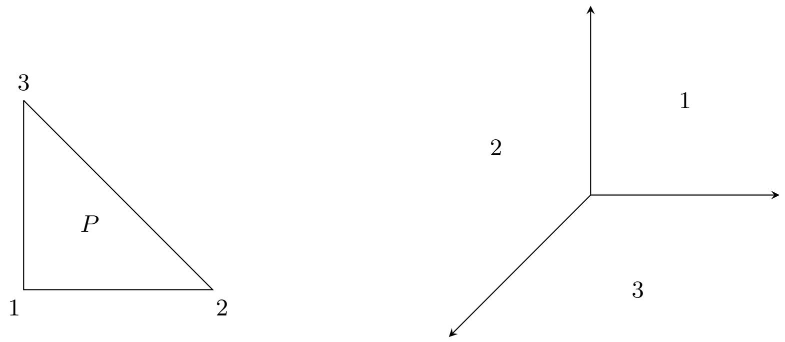

Figure 14: The polyhedron, P, and its normal fan, $\mathcal{N}_P$.

edges of P and Q which lie on the x-axis. Using the definition of $\text{star}_\Sigma(\sigma)$, the cone of $\text{star}_\Sigma(\sigma)$ indexed by P is given by

$$\overline{P} = \{\lambda(x_1 - 1, x_2) \mid \lambda \geq 0,\ (x_1, x_2) \in P\}.$$

So $\overline{P}$ is the area of $\mathbb{R}^2$ bounded below by the x-axis and bounded above by the line $y = 1 - x$. Similarly, $\overline{Q}$ is the area of $\mathbb{R}^2$ above both of these lines. We repeat this for each of the one-dimensional faces and obtain $\text{star}_\Sigma(\sigma)$, as shown in Figure 15.

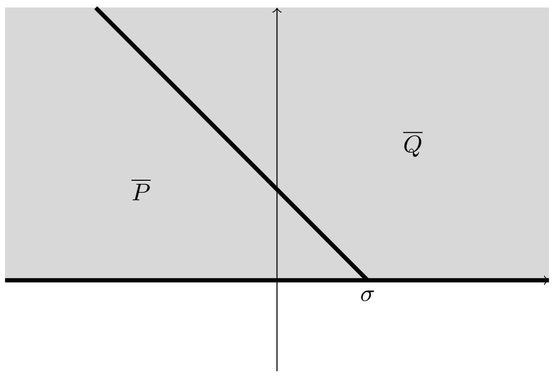

Figure 15: The star of $\sigma := (1, 0)$ in Σ, $\text{star}_\Sigma(\sigma)$.

(2) Towards the end of this section, we show that the tropical hypersurface is a polyhedral complex. So we can define the star of different parts of the tropical hypersurface. Consider the tropical hypersurface in Figure 7, and define $\Sigma = \mathbb{V}_{\text{trop}}(\text{trop}(f))$. Let σ_1 be the vertex $(2, 0)$ and σ_2 be the ray $\{(1 - \lambda, -1) \mid \lambda \geq 0\}$. Then the star of σ_1 is the tropical line and the star of σ_2 is the x-axis. These are pictured in Figure 16.

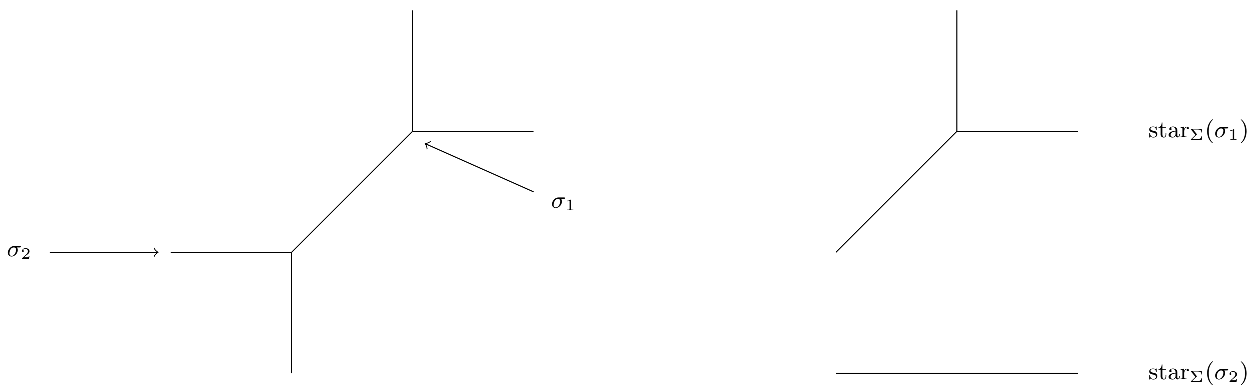

Figure 16: The stars of two polyhedra in a tropical hypersurface.

Definition 4.14. (Regular subdivision). Let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ be vectors in $\mathbb{R}^n$ and fix a weight vector $\boldsymbol{w} = (w_1, \ldots, w_r)$ in $\mathbb{R}^r$. The *regular subdivision* of $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r$ induced by $\boldsymbol{w}$ is a polyhedral fan with support $\mathrm{pos}\{\boldsymbol{v}_1, \ldots, \boldsymbol{v}_r\}$. The cones are given by $\mathrm{pos}\{\boldsymbol{v}_i \mid i \in \rho\}$, for all subsets $\rho \subseteq \{1, \ldots, r\}$, where there exists some $\boldsymbol{c} \in \mathbb{R}^{n+1}$ such that $\boldsymbol{c} \cdot \boldsymbol{v}_i = w_i$ for all $i \in \rho$, and $\boldsymbol{c} \cdot \boldsymbol{v}_i < w_i$ for all $i \notin \rho$.

Remarks 4.15. (1) If the polyhedral fan in Definition 4.14 is simplicial, then we call the subdivision a *regular triangulation.*

(2) We can define the regular subdivision of a polytope,

$$P = \mathrm{conv}\{\boldsymbol{u}_i \mid i \in \{1, \ldots, r\}\} \subset \mathbb{R}^n,$$

induced by some $\boldsymbol{w} = (\boldsymbol{w}_1, \ldots, \boldsymbol{w}_r)$ in $\mathbb{R}^r$, where $\boldsymbol{u}_i \in \mathbb{R}^n$ for all $i \in \{1, \ldots, r\}$. First, define the polytope

$$P_{\boldsymbol{w}} = \mathrm{conv}\{(\boldsymbol{u}_i, w_i) \mid i \in \{1, \ldots, r\}\} \subset \mathbb{R}^{n+1}.$$

The *lower faces* of $P_{\boldsymbol{w}}$ are the faces with an inner normal vector, $\boldsymbol{v} \in (\mathbb{R}^{n+1})^*$, such that the last coordinate of $\boldsymbol{v}$ is positive. We project these lower faces onto P in $\mathbb{R}^n$ and they form a polyhedral complex with support equal to P. This is the regular subdivision of P induced by $\boldsymbol{w}$.

Example 4.16. We return to the polynomial defined in Example 3.13,

$$f = 2xy + x - 4y - 4 \in \mathbb{Q}[x, y],$$

where $\mathbb{Q}$ has the 2-adic valuation. The Newton polytope of f is given by

$$P = \mathrm{Newt}(f) = \mathrm{conv}\{(1, 1), (1, 0), (0, 1), (0, 0)\},$$

which is the unit square in $\mathbb{R}^2$. We wish to find the regular subdivision of these points induced by the vector of valuations of the coefficients of f. So we set $\boldsymbol{w} = (1, 0, 2, 2)$ and consider the polytope

$$P_{\boldsymbol{w}} = \mathrm{conv}\{(1, 1, 1), (1, 0, 0), (0, 1, 2), (0, 0, 2)\}.$$

In Figure 17, the lower faces of $P_{\boldsymbol{w}}$ are shown in grey whilst the Newton polytope is shown below in the xy plane.

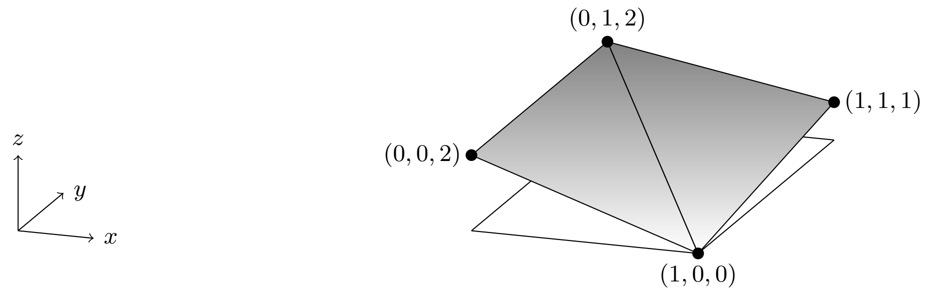

Figure 17: The lower faces of $P_{\boldsymbol{w}}$ above P in $\mathbb{R}^3$.

As in Remark 4.15 (2), we project the lower faces of $P_{\boldsymbol{w}}$ onto P which gives the regular subdivision of $\mathrm{Newt}(f)$ induced by the valuations of the coefficients of f. This is shown in Figure 18. By Remark 4.15 (2), this is a polyhedral complex with support equal to $\mathrm{Newt}(f)$.

Note that the inward pointing normal vector of the left lower face of $P_{\boldsymbol{w}}$ is $(2, 0, 1)$ and that of the right lower face is $(1, -1, 1)$. Projecting these points onto $\mathbb{R}^2$, we obtain the points $(2, 0)$ and $(1, -1)$, which we note are the exact same points that achieve the minimum in $\mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f))$ three times, as seen in Example 3.13. This is not a coincidence. The regular subdivision of $\mathrm{Newt}(f)$ is the planar graph dual to $\mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f))$, which is where we interchange vertices and full dimensional faces between the two graphs. We prove this in the next subsection.

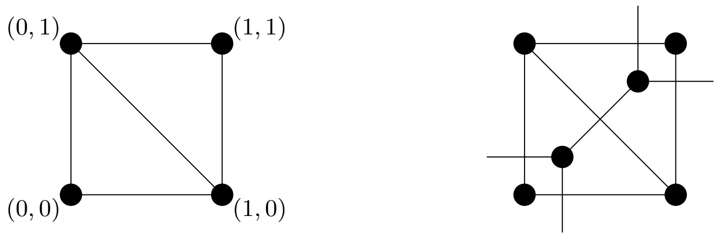

Figure 18: The regular subdivision of $\operatorname{Newt}(f)$ on the left, and its planar graph dual on the right.

4.2 The Structure Theorem

In this subsection, we prove the following theorem:

Theorem 4.17 (The Structure Theorem). *Let* $f = \sum_{\boldsymbol{u}\in\mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$*. Then* $\mathbb{V}_{\mathrm{trop}}(\operatorname{trop}(f))$ *is the support of a balanced, weighted* Γ_{val}*-rational polyhedral complex in* $\mathbb{R}^n$*, which is pure of dimension* $n-1$*.*

We prove this in two parts. First, we show that $\mathbb{V}_{\mathrm{trop}}(\operatorname{trop}(f))$ is the support of a Γ_{val}-rational polyhedral complex, which is pure of dimension $(n-1)$. We then assign weights to this polyhedral complex and show that it's balanced. For the first step, we prove the following proposition, which gives an alternative method for computing tropical hypersurfaces. In the following statement, the term $(n-1)$-*skeleton* of a polyhedral complex Σ refers to the polyhedral complex consisting of all cells $\sigma \in \Sigma$ such that $\dim(\sigma) \le n-1$.

Proposition 4.18. *Let* $f = \sum_{\boldsymbol{u}\in\mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$*. Then* $\mathbb{V}_{\mathrm{trop}}(\operatorname{trop}(f))$ *is the* $(n-1)$ *skeleton of the polyhedral complex dual to the regular subdivision of the Newton polytope of* f*,* $\operatorname{Newt}(f)$*, induced by the weights* $\{\operatorname{val}(c_{\boldsymbol{u}}) \mid c_{\boldsymbol{u}} \neq 0\}$ *on the lattice points of* $\operatorname{Newt}(f)$*.*

Proof. In this proof, we generalise the approach taken in Example 4.16. Let $P = \operatorname{Newt}(f) \subset \mathbb{R}^n$ and $P_{\mathrm{val}} = \operatorname{conv}\{(\boldsymbol{u}, \operatorname{val}(c_{\boldsymbol{u}})) \mid c_{\boldsymbol{u}} \neq 0\} \subset \mathbb{R}^{n+1}$. Let $\pi : \mathbb{R}^{n+1} \longrightarrow \mathbb{R}^n$ be the projection onto the first n coordinates defined by

$$\pi((a_1, \dots, a_n, a_{n+1})) = (a_1, \dots, a_n), \qquad \text{for all } \boldsymbol{a} \in \mathbb{R}^{n+1}.$$

We wish to construct the regular subdivision of P induced by the weights $\{\operatorname{val}(c_{\boldsymbol{u}}) \mid c_{\boldsymbol{u}} \neq 0\}$. Let F be a lower face of P_{val}. As described in Remark 4.15 (2), the lower faces of P_{val} are those with an inner normal vector, $\boldsymbol{v} \in (\mathbb{R}^{n+1})^*$, such that the last coordinate of $\boldsymbol{v}$ is positive. So we have

$$F = \operatorname{face}_{\boldsymbol{v}}(P_{\mathrm{val}}) = \{\boldsymbol{x} \in P_{\mathrm{val}} \mid \boldsymbol{v}\cdot\boldsymbol{x} \le \boldsymbol{v}\cdot\boldsymbol{y} \text{ for all } \boldsymbol{y} \in P_{\mathrm{val}}\}.$$

Define the normal cone of F to be

$$\mathcal{N}(F) = \{\boldsymbol{v} \in (\mathbb{R}^{n+1})^* \mid \operatorname{face}_{\boldsymbol{v}}(P_{\mathrm{val}}) = F\},$$

and the restricted projection of $\mathcal{N}(F)$ to be

$$\tilde{\pi}(\mathcal{N}(F)) = \{\boldsymbol{w} \in (\mathbb{R}^n)^* \mid (\boldsymbol{w}, 1) \in \mathcal{N}(F)\}.$$

These restricted projections, $\tilde{\pi}(\mathcal{N}(F))$, as F varies over all lower faces of P_{val}, form a polyhedral complex in $\mathbb{R}^n$ that is dual to the regular subdivision of P induced by the weights $\{\operatorname{val}(c_{\boldsymbol{u}}) \mid c_{\boldsymbol{u}} \neq 0\}$.

We now show that $\mathbb{V}_{\mathrm{trop}}(\operatorname{trop}(f))$ is the $(n-1)$ skeleton of the dual complex formed by $\tilde{\pi}(\mathcal{N}(F))$. Let $\boldsymbol{v} = (v_1, \dots, v_n, 1) \in \mathcal{N}(F)$. For each vertex, $(\boldsymbol{u}', \operatorname{val}(c_{\boldsymbol{u}'}))$, of F, we have

$$(v_1, \dots, v_n)\cdot\boldsymbol{u}' + \operatorname{val}(c_{\boldsymbol{u}'}) = \boldsymbol{v}\cdot(\boldsymbol{u}', \operatorname{val}(c_{\boldsymbol{u}'})) \le \boldsymbol{v}\cdot(\boldsymbol{u}, \operatorname{val}(c_{\boldsymbol{u}})) = (v_1, \dots, v_n)\cdot\boldsymbol{u} + \operatorname{val}(c_{\boldsymbol{u}}),$$

for all $(\boldsymbol{u}, \operatorname{val}(c_{\boldsymbol{u}})) \in P_{\mathrm{val}}$. So by definition of the tropical hypersurface (Definition 2.7), $\boldsymbol{w} \in \mathbb{V}_{\mathrm{trop}}(\operatorname{trop}(f))$ if and only if $\boldsymbol{w} \in \tilde{\pi}(\mathcal{N}(F))$ for some face F of P_{val} such that F has more than one vertex. This happens if and only if $F = \text{face}_{(\boldsymbol{w},1)}(P_{\text{val}})$ is not a vertex, which happens if and only if $\tilde{\pi}(\mathcal{N}(F))$ does not have dimension n in the dual complex, since the dual graph interchanges vertices and faces of dimension n. Thus, $\boldsymbol{w} \in \mathbb{V}_{\text{trop}}(\text{trop}(f))$ if and only if the face which contains $\boldsymbol{w}$ in the dual complex does not have dimension n.

Therefore, $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is the $(n-1)$ skeleton of the dual complex, and is thus the support of a polyhedral complex with dimension $(n-1)$.

□

We can formally define the polyhedral complex defined in Proposition 4.18.

Definition 4.19. Let $f = \sum_{\boldsymbol{u}\in\mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Define $\Sigma_{\text{trop}(f)}$ to be the coarsest polyhedral complex such that $\text{trop}(f)$ is linear on each cell in $\Sigma_{\text{trop}(f)}$. The facets of $\Sigma_{\text{trop}(f)}$ have the form

$$\sigma = \{\boldsymbol{w} \in \mathbb{R}^n \mid \text{trop}(f)(\boldsymbol{w}) = \text{val}(c_{\boldsymbol{u}}) + \boldsymbol{w} \cdot \boldsymbol{u}\},$$

for each $\boldsymbol{u} \in \mathbb{Z}^n$ such that $c_{\boldsymbol{u}} \neq 0$.

Remarks 4.20. (1) The support of $\Sigma_{\text{trop}(f)}$ is $\mathbb{R}^n$ and $\Sigma_{\text{trop}(f)}$ is Γ_{val}-rational. This implies that $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is also Γ_{val}-rational.

(2) We can add to Figure 7 from Example 3.13, where $f = 2xy + x - 4y - 4 \in \mathbb{Q}[x, y]$. From the above definition, the facets of $\Sigma_{\text{trop}(f)}$ contain points which only achieve the minimum once in $\text{trop}(f)$. So in each region of Figure 7, we can add a label of which function achieves the minimum in that region. This is demonstrated in Figure 19 and can be checked by substituting in points from each region.

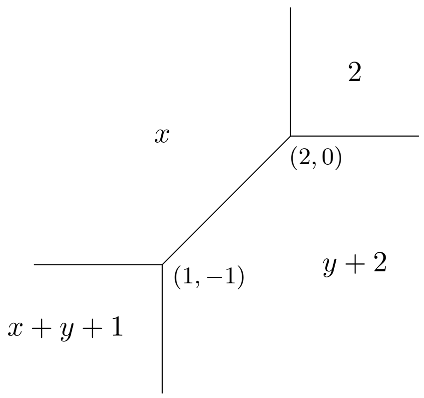

Figure 19: The tropical hypersurface of f, $\mathbb{V}_{\text{trop}}(\text{trop}(f))$, in $\mathbb{R}^2$, with labelled regions of minimality.

So now we need to show that $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is weighted and balanced. We now define a weighted polyhedral complex.

Definition 4.21. A *weighted polyhedral complex* is a pure polyhedral complex, Σ, with a weight w_σ for all facets $\sigma \in \Sigma$.

Example 4.22. There are natural weights which we can assign to a tropical hypersurface. Let $f = \sum_{\boldsymbol{u}\in\mathbb{Z}^n} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$, $P = \text{Newt}(f)$ and Δ be the regular subdivision of P induced by the weight vector $(\text{val}(c_{\boldsymbol{u}}))$ for each $c_{\boldsymbol{u}} \neq 0$. Recall that $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is the $(n-1)$ skeleton of $\Sigma_{\text{trop}(f)}$. Every facet of $\mathbb{V}_{\text{trop}}(\text{trop}(f))$, and so every ridge of $\Sigma_{\text{trop}(f)}$, say σ, corresponds to an edge, $e(\sigma)$, of Δ. Then we define the *weight*, $m(\sigma)$, attached to the facet σ to be the lattice length of $e(\sigma)$, which is one less than the number of lattice points in $e(\sigma)$.

We now define the term *balanced* in the context of a one-dimensional rational fan. We then extend this to a general polyhedral complex via the use of "star", as in Definition 4.12.

Definition 4.23. (Balanced fan). Let Σ be a one-dimensional rational fan with s rays and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_s$ be the first lattice points on the rays of Σ. Assign a weight w_i to the cone containing the lattice point $\boldsymbol{v}_i$. Then Σ is balanced if $\sum_{i=1}^{s} w_i \boldsymbol{v}_i = \mathbf{0}$.

Definition 4.24. (Balanced polyhedral complex). Let Σ be a general polyhedral complex in $\mathbb{R}^n$, pure of dimension d, and let $\tau \in \Sigma$ be polyhedron of dimension $(d-1)$. Let L be the affine span of τ. By definition, $\text{star}_\Sigma(\tau)$ has one cone for each polyhedron $\sigma \in \Sigma$, and has lineality space L. Then $\text{star}_\Sigma(\tau)/L$ is a one dimensional fan which inherits weights from Σ. We say that Σ is *balanced at* τ if $\text{star}_\Sigma(\tau)/L$ is balanced. We says that Σ is *balanced* if Σ is balanced at all cones of dimension $(d-1)$.

The proof of the following proposition completes the proof of the Structure Theorem.

Proposition 4.25. *The polyhedral complex $\mathbb{V}_{\text{trop}}(\text{trop}(f))$, which is pure of dimension $(n-1)$, is balanced for the weights $m(\sigma)$, as defined in Example 4.22.*

Proof. When $n = 1$, the result is trivial since the polyhedral complex has dimension 0. When $n = 2$, we have that $d = 1$ is Definition 4.24. So let $\tau \in \mathbb{V}_{\text{trop}}(\text{trop}(f))$ be a polyhedron of dimension 0, i.e. a vertex. The star of this vertex, $\text{star}_{\mathbb{V}_{\text{trop}}(\text{trop}(f))}(\tau)$, is the tropical line containing τ, and so the lineality space of $\text{star}_{\mathbb{V}_{\text{trop}}(\text{trop}(f))}(\tau)$ is $L = \{(0,0)\}$. So we need to show that $\text{star}_{\mathbb{V}_{\text{trop}}(\text{trop}(f))}(\tau)$ is balanced. Since τ has dimension 0, it is dual to a convex polygon Q in the regular subdivision Δ, from Example 4.22. The cones in $\text{star}_{\mathbb{V}_{\text{trop}}(\text{trop}(f))}(\tau)$, which are indexed by 1-dimensional polyhedra σ containing τ, correspond to edges of Q. Denote theses edges by $e(\sigma)$. The vectors, $\boldsymbol{v}_i$, in Definition 4.23 are the primitive lattice vectors perpendicular to the edges of Q, and the vectors $m(\sigma)\boldsymbol{v}_\sigma$ are exactly the edges of Q rotated by $\frac{\pi}{2}$. Thus,

$$\sum_{\tau \in \sigma} m(\sigma)\boldsymbol{v}_\sigma = \mathbf{0},$$

since the edge vectors of a convex polygon sum to $\mathbf{0}$.

For $n \geq 3$, the result follows by reducing to the $n = 2$ case since $\text{star}_{\mathbb{V}_{\text{trop}}(\text{trop}(f))}(\tau)/L$ has dimension 1 for all $(n-2)$-dimensional cones $\tau \in \mathbb{V}_{\text{trop}}(\text{trop}(f))$. □

Proof of the Structure Theorem. The fact that $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is a Γ_{val}-rational polyhedral complex in $\mathbb{R}^n$, pure of dimension $n-1$, follows from Proposition 4.18. The fact that $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is weighted and balanced is immediate from Proposition 4.25. □

Example 4.26. Consider $f = 2xy + x - 4y - 4 \in \mathbb{Q}[x, y]$ from Example 3.13. The tropical hypersurface $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ has two vertices; $(2, 0)$ and $(1, -1)$. From Example 4.13 (2), the star of each vertex is a tropical line. Consider the vertex $(2, 0)$. The dual to its star is the triangle with inward pointing normal vectors $(-1, 0)$, $(0, -1)$ and $(1, 1)$. The lattice length of each edge is 1 so each cone in the star has a weight of 1. Then,

$$1 \cdot (-1, 0) + 1 \cdot (0, -1) + 1 \cdot (1, 1) = (0, 0),$$

which means the star of $(2, 0)$ is balanced. Similarly, the star of $(1, -1)$ is balanced. Therefore, $\mathbb{V}_{\text{trop}}(\text{trop}(f))$ is balanced.

5 The Fundamental Theorem of Tropical Geometry

In this section, we state and prove the Fundamental Theorem of Tropical Geometry, which gives us different ways of viewing *tropical algebraic sets*. This is a generalisation of Kapranov's Theorem (Theorem 3.11), which is used in the proof as a base case. We also extend our study into initial forms by considering *initial ideals* and an important branch of mathematics called *Gröbner Theory*.

5.1 Gröbner Theory

In Section 3.1, initial forms are defined over the Laurent polynomial ring. However, when defining initial ideals, we must take more care when working over the Laurent polynomial ring. So we first develop the Gröbner Theory over the polynomial ring, and then later extend it to the Laurent polynomial ring.

Definition 5.1 (Initial Ideal). Let (K, val) be a valued field and $I \subset K[x_0, \dots, x_n]$ a homogeneous ideal. The *initial ideal* of I with respect to some $\boldsymbol{w} \in \mathbb{R}^{n+1}$ is an ideal in $\Bbbk[x_0, \dots, x_n]$ given by

$$\mathrm{in}_{\boldsymbol{w}}(I) = \langle\, \mathrm{in}_{\boldsymbol{w}}(f) \mid f \in I \,\rangle \subset \Bbbk[x_0, \dots x_n].$$

A set $\mathcal{G} = \{g_1, \dots, g_s\} \subset I$ is a *Gröbner basis* for I with respect to $\boldsymbol{w} \in \mathbb{R}^{n+1}$ if

$$\mathrm{in}_{\boldsymbol{w}}(I) = \langle\, \mathrm{in}_{\boldsymbol{w}}(g_1), \dots \mathrm{in}_{\boldsymbol{w}}(g_s) \,\rangle.$$

Example 5.2. Let $K = \mathbb{Q}$ with the 2-adic valuation defined in Example 2.12 (2), where $\Bbbk = \mathbb{Z}/2\mathbb{Z}$. Consider the homogeneous ideal

$$I = \langle f, g \rangle = \langle x_0 + 2x_1,\ 3x_1 - 4x_2 \rangle.$$

Then, for $\boldsymbol{w} = (0, 0, 0)$, $\mathrm{in}_{\boldsymbol{w}}(I) = \langle x_0, x_1 \rangle$ and a Gröbner basis for I is $\mathcal{G} = \{f, g\}$. Indeed, the tropicalisation of f with respect to $\boldsymbol{w}$ is given by

$$\mathrm{trop}(f)(\boldsymbol{w}) = \min\{0 + \boldsymbol{w} \cdot (1, 0, 0), 1 + \boldsymbol{w} \cdot (0, 1, 0)\} = 0,$$

and so the initial form of f with respect to $\boldsymbol{w}$ is given by

$$\mathrm{in}_{\boldsymbol{w}}(f) = x_0.$$

Similarly, $\mathrm{trop}(g)(\boldsymbol{w}) = 0$ and so $\mathrm{in}_{\boldsymbol{w}}(g) = x_1$.

The Gröbner basis in this example was computed using Algorithm 2.9. in [CM19]. However, the details of this computation are omitted since we are more interested in the theory behind Gröbner bases than the computations.

This next Lemma shows that we can pick a Gröbner basis for any homogeneous ideal.

Lemma 5.3. *Consider the homogeneous ideal $I \subset K[x_0, \dots, x_n]$ and fix $\boldsymbol{w} \in \mathbb{R}^{n+1}$. Then $\mathrm{in}_{\boldsymbol{w}}(I)$ is homogeneous and we may choose a homogeneous Gröbner basis for I.*

Proof. Let $f = f_0 + \cdots + f_m$ be the homogeneous decomposition of some $f \in I$. Note that the initial form $\mathrm{in}_{\boldsymbol{w}}(f)$ is the sum of the initial forms $\mathrm{in}_{\boldsymbol{w}}(f_i)$ satisfying $\mathrm{trop}(f)(\boldsymbol{w}) = \mathrm{trop}(f_i)(\boldsymbol{w})$. Since I is homogeneous, each f_i lies in I, which implies $\mathrm{in}_{\boldsymbol{w}}(I)$ is generated by polynomials $\mathrm{in}_{\boldsymbol{w}}(g)$ such that g is homogeneous. The polynomial ring is Noetherian so $\mathrm{in}_{\boldsymbol{w}}(I)$ is generated by a finite set of these initial forms, say $\{\mathrm{in}_{\boldsymbol{w}}(g_1), \dots, \mathrm{in}_{\boldsymbol{w}}(g_s)\}$. The initial form of a homogeneous polynomial is homogeneous, therefore $\mathrm{in}_{\boldsymbol{w}}(I)$ is homogeneous and the set $\{g_1, \dots, g_s\}$ is a homogeneous Gröbner basis for I. □

We now give a useful property of initial ideals.

Lemma 5.4. *Consider a homogeneous ideal $I \subset K[x_0, \dots, x_n]$ and fix $\boldsymbol{w} \in \mathbb{R}^{n+1}$. If $g \in \mathrm{in}_{\boldsymbol{w}}(I)$, then $g = \mathrm{in}_{\boldsymbol{w}}(f)$ for some $f \in I$.*

Proof. Let $g = \sum_{\boldsymbol{u}\in\mathbb{N}^{n+1}} a_{\boldsymbol{u}} x^{\boldsymbol{u}} \operatorname{in}_{\boldsymbol{w}}(f_{\boldsymbol{u}}) \in \operatorname{in}_{\boldsymbol{w}}(I)$, where $a_{\boldsymbol{u}} \in \Bbbk^\times$ and $f_{\boldsymbol{u}} \in I$ for all $\boldsymbol{u}$. Recall the definition of the valuation ring,

$$R = \{c \in K \mid \operatorname{val}(c) \geq 0\}.$$

Pick $c_{\boldsymbol{u}} \in R$ for each $\boldsymbol{u}$ such that $\operatorname{val}(c_{\boldsymbol{u}}) = 0$ and $\overline{c_{\boldsymbol{u}}} = a_{\boldsymbol{u}}$. Now, let $W_{\boldsymbol{u}} = \operatorname{trop}(f_{\boldsymbol{u}})(\boldsymbol{w}) + \boldsymbol{w}\cdot\boldsymbol{u}$ for all $\boldsymbol{u}$ and define the polynomial

$$f = \sum_{\boldsymbol{u}} c_{\boldsymbol{u}} t^{-W_{\boldsymbol{u}}} x^{\boldsymbol{u}} f_{\boldsymbol{u}} \in I.$$

Then, it follows from Lemma 3.10 that the tropicalisation of f with respect to $\boldsymbol{w}$ is given by

$$\operatorname{trop}(f)(\boldsymbol{w}) = \min_{\boldsymbol{u}} \left\{\operatorname{val}\left(c_{\boldsymbol{u}} t^{-W_{\boldsymbol{u}}}\right) + \boldsymbol{w}\cdot\boldsymbol{u} + \operatorname{trop}(f_{\boldsymbol{u}})(\boldsymbol{w})\right\}.$$

However,

$$\operatorname{val}(c_{\boldsymbol{u}} t^{-W_{\boldsymbol{u}}}) = \operatorname{val}(c_{\boldsymbol{u}}) + \operatorname{val}(t^{-W_{\boldsymbol{u}}}) = -W_{\boldsymbol{u}} = -\operatorname{trop}(f_{\boldsymbol{u}})(\boldsymbol{w}) - \boldsymbol{w}\cdot\boldsymbol{u},$$

so we have $\operatorname{trop}(f)(\boldsymbol{w}) = 0$, achieved by every term in f. Furthermore,

$$\overline{c_{\boldsymbol{u}} t^{-W_{\boldsymbol{u}}} \cdot t^{-\operatorname{val}(c_{\boldsymbol{u}} t^{-W_{\boldsymbol{u}}})}} = \overline{c_{\boldsymbol{u}} t^{-W_{\boldsymbol{u}}} \cdot t^{W_{\boldsymbol{u}}}} = \overline{c_{\boldsymbol{u}}} = a_{\boldsymbol{u}}.$$

Thus, it follows that

$$\operatorname{in}_{\boldsymbol{w}}(f) = \sum_{\boldsymbol{u}} a_{\boldsymbol{u}} x^{\boldsymbol{u}} \operatorname{in}_{\boldsymbol{w}}(f_{\boldsymbol{u}}) = g.$$

□

We now wish to construct a polyhedral complex from a homogeneous ideal $I \subset K[x_0, \dots, x_n]$. We define the polyhedra in our complex by the Euclidean closure of

$$C_I[\boldsymbol{w}] := \{\boldsymbol{w}' \in \mathbb{R}^{n+1} \mid \operatorname{in}_{\boldsymbol{w}'}(I) = \operatorname{in}_{\boldsymbol{w}}(I)\},$$

for a fixed $\boldsymbol{w} \in \mathbb{R}^{n+1}$.

Definition 5.5. Let $I \subset K[x_0, \dots, x_n]$ be a homogeneous ideal. The *Gröbner complex* of I, $\Sigma(I)$, is the polyhedral complex with polyhedra $\overline{C_I[\boldsymbol{w}]}$ as $\boldsymbol{w}$ varies over $\mathbb{R}^{n+1}$.

Example 5.6. Let $n = 2$ and $K = \mathbb{Q}$ with the 2-adic valuation. Consider the principal ideal

$$I = \langle f \rangle = \langle 2x_0x_1x_2 + x_0^2x_1 + 4x_0^2x_2 + 4x_0^3\rangle.$$

Fix $\boldsymbol{w} = (0, 0, 0)$. The tropicalisation of f with respect to $\boldsymbol{w}$ is given by

$$\operatorname{trop}(f)(\boldsymbol{w}) = \min\{x_0 + x_1 + x_2 + 1, 2x_0 + x_1, 2x_0 + x_2 + 2, 3x_0 + 2\},$$

and so the initial ideal of I with respect to $\boldsymbol{w}$ is given by $\operatorname{in}_{\boldsymbol{w}}(I) = \langle x_0^2x_1\rangle$. Then,

$$\overline{C_I[\boldsymbol{w}]} = \{(v_0, v_1, v_2) \in \mathbb{R}^3 \mid 2v_0 + v_1 \leq \min\{v_0 + v_1 + v_2 + 1, 2v_0 + v_2 + 2, 3v_0 + 2\}\}.$$

Remark 5.7. The Gröbner complex $\Sigma(I)$ is a Γ_{val}-rational polyhedral complex in the quotient space $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$, where $\mathbf{1} = (1, \dots, 1)$ is the all-one vector $\mathbb{R}^{n+1}$. Elements of $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ can be uniquely written as $(0, w_1, \dots, w_n)$. See [MS15, §2.5] for a more in-depth look into Gröbner complexes. In the special case where $I = \langle f \rangle$ is a principal ideal, the Gröbner complex $\Sigma(I)$ in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$ corresponds to the polyhedral complex $\Sigma_{\operatorname{trop}(f)}$ in $\mathbb{R}^2$, from Definition 4.19.

Example 5.8. We return to the principal ideal from Example 5.6. By Remark 5.7, the Gröbner complex of I indicates the regions of linearity of $\operatorname{trop}(f)$. This is pictured in Figure 20.

The ideal has 11 distinct initial ideals, each of them corresponding to a cell in the Gröbner complex of I. There are four cells of dimension 2, five cells of dimension 1, and two cells of dimension 0. Each of these are labelled in Figure 20. Table 1 lists the initial ideals corresponding to each of the labelled cells in the complex, which we obtain by using the same method as in Example 5.6.

Notice that the full dimensional cells, and so the regions where $\operatorname{trop}(f)$ is linear, correspond to the initial ideals whose generators are monomials.

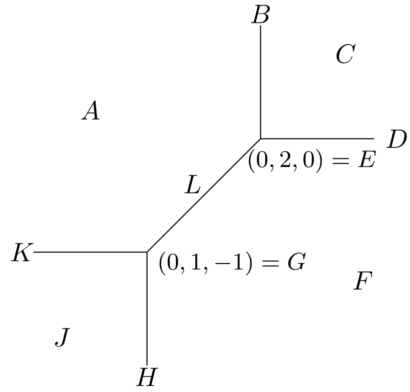

Figure 20: The Gröbner complex of I in $\mathbb{R}^{n+1}/\mathbb{R}\mathbf{1}$.

Table 1: The distinct initial ideals of I.

Cell	Initial Ideal
A	$\langle x_0^2x_1 \rangle$
B	$\langle x_0^2x_1 + 4x_0^3 \rangle$
C	$\langle 4x_0^3 \rangle$
D	$\langle 4x_0^2x_2 + 4x_0^3 \rangle$
E	$\langle x_0^2x_1 + 4x_0^2x_2 + 4x_0^3 \rangle$
F	$\langle 4x_0^2x_2 \rangle$

Cell	Initial Ideal
G	$\langle 2x_0x_1x_2 + x_0^2x_1 + 4x_0^2x_2 \rangle$
H	$\langle 2x_0x_1x_2 + 4x_0^2x_2 \rangle$
J	$\langle 2x_0x_1x_2 \rangle$
K	$\langle 2x_0x_1x_2 + x_0^2x_1 \rangle$
L	$\langle x_0^2x_1 + 4x_0^2x_2 \rangle$

We now have the tools we need to extend the notion of initial ideals to the Laurent polynomial ring. The initial work over the polynomial ring was necessary since for generic choices of $\boldsymbol{w} \in \mathbb{R}^n$, the initial form $\mathrm{in}_{\boldsymbol{w}}(f)$ is a monomial, and thus a unit in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$. This means that the initial ideal $\mathrm{in}_{\boldsymbol{w}}(I)$ is equal to the whole ring itself. However, as seen in the Fundamental Theorem, tropical geometry is concerned with the cases where $\mathrm{in}_{\boldsymbol{w}}(I)$ is a proper ideal, so we must do a bit more work to ensure this is the case.

To find the homogenisation of a Laurent ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$, we need to find the projective closure of $\mathbb{V}_t(I)$. But before that, we need to find the *affine closure* of $\mathbb{V}_t(I)$. This is given by $\mathbb{V}_a(I_{\text{aff}})$, where $I_{\text{aff}} = I \cap K[x_1, \dots, x_n]$, and is proven in [MS15, Proposition 2.2.5]. So then the homogenisation of I is the homogenisation of I_{aff}. Recall from [Cra19], the homogenisation of a polynomial $f \in K[x_1, \dots, x_n]$ is given by

$$\overline{f} = x_0^m \cdot f\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \in K[x_0, x_1, \dots, x_n],$$

where m is the total degree of f.

Definition 5.9. Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent ideal. The *homogenisation* of I is given by

$$\overline{I} = \langle\, \overline{f} \mid f \in I_{\text{aff}} \,\rangle,$$

which is an ideal in $K[x_0, x_1, \dots, x_n]$.

The following result explains how to construct the initial ideal of a Laurent ideal $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ from the initial ideal of its homogenisation, $\overline{I}$.

Proposition 5.10. *Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent ideal and fix $\boldsymbol{w} \in \mathbb{R}^n$. Then $\mathrm{in}_{\boldsymbol{w}}(I)$ is the image of $\mathrm{in}_{(0,\boldsymbol{w})}(\overline{I})$ in $\Bbbk[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ obtained by setting $x_0 = 1$. In particular, every element of $\mathrm{in}_{\boldsymbol{w}}(I)$ is of the form $x^{\boldsymbol{u}}g$ where $g = f(1, x_1, \dots, x_n)$ for some $f \in \mathrm{in}_{(0,\boldsymbol{w})}(\overline{I})$ and Laurent monomial $x^{\boldsymbol{u}}$.*

Proof. Consider a polynomial in I_{aff}, say $f = \sum_{\boldsymbol{u}} c_{\boldsymbol{u}} x^{\boldsymbol{u}}$, and its homogenisation

$$\overline{f} = \sum_{\boldsymbol{u}} c_{\boldsymbol{u}} x^{\boldsymbol{u}} x_0^{m_{\boldsymbol{u}}},$$

where $m_{\boldsymbol{u}} = \max_{\boldsymbol{v}}\{|\boldsymbol{v}| \mid c_{\boldsymbol{v}} \neq 0\} - |\boldsymbol{u}|$. Note that $|\boldsymbol{u}| := \sum_{i=1}^n u_i$ for all $\boldsymbol{u} \in \mathbb{R}^n$. Now, define

$$W := \operatorname{trop}(f)(\boldsymbol{w}) = \min\{\operatorname{val}(c_{\boldsymbol{u}}) + \boldsymbol{w} \cdot \boldsymbol{u}\} = \min\{\operatorname{val}(c_{\boldsymbol{u}}) + (0, \boldsymbol{w}) \cdot (m_{\boldsymbol{u}}, \boldsymbol{u})\} = \operatorname{trop}(\overline{f})((0, \boldsymbol{w})).$$

So by the definition of initial forms,

$$\begin{aligned} \operatorname{in}_{(0,\boldsymbol{w})}(\overline{f})\Big|_{x_0=1} &= \sum_{\operatorname{val}(c_{\boldsymbol{u}}) + \boldsymbol{w}\cdot\boldsymbol{u} = W} \overline{c_{\boldsymbol{u}} t^{-\operatorname{val}(c_{\boldsymbol{u}})}} x^{\boldsymbol{u}} x_0^{m_{\boldsymbol{u}}}\Big|_{x_0=1} \\ &= \sum_{\operatorname{val}(c_{\boldsymbol{u}}) + \boldsymbol{w}\cdot\boldsymbol{u} = W} \overline{c_{\boldsymbol{u}} t^{-\operatorname{val}(c_{\boldsymbol{u}})}} x^{\boldsymbol{u}} \\ &= \operatorname{in}_{\boldsymbol{w}}(f). \end{aligned}$$

Choose polynomials $f_1, \ldots, f_s \in I_{\text{aff}}$ such that $\operatorname{in}_{\boldsymbol{w}}(I) = \langle \operatorname{in}_{\boldsymbol{w}}(f_1), \ldots, \operatorname{in}_{\boldsymbol{w}}(f_s) \rangle$ (possibly after multiplying by Laurent monomials if necessary). By the above, $\operatorname{in}_{\boldsymbol{w}}(f_i) = \operatorname{in}_{(0,\boldsymbol{w})}(\overline{f_i})\big|_{x_0=1}$, and thus, $\operatorname{in}_{\boldsymbol{w}}(I) \subseteq \operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})\big|_{x_0=1}$.

For the opposite inclusion, by Lemma 5.3, we can choose a homogeneous Gröbner basis for $\overline{I}$ with respect to $(0, \boldsymbol{w})$, say $\{g_1, \ldots, g_s\}$. Each generator is of the form $g_i = x_0^m \cdot \overline{f_i}$, where $f_i(x) := g_i(1, x)$ and $m \geq 0$. By the above, $\operatorname{in}_{(0,\boldsymbol{w})}(g_i)\big|_{x_0=1} = \operatorname{in}_{\boldsymbol{w}}(f_i)$ for each $i \in \{1, \ldots, s\}$. Every element of an ideal $J \subset \Bbbk[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ is a Laurent monomial multiplied by an element of $J \cap \Bbbk[x_1, \ldots, x_n]$. Thus, $\operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})\big|_{x_0=1} \subseteq \operatorname{in}_{\boldsymbol{w}}(I)$. □

The following result extends Lemma 5.4 to the Laurent polynomial ring.

Corollary 5.11. *Let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent ideal and fix $\boldsymbol{w} \in \mathbb{R}^n$. If $g \in \operatorname{in}_{\boldsymbol{w}}(I)$, then $g = \operatorname{in}_{\boldsymbol{w}}(f)$ for some $f \in I$.*

Proof. Let $g \in \operatorname{in}_{\boldsymbol{w}}(I)$. By Proposition 5.10, we can write $g = x^{\boldsymbol{u}} \cdot h(1, x_1, \ldots, x_n)$ for some $h \in \operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})$ and some Laurent monomial $x^{\boldsymbol{u}}$. So by Lemma 5.4, we can write $h = \operatorname{in}_{(0,\boldsymbol{w})}(f)$ for some $f \in \overline{I}$. Thus, $x^{\boldsymbol{u}} f\big|_{x_0=1} \in I$ and

$$\operatorname{in}_{\boldsymbol{w}}(x^{\boldsymbol{u}} f\big|_{x_0=1}) = x^{\boldsymbol{u}} \cdot \operatorname{in}_{\boldsymbol{w}}(f\big|_{x_0=1}) = x^{\boldsymbol{u}} \cdot \operatorname{in}_{(0,\boldsymbol{w})}(f)\big|_{x_0=1} = x^{\boldsymbol{u}} \cdot h(1, x_1, \ldots, x_n) = g,$$

as required. □

To close out this subsection, we explore the tropicalisation of a monomial map $\phi : T^n \longrightarrow T^m$. This is a polynomial map where the polynomial functions correspond to monomials in $K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. So, where $\phi^* : K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \longrightarrow K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is the pullback map, we have

$$\phi^*(x_i) = z^{\boldsymbol{a}_i},$$

for all $i \in \{1, \ldots, m\}$ and where $\boldsymbol{a}_i \in \mathbb{Z}^n$. Note that in the last part of this subsection, the vectors we write are column vectors.

Definition 5.12. Let $\phi : T^n \longrightarrow T^m$ be a monomial map and $\phi^* : K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \longrightarrow K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be the pullback as defined above. The *tropicalisation* of ϕ is given by the linear map

$$\operatorname{trop}(\phi) : \mathbb{Z}^n \longrightarrow \mathbb{Z}^m; \quad \boldsymbol{w} \longmapsto A^T \boldsymbol{w},$$

where A is an $n \times m$-matrix with ith column given by $\boldsymbol{a}_i$.

Remarks 5.13. (1) If $I \subset K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ is a Laurent ideal with algebraic set $X = \mathbb{V}_t(I)$ in T^n, then the Zariski closure of the image $\phi(X)$ in T^m is equal to the algebraic set $\mathbb{V}_t({\phi^*}^{-1}(I))$.

(2) The image of the restriction of $\text{trop}(\phi)$ to Γ_{val}^n is contained in Γ_{val}^m. Indeed, let $\boldsymbol{w} \in \Gamma_{\text{val}}^n$ and so there exists some $\boldsymbol{y} \in T^n$ such that $\text{val}(\boldsymbol{y}) = \boldsymbol{w}$. Then,

$$\begin{aligned}\text{trop}(\phi)(\boldsymbol{w}) &= \text{trop}(\phi)(\text{val}(\boldsymbol{y})) \\ &= A^T \text{val}(\boldsymbol{y}) \\ &= (\boldsymbol{a}_1 \cdot \text{val}(\boldsymbol{y}), \ldots, \boldsymbol{a}_m \cdot \text{val}(\boldsymbol{y})) \\ &= (\text{val}(\boldsymbol{y}^{\boldsymbol{a}_1}), \ldots, \text{val}(\boldsymbol{y}^{\boldsymbol{a}_m})) \\ &= \text{val}(\phi(\boldsymbol{y})).\end{aligned}$$

Example 5.14. Let $K = \mathbb{C}\{\{t\}\}$ with natural valuation and let $\phi : T^3 \longrightarrow T^3$ be the monomial map defined by

$$\phi : (t_1, t_2, t_3)^T \longmapsto (t_2t_3, t_1t_3, t_1t_2)^T.$$

We define the pullback $\phi^* : K[x_1^{\pm 1}, x_2^{\pm 1}, x_3^{\pm 1}] \longrightarrow K[z_1^{\pm 1}, z_2^{\pm 1}, z_3^{\pm 1}]$ by

$$\phi^*(x_1) = z_2z_3 = z^{(0,1,1)^T}, \quad \phi^*(x_2) = z_1z_3 = z^{(1,0,1)^T}, \quad \phi^*(x_3) = z_1z_2 = z^{(1,1,0)^T}.$$

So $\text{trop}(\phi)$ is given by the symmetric matrix

$$A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

Now consider $\boldsymbol{y} = (t, t^2, t^3)^T$. Then $\text{val}(\boldsymbol{y}) = (1, 2, 3)^T$. Also, $\phi(\boldsymbol{y}) = (t^5, t^4, t^3)^T$ and so $\text{val}(\phi(\boldsymbol{y})) = (5, 4, 3)^T$. Now,

$$A \, \text{val}(\boldsymbol{y}) = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \\ 3 \end{pmatrix}$$

Thus, we've verified that $\text{trop}(\phi) \, \text{val}(\boldsymbol{y}) = \text{val}(\phi(\boldsymbol{y}))$.

The next result is important in the proof of the Fundamental Theorem.

Lemma 5.15. *Let $\phi : T^n \longrightarrow T^m$ be a monomial map with pullback $\phi^* : K[x_1^{\pm 1}, \ldots, x_m^{\pm 1}] \longrightarrow K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$, as previously defined. Let $I \subseteq K[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]$ be a Laurent ideal and define $I' := {\phi^*}^{-1}(I)$. Then,*

$$\phi^*(\text{in}_{\text{trop}(\phi)(\boldsymbol{w})}(I')) \subseteq \text{in}_{\boldsymbol{w}}(I),$$

for all $\boldsymbol{w} \in \mathbb{R}^n$. In particular, if $\text{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle$, then $\text{in}_{\text{trop}(\phi)(\boldsymbol{w})}(I') \neq \langle 1 \rangle$.

Proof. Let A be the $n \times m$-matrix with ith column given by $\boldsymbol{a}_i$. Then, for all $\boldsymbol{u} = (u_1, \ldots, u_m) \in \mathbb{Z}^n$, we have

$$\begin{aligned}\phi^*(x^{\boldsymbol{u}}) &= \phi^*(x_1^{u_1} \ldots x_m^{u_m}) \\ &= z^{\boldsymbol{a}_1 u_1} \ldots z^{\boldsymbol{a}_m u_m} \\ &= z^{A\boldsymbol{u}},\end{aligned}$$

where we've used the fact that the pullback is a ring homomorphism. Now let $f = \sum c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in I'$, so $\phi^*(f) = \sum c_{\boldsymbol{u}} z^{A\boldsymbol{u}} \in I$. We see that

$$\begin{aligned}W := \text{trop}(f)(A^T\boldsymbol{w}) &= \min_{\boldsymbol{u} \mid c_{\boldsymbol{u}} \neq 0} \{\text{val}(c_{\boldsymbol{u}}) + A^T\boldsymbol{w} \cdot \boldsymbol{u}\} \\ &= \min_{\boldsymbol{u} \mid c_{\boldsymbol{u}} \neq 0} \{\text{val}(c_{\boldsymbol{u}}) + \boldsymbol{w} \cdot A\boldsymbol{u}\} \\ &= \text{trop}(\phi^*(f))(\boldsymbol{w}).\end{aligned}$$

So,

$$\begin{aligned}\phi^*(\mathrm{in}_{\mathrm{trop}(\phi(\boldsymbol{w}))}(f)) &= \phi^*\left(\sum_{\mathrm{val}(c_{\boldsymbol{u}})+\boldsymbol{w}\cdot A\boldsymbol{u}=W} \overline{t^{-\mathrm{val}(c_{\boldsymbol{u}})}c_{\boldsymbol{u}}}x^{\boldsymbol{u}}\right)\\ &= \sum_{\mathrm{val}(c_{\boldsymbol{u}})+\boldsymbol{w}\cdot A\boldsymbol{u}=W} \overline{t^{-\mathrm{val}(c_{\boldsymbol{u}})}c_{\boldsymbol{u}}}z^{A\boldsymbol{u}}\\ &= \mathrm{in}_{\boldsymbol{w}}(\phi^*(f)).\end{aligned}$$

So we must have $\phi^*(\mathrm{in}_{\mathrm{trop}(\phi)(\boldsymbol{w})}(I') \subseteq \mathrm{in}_{\boldsymbol{w}}(I)$. Furthermore, if $\mathrm{in}_{\mathrm{trop}(\phi)(\boldsymbol{w})}(I') = \langle 1\rangle$, then $\mathrm{in}_{\boldsymbol{w}}(I)$ contains $\phi^*(1) = 1$, and thus $\mathrm{in}_{\boldsymbol{w}}(I) = \langle 1\rangle$.

□

Example 5.16. We return to the monomial map defined in Example 5.14. Consider the ideal

$$I = \langle z_1 + z_2\rangle.$$

Then, since $\phi^*(x_1 + x_2) = x_3(x_1 + x_2)$, we have $I' = {\phi^*}^{-1}(I) = \langle x_1 + x_2\rangle$. Let $\boldsymbol{w} = (1, 0, 0)^T$. Then the tropicalisation of the monomial map applied to $\boldsymbol{w}$ is given by

$$\mathrm{trop}(\phi)(\boldsymbol{w}) = \begin{pmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{pmatrix}\begin{pmatrix}1\\0\\0\end{pmatrix} = \begin{pmatrix}0\\1\\1\end{pmatrix}.$$

So the initial ideal of I' with respect to $\mathrm{trop}(\phi)(\boldsymbol{w})$ is given by $\mathrm{in}_{\mathrm{trop}(\phi)(\boldsymbol{w})}(I') = \mathrm{in}_{(0,1,1)^T}(I') = \langle x_1\rangle$. Applying the pullback, we get

$$\phi^*(\mathrm{in}_{\mathrm{trop}(\phi)(\boldsymbol{w})}(I')) = \phi^*(\langle x_1\rangle) = \langle z_2 z_3\rangle.$$

The initial form of the generator of I with respect to $\boldsymbol{w}$ is given by $\mathrm{in}_{\boldsymbol{w}}(z_1 + z_2) = z_2$, which implies $\mathrm{in}_{\boldsymbol{w}}(I) = \langle z_2\rangle$. We note that $\langle z_2 z_3\rangle$ is contained in $\langle z_2\rangle$, and so we've verified Lemma 5.15 in this case.

5.2 The Fundamental Theorem

In this subsection, we state and prove the Fundamental Theorem of Tropical Geometry. To do this, we first have to define what we mean by a *tropical algebraic set.*

Definition 5.17 (Tropical Algebraic Set). Let $I \subset K[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ be a Laurent ideal and $X = \mathbb{V}_t(I)$ be its algebraic set in the algebraic torus T^n. The *tropicalisation* of X, $\mathrm{trop}(X)$, is the intersection of all tropical hypersurfaces defined by elements in I, so

$$\mathrm{trop}(X) = \bigcap_{f\in I} \mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f)) \subseteq \mathbb{R}^n.$$

We call $\mathrm{trop}(X)$ a *tropical algebraic set.*

Example 5.18. Tropicalisations of algebraic sets and intersections don't commute. For example, let $K = \mathbb{Q}$ with the 2-adic valuation, and consider the ideal

$$I = \langle 2x + y + 1, 6x - y\rangle.$$

Then $X := \mathbb{V}_t(I) = \left\{\left(-\frac{1}{8}, -\frac{3}{4}\right)\right\}$ and thus $\mathrm{trop}(X) = \{(-3, -2)\}$. However, when looking at the intersection of the tropical hypersurfaces of the generators, we see

$$\mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(2x + y + 1)) \cap \mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(6x - y)) = \{(x, y) \in \mathbb{R}^2 \mid y = x + 1 \leq 0\},$$

which is shown in Figure 21. So the intersection of the tropical hypersurfaces of the generators does not equal the tropicalisation of the algebraic set.

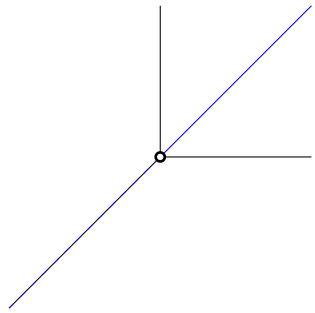

Figure 21: The intersection of two tropical hypersurfaces.

We now state and prove the Fundamental Theorem of Tropical Geometry. This theorem allows us to link tropical algebraic sets to classical algebraic sets, and view them from a polyhedral perspective.

Theorem 5.19 (**The Fundamental Theorem of Tropical Geometry**). *Let (K, val) be an algebraically closed valued field, let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent ideal and $X = \mathbb{V}_t(I)$ be its algebraic set in the algebraic torus T^n. Then the following three sets are equal:*

(1) the tropical algebraic set $\mathrm{trop}(X)$ in $\mathbb{R}^n$;

(2) the set $\{\boldsymbol{w} \in \mathbb{R}^n \mid \mathrm{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle\}$;

(3) the closure of $\mathrm{val}(X) = \{(\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n)) \mid (y_1, \ldots, y_n) \in X\}$ in the Euclidean topology on $\mathbb{R}^n$.

Proof. We denote the set from (1) in the statement of the Fundamental Theorem as set (1), and similarly for (2) and (3). We prove the Fundamental Theorem by showing that set (1) contains set (3), set (2) contains set (1) and set (3) contains set (2).

To show set (1) contains set (3), first note that since $\mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f))$ is closed in the Euclidean topology on $\mathbb{R}^n$ for each $f \in I$, and that the intersection of any collection of closed sets is closed, we have that $\mathrm{trop}(X)$ is closed. So we only need to consider points of the form $\mathrm{val}(\boldsymbol{y}) = (\mathrm{val}(y_1), \ldots, \mathrm{val}(y_n))$ in set (3) for some $\boldsymbol{y} \in X$. If $\boldsymbol{y} \in X$, then $\boldsymbol{y} \in \mathbb{V}_t(f)$ for all $f \in I$. By Theorem 3.11, $\mathrm{val}(\boldsymbol{y}) \in \mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f))$ for all $f \in I$. So,

$$\mathrm{val}(\boldsymbol{y}) \in \bigcap_{f \in I} \mathbb{V}_{\mathrm{trop}}(\mathrm{trop}(f)) = \mathrm{trop}(X).$$

Thus, we have $(3) \subseteq (1)$.

To show set (2) contains set (1), let $\boldsymbol{w} \in \mathrm{trop}(X)$. For any $f = \sum_{\boldsymbol{u}} c_{\boldsymbol{u}} x^{\boldsymbol{u}} \in I$, the minimum in $\mathrm{trop}(f)(\boldsymbol{w})$ is achieved at least twice. So $\mathrm{in}_{\boldsymbol{w}}(f)$ is not a monomial for all $f \in I$. Suppose, for contradiction, that $1 \in \mathrm{in}_{\boldsymbol{w}}(I)$. By Corollary 5.11, there exists some $g \in I$ such that $\mathrm{in}_{\boldsymbol{w}}(g) = 1$. But this contradicts $\mathrm{in}_{\boldsymbol{w}}(g)$ not being a monomial. So we must have that $1 \notin \mathrm{in}_{\boldsymbol{w}}(I)$, which means $\mathrm{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle$. Thus, $\boldsymbol{w}$ lies in set (2) and so $(1) \subseteq (2)$.

To show set (3) contains set (2), we first show that we can reduce to the case where I is prime. Fix $\boldsymbol{w} \in \mathbb{R}^n$. Since $I \subseteq \sqrt{I}$, it follows that $\mathrm{in}_{\boldsymbol{w}}(I) \subseteq \mathrm{in}_{\boldsymbol{w}}(\sqrt{I})$, and in particular $\mathrm{in}_{\boldsymbol{w}}(I) = \langle 1 \rangle$ implies $\mathrm{in}_{\boldsymbol{w}}(\sqrt{I}) = \langle 1 \rangle$. If $\mathrm{in}_{\boldsymbol{w}}(\sqrt{I}) = \langle 1 \rangle$, then by Corollary 5.11 there exists some $f \in \sqrt{I}$ such that $\mathrm{in}_{\boldsymbol{w}}(f) = 1$. By the definition of a radical ideal, there exists some $m \in \mathbb{N}$ such that $f^m \in I$. But by Lemma 3.8,

$$\mathrm{in}_{\boldsymbol{w}}(f^m) = \mathrm{in}_{\boldsymbol{w}}(f)^m = 1^m = 1,$$

so $\mathrm{in}_{\boldsymbol{w}}(I) = \langle 1 \rangle$. So, $\mathrm{in}_{\boldsymbol{w}}(I) = \langle 1 \rangle$ if and only if $\mathrm{in}_{\boldsymbol{w}}(\sqrt{I}) = \langle 1 \rangle$, which means we can reduce to the case where I is radical. By Remark 3.2 (2), we can use the affine $\mathbb{V} - \mathbb{I}$ correspondence to write I as the finite intersection of prime ideals,

$$I = \bigcap_{i=1}^{s} P_i,$$

where each $\mathbb{V}_t(P_i)$ is an irreducible component of $X = \mathbb{V}_t(I)$. We now claim that if $\text{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle$, then there exists some $j \in \{1, \ldots, s\}$ such that $\text{in}_{\boldsymbol{w}}(P_j) \neq \langle 1 \rangle$. Indeed, suppose for contradiction that there is no such $j \in \{1, \ldots, s\}$. By Corollary 5.11 there exists some $f_i \in P_i$ such that $\text{in}_{\boldsymbol{w}}(f_i) = 1$ for all $i \in \{1, \ldots, s\}$. Then, defining the polynomial $f = \Pi_{i=1}^{s} f_i \in I$, Lemma 3.8 implies $\text{in}_{\boldsymbol{w}}(f) = \Pi_{i=1}^{s} \text{in}_{\boldsymbol{w}}(f_i) = 1$, which means $\text{in}_{\boldsymbol{w}}(I) = \langle 1 \rangle$. This is the required contradiction.

So we have shown that if $\boldsymbol{w} \in \mathbb{R}^n$ lies in set (2) for X, then it must lie in set (2) for some irreducible component $\mathbb{V}_t(P_j)$ of X. This means it suffices to show that $\boldsymbol{w} = \text{val}(\boldsymbol{y})$ for some $\boldsymbol{y} \in \mathbb{V}_t(P_j)$ in order to prove $(2) \subseteq (3)$. Similar to the proof of Kapranov's Theorem, we prove this in a separate proposition. □

Proposition 5.20. *Let $X = \mathbb{V}_t(I)$ be a variety of T^n, with $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ a prime ideal. Fix $\boldsymbol{w} \in \Gamma_{\text{val}}^n$ with $\text{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle$ and let $\boldsymbol{\alpha} \in \mathbb{V}_t(\text{in}_{\boldsymbol{w}}(I)) \subset (\Bbbk^\times)^n$. Then there exists some $\boldsymbol{y} \in X$ such that $\text{val}(\boldsymbol{y}) = \boldsymbol{w}$ and $\overline{t^{-\boldsymbol{w}} \boldsymbol{y}} = \boldsymbol{\alpha}$.*

Proof. Let $d = \dim(X)$. The case $n = 1$ follows from the base case of Proposition 3.14 and the case where $d = n - 1$ corresponds to when X is a hypersurface, which is Proposition 3.14. So we assume $0 \leq d \leq n - 2$, and use induction on n.

As shown by Maclagan and Sturmfels in [MS15, Proposition 3.2.11], it is possible to choose a monomial projection $\phi : T^n \longrightarrow T^{n-1}$ such that $\phi(X)$ is closed, and that we can uniquely recover $\boldsymbol{w}$ from its image under $\text{trop}(\phi)$. Now, consider the ideal $I' = {\phi^*}^{-1}(I) = I \cap K[x_1^{\pm 1}, \ldots, x_{n-1}^{\pm 1}]$ and its corresponding variety $X' = \mathbb{V}_t(I')$. Since we assumed $\phi(X)$ is closed, this means that $X' = \mathbb{V}_t({\phi^*}^{-1}(I)) = \phi(X)$, by Remark 5.13 (1). Also, since $\text{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle$, Lemma 5.15 implies that $\text{in}_{\text{trop}(\phi)(\boldsymbol{w})}(I') \neq \langle 1 \rangle$. So we can use the inductive hypothesis to obtain some $\boldsymbol{y}' = (y_1, \ldots, y_{n-1}) \in X'$ such that $\text{val}(y_i) = w_i$ and $\overline{t^{-w_i} y_i} = \alpha_i$ for all $i \in \{1, \ldots, n-1\}$.

Now we wish to find a point $y_n \in K^\times$ such that the desired properties hold. We show that this is possible. Indeed, define the ideal $J = \langle f(y_1, \ldots, y_{n-1}, x_n) \mid f \in I \rangle$ which is an ideal in $K[x_n^{\pm 1}]$. But $K[x_n^{\pm 1}]$ is a principal ideal domain so there exists a polynomial $f \in I$ such that $J = \langle f(y_1, \ldots, y_{n-1}, x_n) \rangle$. We can write f in such a way to ensure that $J \neq \langle 1 \rangle$ and so $\mathbb{V}_t(J)$ is nonempty (see [MS15, Proposition 3.2.7]). So if we can find a point in $y_n \in K^\times$ such that $f(y_1, \ldots, y_n) = 0$, then we will have shown that $(y_1, \ldots, y_n) \in X$ such that the desired properties hold.

By [MS15, Proposition 3.2.7], we can write f as a polynomial in x_n with monomial coefficients, $f = \sum_i c_i x^{\boldsymbol{u}_i} x_n^i$, where $\boldsymbol{u}_i \in \mathbb{Z}^n$ for each $i \in \{1, \ldots, n\}$. Let

$$g = f(y_1, \ldots, y_{n-1}, x_n) = \sum_i c_{\boldsymbol{u}} y^{\boldsymbol{u}_i} x_n^i.$$

Then it follows from the proof of Proposition 3.14 that $\text{in}_{\boldsymbol{w}}(f)(\alpha_1, \ldots, \alpha_{n-1}, x_n) = \text{in}_{w_n}(g)(x_n)$ and so $\text{in}_{w_n}(g)(\alpha_n) = 0$, since $\boldsymbol{\alpha} \in \mathbb{V}_t(\text{in}_{\boldsymbol{w}}(I))$. The element α_n is a unit, which means that $\text{in}_{w_n}(g)$ is not a monomial. So by Proposition 3.14, there exists some $y_n \in K^\times$ such that $g(y_n) = 0$, $\text{val}(y_n) = w_n$ and $\overline{t^{-w_n} y_n} = \alpha_n$. Thus, it follows from the fact that we can uniquely recover $\boldsymbol{w}$ from $\text{trop}(\phi)$ that there exists some $\boldsymbol{y} = (y_1, \ldots, y_n) \in X$ such that $\text{val}(\boldsymbol{y}) = \boldsymbol{w}$ and $\overline{t^{-\boldsymbol{w}} \boldsymbol{y}} = \boldsymbol{\alpha}$, as required.

Therefore, set (3) contains set (2), and the proof of the Fundamental Theorem is complete. □

The polyhedral perspective of tropical algebraic sets comes from set (2) in the Fundamental Theorem. This set is the support of a polyhedral complex contained in the Gröbner complex $\Sigma(\overline{I})$.

Proposition 5.21. *Let $I \subset K[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$ be a Laurent ideal, and let $X = \mathbb{V}(I)$ be its algebraic set. Then the set $\{\boldsymbol{w} \in \mathbb{R}^n \mid \operatorname{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle\}$ is the support of a polyhedral complex contained in the Gröbner complex $\Sigma(\overline{I})$. Furthermore,* $\operatorname{trop}(X)$ *is the support of a Γ_{val}-rational polyhedral complex.*

Proof. By Remark 5.7, $\Sigma(\overline{I})$ is a Γ_{val}-rational polyhedral complex. By Proposition 5.10, we have $1 \in \operatorname{in}_{\boldsymbol{w}}(I)$ if and only if $1 \in \operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})|_{x_0=1}$. This occurs if and only if a monomial in $x_1, \ldots, x_n$ is contained in $\operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})|_{x_0=1}$, which in turn occurs if and only if a polynomial in x_0 multiplied by a monomial in $x_1, \ldots, x_n$ is contained in $\operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})$. Since $\operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})$ is homogeneous by Lemma 5.3, this occurs if and only if a monomial in $x_0, x_1, \ldots, x_n$ is contained in $\operatorname{in}_{(0,\boldsymbol{w})}(\overline{I})$. So we have

$$\{\boldsymbol{w} \in \mathbb{R}^n \mid \operatorname{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle\} = \{\boldsymbol{w} \in \mathbb{R}^n \mid \operatorname{in}_{(0,\boldsymbol{w})}(\overline{I}) \text{ does not contain a monomial } \}.$$

This is a union of all cells in the Gröbner complex $\Sigma(\overline{I})$ which aren't of full dimension and thus form a skeleton of $\Sigma(\overline{I})$. Therefore, $\{\boldsymbol{w} \in \mathbb{R}^n \mid \operatorname{in}_{\boldsymbol{w}}(I) \neq \langle 1 \rangle\}$ is the support of a polyhedral complex contained in $\Sigma(\overline{I})$. By Remark 5.7 and the Fundamental Theorem, Theorem 5.19, it follows that $\operatorname{trop}(X)$ is the support of a Γ_{val}-rational polyhedral complex.

□

References

[BS14] Erwan Brugallé and Kristin Shaw. A bit of tropical geometry. *Amer. Math. Monthly*, 121(7):563–589, 2014.

[CM19] Andrew J. Chan and Diane Maclagan. Gröbner bases over fields with valuations. *Math. Comp.*, 88(315):467–483, 2019.

[Cra19] Alastair Craw. Lecture notes in algebraic curves. University of Bath, 2019.

[EKL06] Manfred Einsiedler, Mikhail Kapranov, and Douglas Lind. Non-Archimedean amoebas and tropical varieties. *J. Reine Angew. Math.*, 601:139–157, 2006.

[MR18] Diane Maclagan and Felipe Rincón. Tropical ideals. *Compos. Math.*, 154(3):640–670, 2018.

[MS15] Diane Maclagan and Bernd Sturmfels. *Introduction to tropical geometry*, volume 161 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2015.

[Stu20] Bernd Sturmfels. Twelve lectures on tropical geometry, lecture 5 - fundamental theorem. Max Planck Institute for Mathematics in the Sciences, Leipzig, Germany, 2020. Available from: `https://www.youtube.com/watch?v=7EkQdaWKnaQ&t=3200s`.

[Zie95] Günter M. Ziegler. *Lectures on polytopes*, volume 152 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1995.