Linking Groebner Bases and Toric Varieties

Masters Project

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Abstract: This article describes a link between two apparently different constructions, namely, a Groebner fan, and the minimal resolution of a toric variety.

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5CM - March 2009

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1. Introduction

Many of the important results in Mathematics are obtained by finding links between apparently different mathematical objects. If we can establish a correspondence between two such objects, we can use the information we have about one of them to obtain new results about the other. In this project we will consider one such link, namely between Groebner fans and toric varieties, which will be defined in due course. We will also give a brief discussion of the G-Hilbert Scheme, and state that there is also a correspondence between the toric variety $\mathbb{C}^2/G$ and the G-Hilbert Scheme, which enables us to find a description of $\mathbb{C}^2/G$ more directly in terms of G.

We begin by a discussion of the Ideal Membership Problem, and the solution of it via Groebner bases. We will prove their existence, and give the algorithm to find them, known as Buchberger's algorithm. This discussion leads naturally to the consideration of Groebner fans, which we will construct, beginning with some basic geometric definitions, and then considering some examples. We will also consider the Groebner Walk, which is a useful algorithm to obtain the Groebner bases that are more difficult to obtain via Buchberger's algorithm.

In the second half of this project, we will introduce some toric geometry with the discussion of toric varieties, which we will illustrate with several examples. We then restrict attention to toric surfaces, and show that each affine subvariety of a toric variety is isomorphic to the cyclic quotient singularity of type $\frac{1}{r}(1, a)$. We then proceed to consider the minimal resolution of these singularities, and show that the resulting fan is isomorphic to the Groebner fan of the ideal I_G defined by the free G-orbit.

In the final section, we will conclude by giving an informal discussion of the G-Hilbert Scheme, considered as the set of all G-clusters, and will show that in our special case it carries the structure of a toric variety, which is isomorphic to the minimal resolution of $\mathbb{C}^2/G$. We will also illustrate this with an example.

2. Groebner Bases

2.1. **The Ideal Membership Problem.** We begin be recalling the following basic definitions, and the Hilbert basis theorem. Let $S := k[x_1, x_2, ...x_n]$ be a polynomial ring in n variables over a field k.

- A *polynomial ideal* I of S is a subset of S generated by polynomials, such that, $\forall f, g \in I$, and $h \in S$, $f - g, hf \in I$.
- A polynomial ideal I of S is said to be *finitely generated* over k, if $\exists f_1, f_2, ..., f_r \in I$, such that, $\forall f \in I, \exists h_1, ..., h_r \in S$, such that $f = \Sigma h_i f_i$.

Theorem 2.1. *(Hilbert's Basis Theorem). Every polynomial ideal is finitely generated.*

Proof. See [9], pages 48-49. □

Given such a basis $\{f_1, ..., f_r\}$ for I, we write $I = \langle f_1, ..., f_n \rangle$. We denote the monomial $x_1^{a_1}...x_n^{a_n}$ of S as $\mathbf{x}^{\mathbf{a}}$, where $\mathbf{a} = (a_1, ..., a_n) \in \mathbb{N}^n$.

For an ideal $I = \langle f_1, f_2, ..., f_n \rangle$ and a polynomial g in $k[x_1, ..., x_n]$, how can we determine whether or not $g \in I$? This is known as the *Ideal Membership Problem*. In this section we will show how Groebner bases can be used to solve it.

Principally generated ideals $\langle f \rangle$ are simple and easy to understand: their elements are all polynomials of the form fg, for $g \in k[x_1, ..., x_n]$. For all ideals $I \subset k[x]$, the problem takes this form, since $k[x]$ is a principal ideal domain. When considering ideals of a polynomial ring in more than 1 variable, ideals may have more than one generator, and whether or not a specific polynomial lies in the ideal can be more difficult to determine. We now have many different monomials with the same degree. In $k[x, y]$, there are 5 monomials of degree 4, namely x^4, x^3y, x^2y^2, xy^3, and y^4, as compared to only 1 in $k[x]$. This causes the ordinary division algorithm to fail. For example, taking $g = xy + 3y^2 - 1$, and $f = 2x$, applying the division algorithm gives

$$g = \frac{1}{2}yf + 3y^2 - 1.$$

But the degree of the remainder is the same as the degree of g. Hence the process will not necessarily terminate satisfactorily. To try and mimic the division algorithm in multi variable polynomial rings, we have to introduce the idea of monomial ordering.

Definition 2.2. A *monomial order* $\prec$ on a polynomial ring $S = k[x_1, ..., x_n]$ is a total order on monomials such that,

- For all monomials $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}, \mathbf{x}^{\mathbf{c}}$, if $\mathbf{x}^{\mathbf{a}} \prec \mathbf{x}^{\mathbf{b}}$ then $\mathbf{x}^{\mathbf{a}}\mathbf{x}^{\mathbf{c}} \prec \mathbf{x}^{\mathbf{b}}\mathbf{x}^{\mathbf{c}}$.
- Any arbitrary set of monomials $\{\mathbf{x}^{\mathbf{a}}\}_{\mathbf{a} \in A}$ has a least element.

The *leading monomial* $LM(f)$ of a polynomial f is defined to be the monomial of f with the highest order. The *leading term* $LT(f)$ is simply $LM(f)$ multiplied by its coefficient in f.

Being a total order implies that the orders of any 2 monomials are comparable, their orders are either equal, or one is bigger than the other. The 2nd condition above is equivalent to saying that, for any $\mathbf{a} \in \mathbb{R}^n_{\geq 0} \backslash \{0\}$, $\mathbf{x}^\mathbf{a} \prec \mathbf{x}^\mathbf{0} = 1$. This property of a monomial order is necessary to prove that our algorithm will terminate.

Example 2.3. A simple example of a term order in $k[x_1, ..., x_n]$ is *lexicographic order*, which is similar to the way that words are ordered in a dictionary. It is given by the rule

$$\mathbf{x}^\mathbf{a} \succ_{lex} \mathbf{x}^\mathbf{b}$$

if and only if the first non-zero term of $(a_1 - b_1, ..., a_n - b_n)$ is positive.

Another example is *graded reverse lexicographic order*, where $\mathbf{x}^\mathbf{a} \succ_{grelex} \mathbf{x}^\mathbf{b}$ if $deg(\mathbf{x}^\mathbf{a}) > deg(\mathbf{x}^\mathbf{b})$, or if $deg(\mathbf{x}^\mathbf{a}) = deg(\mathbf{x}^\mathbf{b})$ and the last non-zero term of $(a_1 - b_1, ..., a_n - b_n)$ is negative.

We now have a method of ordering all the monomials in $k[x_1, ..., x_n]$. The following algorithm is a generalisation of the ordinary division algorithm, which at each step gives a remainder whose leading term has lower order than the polynomial we started with.

Algorithm 2.4 (Naive Algorithm)**.** *Given a polynomial g, and an ideal $I = \langle f_1, ...f_r \rangle$, we set $g = g_0$. If there does not exist an f_i with $LM(f_i)|LM(g_0)$, then we stop. Otherwise we set*

$$g_1 := g_0 - f_i \frac{LT(g_0)}{LT(f_i)}.$$

We repeat the process for g_1 and $\{f_1, ..., f_r\}$ until the algorithm stops, either with $LT(g_n)$ not divisible by any of the $LT(f_i)$, or $g_n = 0$. The algorithm must stop after a finite number of steps by the well ordering property of the monomial order. This is because g_{i+1} is a polynomial contained in I that has lower order than g_i.

Clearly if $g_n = 0$ we can conclude that g lies in I, since the algorithm provides an expression for g as a sum of multiples of the generators of I. However, the converse is not true in general, since the algorithm may stop prematurely, giving a non-zero remainder for polynomials that are contained in the ideal. So $g_n \neq 0$ does not necessarily imply that g is not in I. For example, if $I = \langle f_1 := x^2 - x + 1,\ f_2 := x^2 + 2 \rangle$, and $g = x + 1$, then $LM(g)$ does not divide $LM(f_i)$, for $i = 1, 2$, but g is contained in I. If our algorithm is to work successfully, we need to choose a special basis for our ideal.

Definition 2.5. Let $\prec$ be a fixed monomial order, and I a polynomial ideal.

- The *ideal of leading terms* of I is defined to be
$$LT(I) := \langle LT(g) : g \in I \rangle.$$
- A *Groebner basis* for I is a collection of non-zero polynomials
$$\{f_1, ..., f_r\} \subseteq I,$$
such that $LT(f_1), ..., LT(f_r)$ generate $LT(I)$.

The existence of Groebner bases follows trivially from Hilbert's Basis Theorem, since it implies that we can find a finite generating set for $LT(I)$. Suppose $\{\mathbf{x}^{\mathbf{a_1}}, ..., \mathbf{x}^{\mathbf{a_r}}\}$ is such a set. Then we can find a unique minimal generating set by eliminating monomials that are divisible by other monomials in the set. The minimal generating set of a monomial ideal in two variables can be illustrated nicely by a 'staircase diagram'. By considering the lattice $\mathbb{Z}^2$ of points with (r, s) corresponding to the monomial $x^r y^s$, if we plot the points in the generating set, all the monomials in the ideal are given by points to the right or above a point in the generating set, and the minimal generators are the points with no other points to their left or below them. For example, if $I = \langle x^6y^3, x^4y^6, xy^5, y^7, x^2y^4, x^5y^5 \rangle$, then the monomials in I and it's minimal generators are shown in the diagram below.

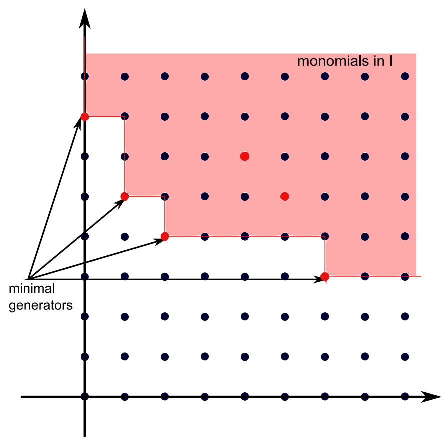


FIGURE 1. The minimal generators of I.

This has a generalisation in higher dimensions, but it is more difficult to draw. At this point, it is not clear why a Groebner basis is useful, or even whether it generates the ideal at all. The following proposition shows us how Groebner bases solve the ideal membership problem.

Proposition 2.6. *Let I be an ideal as above, and let $\{f_1, ..., f_r\}$ be a Groebner basis for I. Then $I = \langle f_1, ..., f_r \rangle$, and for all $g \in S$, the division algorithm terminates either with*

$$g = h_1 f_1 + ... + h_r f_r,$$

with $h_i \in S$, and hence $g \in I$, or

$$g = h_1 f_1 + ... + h_r f_r + g_r,$$

where $LT(g_r)$ is not contained in $\langle LT(f_1), ..., LT(f_r) \rangle$, so g_r,and hence also g are not in I.

Proof. We have already seen that whenever the algorithm yields a zero remainder we obtain an expression

$$g = h_1 f_1 + ... + h_r f_r,$$

and this implies $g \in I$. We have to show that this algorithm always gives a zero remainder for $g \in I$. Suppose for a contradiction that this is not the case. Then there exists a $g \in I$ such that $LT(g_i)$ is not divisible by any of the $LT(f_j)$'s for some $i \in \mathbb{N}$. But this implies that $LT(g_i)$ is not contained in $\langle LT(f_1), ..., LT(f_r) \rangle$ since a monomial in a monomial ideal must be a multiple of at least one of the monomial generators. But

$$\langle LT(f_1), ..., LT(f_r) \rangle = in_{\prec}(I)$$

and so g_i is not in I. This implies that g is not contained in I. This is a contradiction, and hence each $LT(g_i)$ must be divisible by a $LT(f_j)$, and so by the well ordering property we obtain a zero remainder whenever $g \in I$. Since $g \in I$ if and only if $g = h_1 f_1 + ... + h_r f_r$ for $h_i \in S$, then it follows trivially that $I = \langle f_1, ..., f_r \rangle$. □

We now show how we can determine via a Groebner basis whether or not a polynomial g belongs to I.

Algorithm 2.7. *(Algorithm 2.4 with a Groebner basis.) Begin by fixing a monomial order $\prec$ and a Groebner basis $\{f_1, ..., f_r\}$ of I. Now set $g = g_0$. If there does not exist a term $c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$ of g such that $LM(f_i) \mid c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}}$, for any f_i, then we stop. Otherwise we set*

$$g_1 := g_0 - c_{\boldsymbol{\alpha}} \mathbf{x}^{\boldsymbol{\alpha}} \frac{f_i}{LT(f_i)},$$

We repeat the process for each g_i, for $i \geq 1$ and $\{f_1, ..., f_r\}$ until the algorithm stops.

This algorithm yields the expression

$$g = h_1 f_1 + ... + h_r f_r + g_r,$$

where g_r is called the *normal form* of g with respect to $\{f_1, ...f_r\}$. By the above proposition, g is contained in I if and only if it's normal form is zero. Hence the Ideal Membership problem for multi-variate ideals is solved.

2.2. **Buchberger's Algorithm.** We have now established the existence of Groebner bases, but our proof does not tell us how to find one. An algorithm for calculating the Groebner basis of an ideal I is given below. Let $I = \langle f_1, ...f_n\rangle \subseteq k[x_1, ..., x_n]$, and let $\prec$ be a monomial order. For each i, j, define $\mathbf{x}^{\boldsymbol{\gamma}(ij)}$ to be the lowest common multiple of $LM(f_i)$ and $LM(f_j)$. Then the *S-polynomial*

$$S(f_i, f_j) := \frac{\mathbf{x}^{\boldsymbol{\gamma}(ij)}}{LT(f_i)} f_i - \frac{\mathbf{x}^{\boldsymbol{\gamma}(ij)}}{LT(f_j)} f_j.$$

The S-polynomial is constructed so that the leading terms of both expressions will cancel, so we obtain a member of I with order less than the least common multiple of $LM(f_i)$ and $LM(f_j)$.

Theorem 2.8 (Buchberger's Criterion)**.** *The generating set $\{f_1, ...f_n\}$ forms a Groebner basis for I if and only if every S-polynomial $S(f_i, f_j)$ gives a zero remainder on application of the division algorithm 2.4.*

Proof. Suppose first that $\{f_1, ..., f_r\}$ is a Groebner basis for I. Then every S-polynomial is an element of I, and so Algorithm 2.4 terminates with

$$S(f_i, f_j) = \sum_{l=1}^{r} h(ij)_l f_l$$

for all $1 \leq i, j \leq r$. This has zero remainder as required. Now suppose that each S-polynomial gives remainder zero, and assume for a contradiction that $\{f_1, ..., f_r\}$ is not a Groebner basis. Then there exists some polynomial $h \in I$ such that $LT(h)$ is not contained in $\langle LT(f_1), ..., LT(f_r)\rangle$. We can choose a representation $h = h_1 f_1 + ... + h_r f_r$ such that the monomial

$$\mathbf{x}^{\boldsymbol{\alpha}} := max\{LM(h_i)LM(f)_i); 1 \leq i \leq r\}$$

is minimal, and the number of indices i such that $LM(h_i)LM(f_i) = \mathbf{x}^{\boldsymbol{\alpha}}$ is minimal. After reordering of indices we may assume that

$$\mathbf{x}^{\boldsymbol{\alpha}} = LM(h_1 f_1) = ... = LM(h_m f_m),$$

and $LM(h_j f_j) \prec \mathbf{x}^{\boldsymbol{\alpha}}$ for $j > m$. It is clear that $m \geq 2$ since the $\mathbf{x}^{\boldsymbol{\alpha}}$ terms must cancel since $\mathbf{x}^{\boldsymbol{\alpha}} \in \langle LT(f_1), ..., LT(f_r)\rangle$ but $LT(h)$ is not. By assumption $S(f_1, f_2) = \Sigma h(12)_l f_l$ with $LM(S(f_1, f_2)) \geq LM(h(12)_l f_l)$ for each l. This implies

$$\frac{\mathbf{x}^{\boldsymbol{\gamma}(12)}}{LT(f_1)} f_1 - \frac{\mathbf{x}^{\boldsymbol{\gamma}(12)}}{LT(f_2)} f_2 - \Sigma h(12)_l f_l = 0$$

by definition of the S-polynomial. Since $LM(h_1f_1) = LM(h_2f_2) = \mathbf{x}^{\boldsymbol{\alpha}}$, and $\mathbf{x}^{\boldsymbol{\gamma}(12)} = LCM(f_1, f_2)$, then $\mathbf{x}^{\boldsymbol{\gamma}(12)} \mid \mathbf{x}^{\boldsymbol{\alpha}}$, and hence $\mu\mathbf{x}^{\boldsymbol{\gamma}(12)} = \mathbf{x}^{\boldsymbol{\alpha}}$ for some monomial μ. So

$$h = h_1f_1 + ... + h_rf_r - \mu\Big(\frac{\mathbf{x}^{\boldsymbol{\gamma}(12)}}{LT(f_1)}f_1 - \frac{\mathbf{x}^{\boldsymbol{\gamma}(12)}}{LT(f_2)}f_2 - \sum_{l=1}^{r} h(12)_l f_l\Big) = \tilde{h_1}f_1 + ...\tilde{h_1}f_r.$$

By adding on a multiple of zero we have obtained an expression for h with $\mathbf{x}^{\boldsymbol{\alpha}} > LM(\tilde{h_j}f_j)$ for $j > m$ and $j = 1$. This contradicts our second minimality assumption, and hence the result follows. □

Algorithm 2.9 (Buchberger's Algorithm)**.** *Let $I = \langle f_1, ...f_n\rangle$. For each i, j, apply the Algorithm 2.7 to $S(f_i, f_j)$ to obtain the expression*

$$S(f_i, f_j) = \Sigma_{i=1}^{r} h(ij)_l f_l + r(ij),$$

where $LM(r(ij))$ is not divisible by any $LM(f_i)$. If all the $r(ij)$ are zero, then by Buchberger's criterion $\{f_1, ..., f_r\}$ is already a Groebner basis. Otherwise let $f_{r+1}, ..., f_{r+s}$ be the non-zero $r(ij)$. We adjoin these to the basis to get a new set of generators. $\{f_1, ..., f_{r+s}\}$.

We now repeat the process with our new basis $\{f_1, \ldots f_{r+s}\}$. The algorithm will eventually terminate with a Groebner basis for I.

Proof. We begin with a set of generators $S_1 = \{f_1, ..., f_r\}$ of I. We define $J_1 = \langle LT(f_1), ..., LT(f_r)\rangle$. If $J_1 = LT(I)$, then S_1 is Groebner basis and we're done. Otherwise define $S_2 = \{f_1, ..., f_r, ..., f_{r+s}\}$, and let $J_2 = \langle LT(f_1), ..., LT(f_{r+s})\rangle$. By iterating this procedure we obtain an increasing chain of monomial ideals

$$J_1 \subset J_2 \subset ... \subseteq LT(I),$$

and as long as J_m is properly contained in $LT(I)$, then Buchberger's criterion guarantees than J_m is properly contained in J_{m+1}. But $k[x_1, ..., x_n]$ is a Noetherian ring, and hence every chain of ascending ideals must stabilise. This implies that for some $N \in \mathbb{N}$ we have

$$J_N = J_{N+1} = ...$$

and hence $J_N = LT(I)$. Therefore the algorithm terminates after a finite number steps with a Groebner basis S_N for I. □

Definition 2.10. A Groebner basis $\{f_1, ..., f_r\}$ is *minimal* if $\{LT(f_1), ..., LT(f_r)\}$ is the unique minimal generating set for $LT(I)$. A Groebner basis is *reduced* if no non-initial terms are divisible by any monomial in $LT(I)$.

Example 2.11. Let $I = \langle f_1 = x^3y^2 - 1, f_2 = x^7 - y, f_3 = x^4 - y^3\rangle$, and let $\prec$ be lexicographic order, which is defined in Example 2.3. We can use Buchberger's Algorithm to calculate a Groebner basis for I.

- $S(f_3, f_2) = x^3(x^4 - y^3) - (x^7 - y) = x^3y - y = y(x^3y^2 - 1)$

$S(f_3, f_2) \in I$, so we have no new basis elements.

- $S(f_3, f_1) = y^2(x^4 - y^3) - x(x^3y^2 - 1) = x - y^5$

$LT(x - y^5)$ is not divisible by the leading terms of the f_i, so we define $f_4 := x - y^5$.

- $S(f_2, f_1) = y^2(x^7 - y) + x^4(x^3y^2 - 1) = -y^3 - x^4$
- $S(f_1, f_4) = (x^3y^2 - 1) - x^2y^2(x - y^5) = xy^7(x - y^5) + y^{12}(x - y^5) + y^{17} - 1$

Set$f_5 = y^{17} - 1$. Similar calculations give that

- $S(f_2, f_4) = x^2y^5(x^4 - y^3) + (xy^8 + y^{13})(x - y^5) + y(y^{17} - 1)$
- $S(f_3, f_4) = y^3(x^3y^2 - 1)$
- $S(f_1, f_4) = (xy^5 + y^{12})(x - y^5) + y^{17} - 1$

Hence $\{f_1, ..., f_5\}$ is a Groebner basis of I, but the leading terms of f_1, f_2, f_3 are divisible by the leading term of f_4, and are hence unnecessary. So the reduced Groebner basis for I with respect to lexicographic order is $\{x - y^5, y^{17} - 1\}$.

Now let $\prec$ be graded reverse lexicographic order, also defined in Example 2.3.

- $S(f_1, f_2) = x^4(x^3y^2 - 1) - y^2(x^7 - y) = -x^4 + y^3 = f_3$
- $S(f_1, f_3) = x(x^3y^2 - 1) - y^2(x^4 - y^3) = x - y^5$

Set $f_4 := x - y^5$.

- $S(f_2, f_3) = (x^7 - y) - x^3(x^4 - y^3) = y(x^3y^2 - 1)$
- $S(f_1, f_4) = y^3(x^3y^2 - 1) + x^3(x - y^5) = x^4 - y^3$
- $S(f_2, f_4) = y^5(x^7 - y) + x^7(x - y^5) = x^4(x^4 - y^3) + y^3(x^4 - y^3)$
- $S(f_3, f_4) = y^5(x^4 - y^3) + x^4(x - y^5) = y^3(x - y^5) + x(x^4 - y^3)$

Similarly, a reduced Groebner basis for I with respect to $\prec_{grelex}$ is

$$\{x^3y^2 - 1, x^4 - y^3, x - y^5\}.$$

3. The Groebner Fan.

In the previous section we discussed how, given a polynomial ideal I and a monomial order $\prec$, we could calculate a Groebner basis for I and hence solve the Ideal Membership Problem. Clearly this Groebner basis is determined by the monomial order that is chosen, as a different monomial order will give different leading terms for the polynomials in I. Hence for each possible monomial order we can find a (possibly different) Groebner basis. In this section we will show how, by considering different monomial orders, we can encode the combinatorial data of an ideal I into a *fan*, see Definition 3.2. We will begin by revising some basic geometric definitions.

3.1. **Polyhedral Basics.**

Definition 3.1. Let $\mathcal{U}, \mathcal{V} \subseteq \mathbb{R}^d$.

- $\mathcal{U}$ is *convex*, if for all $\mathbf{u}, \mathbf{v} \in \mathcal{U}$, and $\lambda \in [0, 1]$, $\lambda\mathbf{u} + (1 - \lambda)\mathbf{v} \in \mathcal{U}$.

- A *convex polyhedron* is a convex set obtained as the intersection of finitely many half spaces.
- The *convex hull* of a set $\mathcal{V}$ is defined to be the intersection of all convex subsets of $\mathbb{R}^d$ containing it.
- $\mathcal{U}$ is called a *polytope* if it is a bounded polyhedron. Every polytope is the convex hull of a finite set of points.
- A *face* of a convex polyhedron P is the intersection of P with any hyperplane that touches it. Every face of P is a subset of the form $face_{\mathbf{c}}(P) := \{\mathbf{x} \in P : \mathbf{c}.\mathbf{x} \geq \mathbf{c}.\mathbf{y}, \forall \mathbf{y} \in P\}$, where $\mathbf{c}$ is any vector in $\mathbb{R}^d$.
- The *dimension* of a face F of P is defined to be the dimension of the subspace $\mathbf{v} + H$ of $\mathbb{R}^d$, where $\mathbf{v} \in F$, and H is spanned by the vectors $\mathbf{u} - \mathbf{v}$ for $\mathbf{u} \in F$. $\mathbf{v} + H$ is called the *affine span* of F.
- The $(d-1)$-dimensional faces of a d-dimensional polytope are called *facets.*
- A *cone* C is a convex polyhedron in $\mathbb{R}^d$ such that, for all $\mathbf{u}, \mathbf{v} \in C, \lambda \in \mathbb{R}_{\geq 0}, \mathbf{u} + \mathbf{v}, \lambda\mathbf{u} \in C$.
- Given a set $\mathcal{V} = \{\mathbf{v_1}, ...\mathbf{v_n}\} \subseteq \mathbb{R}^d$, then its *positive hull*
$$pos(\mathcal{V}) := \{\Sigma_i \lambda_i \mathbf{v}_i : \lambda_i \geq 0\}.$$
- A cone is called *polyhedral* if it has finitely many generators.
- Given a subset $\mathcal{V}$ of $\mathbb{R}^d$, the *relative interior* of $\mathcal{V}$ is the interior of $\mathcal{V}$ inside its affine span. We can think of this as $\mathcal{V}$ without it's boundary, see Figure 3.1. $\mathcal{V}$ is *relatively open* if it is equal to it's own relative interior.

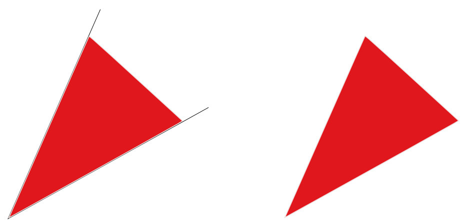


FIGURE 2. A polyhedral cone and it's relative interior.

Definition 3.2.
- A *polyhedral fan* $\mathcal{F}$ is a collection of polyhedral cones in $\mathbb{R}^n$ such that the intersection of any collection of these cones is a face of each, and for all cones $C \in \mathcal{F}$, all the faces of C are also cones in $\mathcal{F}$.
- A fan $\mathcal{F}$ is *complete* if the union if its cones cover $\mathbb{R}^n$. $\mathcal{F}$ is *simplicial* if each d-dimensional cone is the positive hull of d vectors.

- Let P be a polytope. Then the *outer normal fan* of P is the fan with cones corresponding to the faces F of P. $C_F = \{\mathbf{c} \in \mathbb{R}^n : face_{\mathbf{c}}(P) = F\}$. A fan is *polytopal* if it is the outer normal fan of some polytope P.

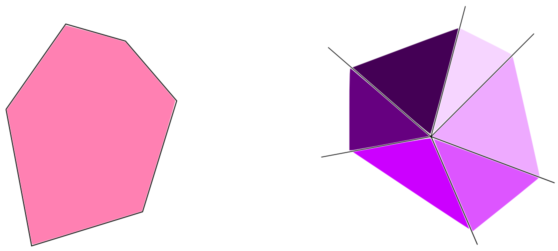


FIGURE 3. A polytope and its outer normal fan.

3.2. **Constructing the Groebner Fan.** The fundamental idea behind the Groebner fan is that every term order can be represented by a weight vector $\mathbf{w} = (w_1, ..., w_n) \in \mathbb{R}^n_{\geq 0}$, where w_i is the weight given to the variable x_i in the monomial order. To give a monomial order in which every monomial has a different order, we require that the w_i are independent and transcendental, but given a monomial order $\prec$, lexicographic order for example, then we can think of any vector in $\mathbb{R}^n_{\geq 0}$ as representing a monomial order in the following way: given $\mathbf{x}^{\mathbf{a}}, \mathbf{x}^{\mathbf{b}}$ in S, we say $\mathbf{x}^{\mathbf{a}} \prec_{\mathbf{w}} \mathbf{x}^{\mathbf{b}}$ if $\mathbf{a}.\mathbf{w} < \mathbf{b}.\mathbf{w}$, or if $\mathbf{a}.\mathbf{w} = \mathbf{b}.\mathbf{w}$ and $\mathbf{x}^{\mathbf{a}} \prec_{lex} \mathbf{x}^{\mathbf{b}}$.

Clearly there are infinitely many weight vectors in $\mathbb{R}^n_{\geq 0}$, does that mean that there are infinitely many Groebner bases for an ideal I? The answer to this is given by the following theorem. To prove it we first need a lemma.

Lemma 3.3. *Let M be an initial ideal of $I \subseteq S$ with respect to a term order $\prec$. Then the monomials of S not in M form a vector space basis for S/I.*

Proof. It is clear that the monomials in $S \setminus M$ span S/M, since every polynomial $f \in S$ has a normal form g such that $f - g \in I$ and g is a linear combination of monomials not in M. For linear independence, note that any dependence relation would give a polynomial $f \in I$ none of whose terms lie in M, which is impossible since M is an initial ideal of I, and so $in_{\prec}(f) \in M$. $\square$

Theorem 3.4. *Given a polynomial ideal I, there are only finitely many initial ideals LT(I).*

Proof. Suppose that I is an ideal with an infinite number of initial ideals, and let Σ_0 denote the set of all of them. Since Σ_0 is infinite, $I \neq \{0\}$. So we can choose an element $f_1 \in I$. f_1 has finitely many terms, and for each monomial ideal M in Σ_0, M contains one of the terms of f_1. Hence there must be one term m_1 of f_1 that is contained in infinitely many initial ideals of I. Let

$$\Sigma_1 := \{M \in \Sigma_0 : m_1 \in M\},$$

and let $J_1 = \langle m_1 \rangle$. Since infinitely many initial ideals contain J_1 there must be some initial ideal that properly contains it. By Lemma 3.3, since J_1 is not an initial ideal, the monomials of S outside J_1 are linearly dependent modulo I, so the linear dependence relation gives a polynomial f_2 in I with no term lying in in J_1. Similarly, there is a term m_2 of f_2 that is contained in infinitely many initial ideals in Σ_1. We now define

$$\Sigma_2 := \{M \in \Sigma_1 : m_2 \in M\},$$

and let $J_2 := J_1 + \langle m_2 \rangle$. Iterating this procedure gives an infinite properly increasing chain of ideals

$$J_1 \subset J_2 \subset J_3 ...,$$

but since S is Noetherian this is impossible. Hence there can only be finitely many initial ideals for I. □

Since the Groebner basis depends on the monomial order only via it's effect on the initial ideal, we can hence deduce that there are only finitely many Groebner bases for I.

Definition 3.5. The *universal Groebner basis* of a polynomial ideal I is the finite union of the reduced Groebner bases for all possible monomial orders.

This union is finite, since, by the above theorem, all the weight vectors only produce a finite number of distinct Groebner bases. We now proceed to associate a polyhedral fan to an ideal I, where each top dimensional cone corresponds to a different initial ideal of I, (see Theorem 3.11). We first establish several preliminary results.

Definition 3.6. Let $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$, and let $f = \Sigma c_{\mathbf{v}} \mathbf{x}^{\mathbf{v}}$ be a polynomial in S. The *initial form* of f with respect to $\mathbf{W}$ is defined to be

$$in_{\mathbf{w}}(f) = \Sigma_{\mathbf{v}'} c_{\mathbf{v}'} \mathbf{x}^{\mathbf{v}'},$$

where the $c_{\mathbf{v}'} \mathbf{x}^{\mathbf{v}'}$ are the terms of f on which $\mathbf{w}.\mathbf{v}$ is maximised. The *initial ideal of* I *with respect to* $\mathbf{w}$ is $in_{\mathbf{w}}(I) := \langle in_{\mathbf{w}}(f) : f \in I \rangle$. If the choice of $\mathbf{w}$ is clear, then we refer to this as the *initial ideal*.

The definition of the initial form of a polynomial is similar but not identical to the definition of the leading term. The difference is that the leading term is always a monomial, whereas the initial form needn't always be. Similarly, $in_{\prec}(I)$ is always a monomial ideal, whereas that is not always true of $in_{\mathbf{w}}(I)$. For example, if $I = \langle x^3 - 1, x - y^2 \rangle$, and $\mathbf{w} = (2, 1)$, then $in_{\mathbf{w}}(I) = \langle x^3, x -$

$y^2\rangle$, which is not a monomial ideal. However for $\mathbf{w}' = (1,1)$ the initial ideal $in_{\mathbf{w}'}(I) = \langle x^3, y^2\rangle$ which is a monomial ideal.

Note also that these definitions make sense for $\mathbf{w} \in \mathbb{R}^n$, but $\mathbf{w}$ does not define a term order when any of the w_i are negative, since $\mathbf{x^1} = \mathbf{0}$ is not the smallest monomial.

To prove the next result we require the following Lemma.

Lemma 3.7. *[Farkas' Lemma.] Let $A \in \mathbb{R}^{d\times n}$ and $\mathbf{z} \in \mathbb{R}^d$. Then either there exists an $\mathbf{x} \in \mathbb{R}^n$ with $A\mathbf{x} \leq \mathbf{z}$ or a $\mathbf{c} \in \mathbb{R}^d$ such that $\mathbf{c}^T A = 0$ and $\mathbf{c} \cdot \mathbf{z} < 0$, but not both.*

Proof. See [10], Section 1.4. □

Proposition 3.8. *Let I be a fixed ideal of S, and let $\prec$ be a monomial order. Then there exists a weight vector $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ such that $in_{\mathbf{w}}(I) = in_{\prec}(I)$.*

Proof. Let $\mathcal{G} = \{g_1, ..., g_r\}$ be a Groebner basis for I with respect to some monomial order $\prec$. Write

$$g_i = \sum_j d_{ij} \mathbf{x}^{\mathbf{u_{ij}}},$$

where $in_{\prec}(g_i) = d_{i1}\mathbf{x}^{\mathbf{u_{i1}}}$. Let

$$C_{\prec} := \{\mathbf{w} \in \mathbb{R}^n_{\geq 0} : \mathbf{w}.\mathbf{u_{i1}} > \mathbf{w}.\mathbf{u_{ij}}, \forall j \geq 2, 1 \geq i \geq r\}.$$

For any weight vector $\mathbf{w} \in C_{\prec}$ we have $in_{\prec}(I) \subseteq in_{\prec_{\mathbf{w}}}(I)$. Now suppose for a contradiction that $in_{\prec}(I)$ is properly contained in $in_{\prec_{\mathbf{w}}}(I)$. Then there exists a monomial $\mathbf{x^u}$ in $in_{\prec_{\mathbf{w}}}(I) \setminus in_{\prec}(I)$. By Lemma 3.3 the monomials not in $in_{\prec_{\mathbf{w}}}(I)$ form a basis for S/I. So there is a polynomial $g \in S$, none of whose terms lie in $in_{\prec \mathbf{w}}(I)$, and for which $\mathbf{x^u} - g$ lies in I. But none of these terms lie in $in_{\prec}(I)$, which is a contradiction of the definition of the initial ideal. So $in_{\prec}(I) = in_{\prec_{\mathbf{w}}}(I)$.

It now remains to show that $C_{\prec}$ is non-empty. Suppose for a contradiction that it is empty, and let U be the $s \times n$ matrix whose rows are given by the vectors $\mathbf{u_{i1}} - \mathbf{u_{ij}}$ for $j > 1$ and $1 \leq i \leq r$. Then there does not exist any $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$ with $U\mathbf{w} > 0$, where the inequality is termwise. Equivalently we can say there is not $\mathbf{w}' \in \mathbb{R}^n$ with

$$U'\mathbf{w}' = \begin{pmatrix} -U \\ -I \end{pmatrix} \mathbf{w}' \leq (\mathbf{0}, \mathbf{1})^T,$$

where the vector $\mathbf{0}$ is of length n, and the vector $\mathbf{1}$ has length s. Hence Lemma 3.7 gives that there must exist some $\mathbf{c} \in \mathbb{R}^{s+n}_{\geq 0} \setminus \{0\}$ with $\mathbf{c}^T U' = 0$. U' has intergral entries so we can choose $\in \mathbb{N}^{s+n}$. If c_{im} is the component of $\mathbf{c}$ corresponding to the row $\mathbf{u_{im}} - \mathbf{u_{i1}}$ of U', then we conclude that

$$\sum_{i,m} c_{im}(\mathbf{u_{im}} - \mathbf{u_{i1}}) \geq 0$$

since when this is subtracted from the sum $\mathbf{c}U' = \mathbf{0}$, all that is left is a sum of nonpositive coordinates. Thus

$$\prod_{i,m} (\mathbf{x}^{\mathbf{u_{i1}}})^{c_{im}} \text{ divides } \prod_{i,m} (\mathbf{x}^{\mathbf{u_{im}}})^{c_{im}}$$

so

$$\prod_{i,m} (\mathbf{x}^{\mathbf{u_{i1}}})^{c_{im}} \preceq \prod_{i,m} (\mathbf{x}^{\mathbf{u_{im}}})^{c_{im}}.$$

But for all i, m we already have $\mathbf{x}^{\mathbf{u_{im}}} \prec \mathbf{x}^{\mathbf{u_{i1}}}$ so

$$\prod_{i,m} (\mathbf{x}^{\mathbf{u_{i1}}})^{c_{im}} \succ \prod_{i,m} (\mathbf{x}^{\mathbf{u_{im}}})^{c_{im}}.$$

This gives a contradiction, and hence we deduce that $C_{\prec}$ is non empty as required. □

The previous proposition associated at least one vector in $\mathbb{R}^n_{\geq 0}$ to each initial ideal. The following proposition associates a cone to it.

Proposition 3.9. *Let I be as above, and for each $\boldsymbol{w} \in \mathbb{R}^n_{\geq 0}$, let*

$$C[\mathbf{w}] := \{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{w}'}(I) = in_{\mathbf{w}}(I)\}.$$

Then $C[\mathbf{w}]$ is the relative interior of a polyhedral cone in $\mathbb{R}^n_{\geq 0}$.

Proof. Let $\mathbf{w} \in \mathbb{R}^n_{\geq 0}$, $J =\in_{\mathbf{w}} (I)$ and $\mathcal{G} = \{g_1, ..., g_r\}$ be a reduced Groebner basis for I with respect to $\prec_{\mathbf{w}}$. We write

$$g_i = \Sigma_j c_{ij} \mathbf{x}^{\mathbf{a_{ij}}} + \Sigma_j c'_{ij} \mathbf{x}^{\mathbf{b_{ij}}}$$

where $in_{\mathbf{w}}(g_i) = \Sigma_j c_{ij} \mathbf{x}^{\mathbf{a_{ij}}}$. It suffices to show that

$$C[\mathbf{w}] = \{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{w}'}(g) = in_{\mathbf{w}}(g) \forall g \in \mathcal{G}\},$$

since this is just

$$\{\mathbf{w}' \in \mathbf{R}^n_{\geq 0} : \mathbf{w}' \cdot \mathbf{a_{ij}} = \mathbf{w}' \cdot \mathbf{a_{ik}}, \mathbf{w}' \cdot \mathbf{a_{ij}} > \mathbf{w}' \cdot \mathbf{b_{ik}} \text{ for all } j, k \text{ and } i = 1, ..., r\},$$

which is the relative interior of a polyhedral cone by definition. Let $\mathbf{w}' \in \{\mathbf{u} \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{u}}(g) = in_{\mathbf{w}}(g) \forall g \in \mathcal{G}\}$. Then $in_{\mathbf{w}}(I) \subseteq in_{\mathbf{w}'}(I)$, since $in_{\mathbf{w}}(I)$ is generated by the $in_{\mathbf{w}}(g_i)$, and these all lie in $in_{\mathbf{w}'}(I)$. These may not be monomial ideals, but the containment (and whether or not it is proper) must be preserved for taking initial ideals with respect to any arbitrary order $\prec$. So we have $in_{\prec}(in_{\mathbf{w}}(I)) \subseteq in_{\prec}(in_{\mathbf{w}'}(I))$ By the proof of Proposition 3.8 this inclusion cannot be proper, so we conclude that $in_{\mathbf{w}}(I) = in_{\mathbf{w}'}(I)$. Hence

$$\{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{w}'}(g) = in_{\mathbf{w}}(g) \forall g \in \mathcal{G}\} \subseteq C[\mathbf{w}].$$

Now let $\mathbf{w}' \in C[\mathbf{w}]$. Since $in_{\prec_{\mathbf{w}}}(I) = in_{\prec}(in_{\mathbf{w}}(I))$ it follows that $in_{\mathbf{w}}(\mathcal{G}) = \{in_{\mathbf{w}}(g) : g \in \mathcal{G}\}$ is a Groebner basis for $in_{\mathbf{w}}(I) = in'_{\mathbf{w}}(I)$ with respect to $\prec$.

Fix some $g \in \mathcal{G}$. Then $in_{\mathbf{w}'}(g) = \Sigma_i h_i in_{\mathbf{w}}(g_i)$ for some $h_i \in S$. Now $m := in_{\prec_{\mathbf{w}}}(g)$ is the only term of g which is divisible by the leading term with respect to $\prec$ of a polynomial in $in_{\mathbf{w}}\mathcal{G}$, so it must occur in $in_{\mathbf{w}'}(g)$ for the reduction to be possible. Hence we can write $in_{\mathbf{w}}(g) = m + h$ and $in_{\mathbf{w}'}(g) = m + h'$. By the choice of m we know that both h and h' have no terms that occur in $in_{\prec_{\mathbf{w}}}(I)$. But $in_{\mathbf{w}}(g) - in_{\mathbf{w}'}(g) = h - h' \in in_{\mathbf{w}}(I)$, so $in_{\prec}(h - h') \in in_{\mathbf{w}}(in_{\prec_{\mathbf{w}}}(I)) = in_{\prec_{\mathbf{w}}}(I)$. This is only possible if $h - h' = 0$, so $in_{\mathbf{w}}(g) = in_{\mathbf{w}'}(g)$. Hence $\mathbf{w}' \in \{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{w}'}(g) = in_{\mathbf{w}}(g) \forall g \in \mathcal{G}\}$ so

$$C[\mathbf{w}] = \{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{w}'}(g) = in_{\mathbf{w}}(g) \forall g \in \mathcal{G}\}$$

as required. □

The following lemma was used in the above proof.

Lemma 3.10. *Let I be an ideal in S, $\mathbf{w}$ some weight vector, and $\prec$ some monomial order. Then $in_{\prec_{\mathbf{w}}}(I) = in_{\prec}(in_{\mathbf{w}}(I))$.*

Proof. For every $f \in I$ we have $in_{\prec}(in_{\mathbf{w}}(f)) = in_{\prec_{\mathbf{w}}}(f)$ by definition of $\prec_{\mathbf{w}}$. Since every monomial in the ideal $in_{\prec_{\mathbf{w}}}(I)$ is of the form $in_{\prec_{\mathbf{w}}}(f)$ for some $f \in I$, this means that $in_{\prec_{\mathbf{w}}}(I) \subseteq in_{\prec}(in_{\mathbf{w}}(I))$. To see the reverse inclusion note that the vector $\mathbf{w}$ gives an $\mathbb{R}$-grading of S by $deg(x_i) = w_i$. Since the group generated by the w_i is isomorphic to $\mathbb{Z}^k$ for some $k \leq n$ this is a grading of a finitely generate abelian group. The ideal $in_{\mathbf{w}}(I)$ is homogeneous with respect to this grading (i.e. every term appearing in the generators of $in_{\mathbf{w}}(I)$ has the same total degree with respect to $\mathbf{w}$), and thus so is the reduced Groebner basis for $in_{\mathbf{w}}(I)$ with respect to $\prec$. This is clear if we consider how we obtain a Groebner basis from a generating set of an ideal. Now let $\mathbf{x}^{\mathbf{u}}$ be a minimal generator of $in_{\prec}(in_{\mathbf{w}}(I))$. So $\mathbf{x}^{\mathbf{u}} = in_{\prec}(g)$ for some $\mathbf{w}$-homogeneous $g \in in_{\mathbf{w}}(I)$. Now $g \in in_{\mathbf{w}}(I)$, so $g = \Sigma_i h_i in_{\mathbf{w}}(g_i)$ where $g_i \in \mathcal{G}$ and h_i is a monomial in S for each i. Hence $g = \Sigma_i in_{\mathbf{w}}(g_i h_i)$. We have as few terms as possible, so no cancellation occurs, and hence since g is $\mathbf{w}$-homogeneous, so also are all the $in_{\mathbf{w}}(g_i h_i)$. This implies that $g = in_{\mathbf{w}}(\Sigma_i h_i g_i)$. This proves the claim for $f = \Sigma_i h_i g_i \in I$. This means that $in_{\prec}(g) = in_{\prec}(in_{\mathbf{w}}(f)) = in_{\prec_{\mathbf{w}}}(f)$ and therefore $in_{\prec}(in_{\mathbf{w}}(I)) \subseteq in_{\prec_{\mathbf{w}}}(I)$. This implies $in_{\prec}(in_{\mathbf{w}}(I)) = in_{\prec_{\mathbf{w}}}(I)$, and since $in_{\mathbf{w}}(\mathcal{G}$ is a basis for $in_{\mathbf{w}}(I)$ such that $in_{\prec}(in_{\mathbf{w}}(I))$ is generated by the $in_{\prec}(in_{\mathbf{w}}(g_i))$, we conclude that $in_{\mathbf{w}}\mathcal{G}$ is a Groebner basis for $in_{\mathbf{w}}(I)$. □

For example, take the polynomial ideal $I = \langle x + y, x^2 + 1 \rangle$. There are two possible initial ideals for this, namely $\langle x, y^2 \rangle$ and $\langle y, x^2 \rangle$. Taking $\mathbf{w} = (2, 1)$, we obtain $in_{\mathbf{w}}(I) = \langle x \rangle$. Then $C[\mathbf{w}]$ is the set of vectors in $\mathbb{R}^2$ that give the same initial ideal, namely all the vectors that give x more weight than y. This is the relative interior of the cone spanned by $(1, 1)$ and $(1, 0)$, as shown below.

Theorem 3.11. *The set $\{C[\boldsymbol{w}] : \boldsymbol{w} \in \mathbb{R}^n_{\geq 0}\}$ of closures of the above cones forms a polyhedral fan.*

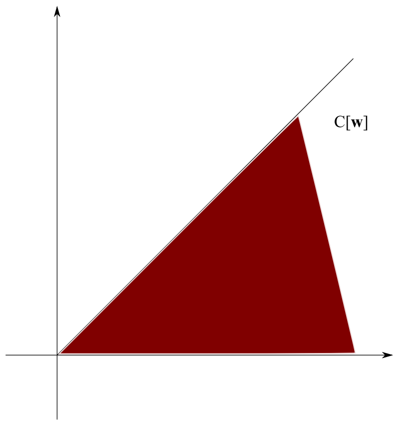


FIGURE 4. Polyhedral cone C[w].

Proof. We begin by showing that if $\mathbf{w}'$ lies in a face of $C[\mathbf{w}]$ with $\mathbf{w}'$ not in $C[\mathbf{w}]$, then $\overline{C[\mathbf{w}']}$ is a face of $C[\mathbf{w}]$. Let $\prec$ be some fixed monomial order, and let $\mathcal{G}$ be the reduced Groebner basis for I with respect to $\prec_{\mathbf{w}}$. Fix some $\mathbf{w}'$ on a face of $C[\mathbf{w}]$, and let $J := \langle in_{\mathbf{w}'}(g) : g \in \mathcal{G}\rangle$. From the proof of 3.9 we know that $C[\mathbf{w}] = \{\mathbf{w}' \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{w}'}(g) = in_{\mathbf{w}}(g) \forall g \in \mathcal{G}\}$. Since the faces of $C[\mathbf{w}]$ will be spanned by vectors $\mathbf{u}$ such that $in_{\mathbf{u}}(g_i)$ will be $in_{\mathbf{w}}(g_i) + h_i$, so $in_{\mathbf{w}}(in_{\mathbf{u}}(g_i)) = in_{\mathbf{w}}(g_i)$. This means that $in_{\prec}(in_{\mathbf{w}}(I)) \subseteq in_{\prec}(in_{\mathbf{w}}(J)$, and by Lemma 3.10 $in_{\prec_{\mathbf{w}}}(I) = in_{\prec}(in_{\mathbf{w}}(I))$, so $in_{\prec_{\mathbf{w}}}(I) \subseteq in_{\prec_{\mathbf{w}}}(J)$. Suppose now that J is a proper subset of $in_{\mathbf{w}'}(I)$. Then $in_{\prec_{\mathbf{w}}}(J)$ is a proper subset of $in_{\prec_{\mathbf{w}}}(in_{\mathbf{w}'}(I)) = in_{\prec_{\mathbf{w},\mathbf{w}'}}(I)$, where the monomial order $\prec_{\mathbf{w},\mathbf{w}'}$ is the term order that first compares monomials with $\mathbf{w}'$ and then breaks ties with $\prec_{\mathbf{w}}$. But then we have a proper inclusion of initial ideals

$$in_{\prec_{\mathbf{w}}}(I) \subset in_{\prec_{\mathbf{w}}}(J)$$

which cannot occur. Thus $J = in_{\mathbf{w}'}(I)$. Since $in_{\prec_{\mathbf{w}}}(I) \subseteq in_{\prec_{\mathbf{w}}}(J) = in_{\prec_{\mathbf{w},\mathbf{w}'}}(I)$, by the same argument

$$in_{\prec_{\mathbf{w}}}(I) = in_{\prec_{\mathbf{w},\mathbf{w}'}}(I.$$

This implies that $\mathcal{G}$ is a reduced Groebner basis for $\prec_{\mathbf{w},\mathbf{w}'}$, and hence $C[\mathbf{w}'] = \{\mathbf{w}'' \in \mathbb{R}^n_{\geq 0} : in_{\mathbf{w}''}(g) = in_{\mathbf{w}'} \forall g \in \mathcal{G}\}$ is a face of $C[\mathbf{w}]$ as required. □

Definition 3.12. The polyhedral fan defined in the above proposition is called the *Groebner fan* for the ideal I.

Example 3.13. Consider the ideal $I = \langle x + y, x^2 + 1\rangle$ as above. This has 2 different initial ideals, namely $\langle x, y^2\rangle, \langle x^2, y\rangle$. These correspond to the relative interior of the cones spanned by $(1, 0), (1, 1)$ and $(1, 1), (0, 1)$ respectively. These form the fan shown below.

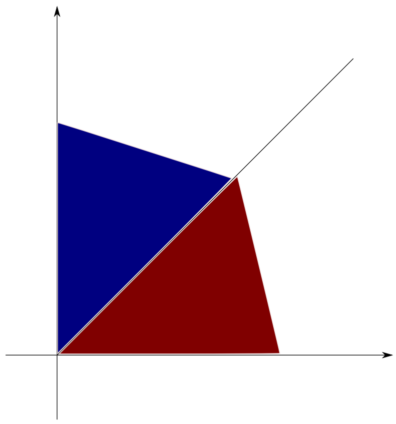


FIGURE 5. Groebner fan for $\langle x+y\rangle$.

Example 3.14. Let $I = \langle x^7 - y, x^4 - y^3, x^3y^2 - 1\rangle$. This has 4 different initial ideals, namely

- $\langle y^{17}, x\rangle$
- $\langle x^3y^2, y^5, x^4\rangle$
- $\langle x^3y^2, y^3, x^7\rangle$
- $\langle x^{17}, y\rangle$

These correspond to the cones shown on the diagram below, starting at the x-axis, and moving anticlockwise. Lexicographic order gives higher weight to

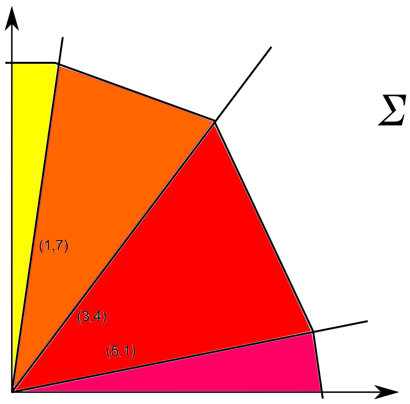


FIGURE 6. Groebner fan of I.

x than to y, so it corresponds to the cone of weight vectors closest to the x-axis. The Groebner bases corresponding to the above cones are

- $\{y^{17} - 1, x - y^5\}$
- $\{x^3y^2 - 1, y^5 - x, x^4 - y^3\}$
- $\{x^3y^2 - 1, y^3 - x^4, x^7 - y\}$
- $\{x^{17} - 1, y - x^7\}$

respectively, and hence the universal Groebner basis for I is

$$\{x^{17} - 1, x^3y^2 - 1, y^5 - x, x^4 - y^3, y^{17} - 1\}.$$

It is important to understand how the cones in the fan are glued together. The two dimensional cones overlap in a one dimensional line. This line is generated by a vector $\mathbf{w}$, where $in_{\mathbf{w}}(I)$ is not a monomial ideal. Two adjacent cones correspond to equivalence classes of weight vectors that change the leading term in just one of the elements of the universal Groebner basis.

For example, let I be an ideal with universal Groebner basis $\{f_1, ...f_n\}$. For each pair of adjacent cones in the Groebner fan C_i, C_{i+1}, the line in between them corresponds to the ideal generated by the monomials the two initial ideals have in common and by $c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}} + c_{\mathbf{u}'}\mathbf{x}^{\mathbf{u}'}$, where $c_{\mathbf{u}}\mathbf{x}^{\mathbf{u}}, c_{\mathbf{u}'}\mathbf{x}^{\mathbf{u}'}$ are the two terms that were only contained in one ideal. For example, the line through $(1, 1)$ in example 3.13 above corresponds to the initial ideal $\langle x + y, x^2, y^2\rangle$.

3.3. **The Groebner Walk.** One important application of the Groebner fan is the Groebner walk, an algorithm for converting one Groebner basis into another. This is a highly useful technique, as some Groebner bases are much easier to compute than others. For example, the Grobner basis for lexicographic order is very difficult to compute, but the Groebner walk gives a simple method for finding it from another Groebner basis that is easier to compute.

Proposition 3.15. *Let I be an ideal and let $\mathcal{G}$ be a Groebner basis for I with respect to a monomial order $\prec_{\mathbf{w}}$. If ω is a point on the boundary of the cone $C[\mathbf{w}]$, then*

- *The reduced Groebner basis for $in_\omega(I)$ over $\prec_{\mathbf{w}}$ is $G_\omega = \{in_\omega(g) | g \in \mathcal{G}\}$.*
- *If H is the reduced Groebner basis for $in_\omega(I)$ over $\prec_{\mathbf{w}'}$, then*

  $$\{f - f^G | f \in H\}$$

  is a minimal Groebner basis for I over $\prec_{\mathbf{w}'\omega}$, which is the monomial order for the weight vector ω, breaking ties with $\prec_{\mathbf{w}'}$, and f^G is the remainder obtained by dividing f modulo G.
- *The reduced Groebner basis for I over $\prec_{\mathbf{w}'\omega}$ coincides with the reduced Groebner basis for I over $\prec_{\mathbf{w}'}$.*

Proof. See [3], Proposition 3.2. □

The following algorithm uses the above proposition to find one Groebner basis from another.

Algorithm 3.16. *Let I be a polynomial ideal, and let $\mathcal{G}_i = \{f_1, ... f_r\}$ be a Groebner basis for I. Then we can find a weight vector $\mathbf{w_s}$ such that $\mathcal{G}_i$ is a Groebner basis with respect to the monomial order $\prec_{\mathbf{w_s}}$. Suppose we want to find the Groebner basis for I with respect to another weight vector $\mathbf{w_t}$. These two vectors can be drawn onto the Groebner fan, and a path can be drawn between them that only passes through cones of co-dimension 1. In the simplest cases this can be taken to be a straight line.*

Step 1. *Supposing $\mathbf{w_s}$ is contained in a cone C_i, we can find the last point of the path contained in the cone $\overline{C_i}$ by intersecting the path with the facets of C_i. Denote this point by $\mathbf{w_{new}}$.*

Step 2. *We now consider the ideal $\langle in_{\mathbf{w_{new}}}(\mathcal{G}_i)\rangle$ of initial forms with respect to $\mathbf{w_{new}}$. This will not be a monomial ideal, as $\mathbf{w_{new}}$ is a boundary point, but will usually be generated by mostly monomials. We can define a new term order $\prec_{i+1}$ by $\mathbf{w_{new}}$, breaking ties with our target order $\prec_t$. This is the monomial order for the other cone with boundary $\mathbf{w_{new}}$, and we can use the above proposition to find its Groebner basis.*

By repeating this process a finite number of times, we will eventually find the Groebner basis for I with respect to $\prec_t$.

Example 3.17. Let $I = \langle x^7 - y, x^4 - y^3, x^3y^2 - 1\rangle$, and let $\prec$ be lexicographic order. We calculated above that the Groebner basis for $\prec_{(6,1)}$ is $\{y^{17} - 1, x - y^5\}$. We want $\prec_{(1,1)}$. Drawing a line from $(6,1)$ to $(1,1)$ crosses a line of the Groebner fan at $k(5,1)$ for some $k \in \mathbb{R}$.

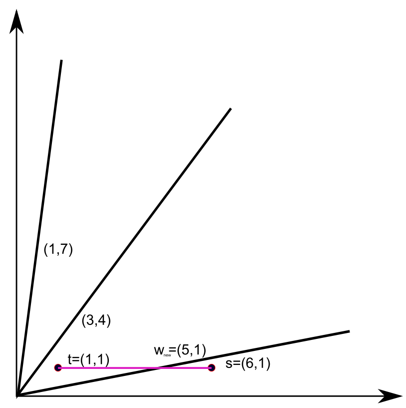


FIGURE 7. Path for the Groebner Walk

$in_\omega(I) = \langle h_1 = y^{17}, h_2 = x - y^5\rangle$

- $S(h_1, h_2) = y^{17} + y^{12}(x - y^5) = y^{12}x = (y^7x + y^2x^2)(y^5 - x) + y^2x^3$.

Set $h_3 := y^2x^3$.

- $S(h_1, h_3) = x^3y^{17} - y^{15}(y^2x^3) = 0$
- $S(h_2, h_3) = x^3(y^5 - x) - y^3(x^3y^2) = x^4$

Set $h_4 := x^4$.

- $S(h_1, h_4) = x^4(y^{17}) - y^{17}(x^4) = 0$.
- $S(h_2, h_4) = x^4(y^5 - x) - y^5x^{64} = x^5 = x(x^4)$.
- $S(h_3, h_4) = x(x^3y^2) - y^2(x^4) = 0$

So $H = \{y^{17}y^5 - x, y^2x^3, x^4\}$ is a Groebner basis for I with respect to $\prec_{(5,1)}$, breaking ties with $(1, 1)$, but y^{17} is redundant. $\{h - h^G : h \in H\}$ is a Groebner basis for I, and $G = \{y^{17} - 1, y^5 - x\}$.

- $y^2x^3 = (y^2x^2 + y^7x + y^{12})(x - y^5) + (y^{17} - 1) + 1$, so $h_1^g = 1$.
- $x^4 = (x^3 + x^2y^5 + xy^{10} + y^{15})(x - y^5) + y^3(y^{17} - 1) + y^3$, so $(x^4)^G = y^3$.

Hence $\mathcal{G}' = \{y^5 - x, y^2x^3 - 1, x^4 - y^3\}$. Looking back to Example 3.14 we see that this is the same as the Groebner basis for the cone generated by the vectors $(3, 4)$ and $(5, 1)$. Especially in more complicated examples this method of finding a Groebner basis will be considerably less work than finding it via Buchberger's Algorithm.

4. Toric Geometry.

4.1. **Toric Ideals.**

In this section we give a brief description of a special class of polynomial ideals, called toric ideals. These are the defining ideals of affine toric varieties, which will be discussed below.

Definition 4.1. Let S be a semigroup. The *semigroup algebra* $k[S]$ is the set of all finite linear combinations of elements $\mathbf{x}^\mathbf{s}$, where $s \in S$, with coefficients in k. Since S is a semigroup, it is closed under addition. Hence we can give $k[S]$ an algebra structure by defining addition, multiplication and scalar multiplication as follows, where $a_i, b_i, c \in k$.

$$\sum_{\mathbf{s}\in S} a_\mathbf{s}\mathbf{x}^\mathbf{s} + \sum_{\mathbf{s}\in S} b_\mathbf{s}\mathbf{x}^\mathbf{s} = \sum_{\mathbf{s}\in S}(a_\mathbf{s} + b_\mathbf{s})\mathbf{x}^\mathbf{s}$$

$$c\sum_{\mathbf{s}\in S} a_\mathbf{s}\mathbf{x}^\mathbf{s} = \sum_{\mathbf{s}\in S} ca_\mathbf{s}\mathbf{x}^\mathbf{s}$$

$$\sum_{\mathbf{s}\in S} a_\mathbf{s}\mathbf{x}^\mathbf{s} \cdot \sum_{\mathbf{s}\in S} b_\mathbf{s}\mathbf{x}^\mathbf{s} = \sum_{\mathbf{s}\in S,\mathbf{t}\in S} a_\mathbf{s}b_\mathbf{t}\mathbf{x}^{\mathbf{s}+\mathbf{t}}$$

For $d, n \in \mathbb{N}$, let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ denote a nonempty subset of $\mathbb{Z}^d \setminus \{0\}$. Then we can construct a $d \times n$ matrix $A = [\mathbf{a_1}\, \mathbf{a_2} ... \mathbf{a_n}]$, which we assume has rank d. Then there is a semigroup homomorphism

$$\pi : \mathbb{N}^n \to \mathbb{Z}^d; \mathbf{u} = (u_1, ... u_n) \mapsto \Sigma_{i=1}^n \mathbf{a}_i u_i = A\mathbf{u}.$$

This extends to a homomorphism from the semigroup algebra of $\mathbb{N}^n$ to the semigroup algebra of $\mathbb{Z}^d$,

$$\hat{\pi} : k[x_1, ...x_n] \longrightarrow k[t_1^{\pm 1}, ...t_d^{\pm 1}]; x_j \mapsto \mathbf{t}^{\mathbf{a_j}} = t_1^{a_{1j}}...t_d^{a_{dj}},$$

where the semigroup algebra $k[S]$ is the set of linear combinations of finitely many elements of S with coefficients in k.

Definition 4.2. The *toric ideal* of $\mathcal{A}$, denoted as $I_{\mathcal{A}}$, is the kernel of $\hat{\pi}$. An *affine variety* V is *toric* if

$$V = \mathrm{Spec}(k[x_1, ..., x_n]/I_{\mathcal{A}}),$$

for some toric ideal $I_{\mathcal{A}}$.

The following simple examples illustrate the above definition.

Example 4.3. Let $\mathcal{A} = \{(1, 0), (0, 1)\}$. Then this gives the homomorphism

$$\pi : \mathbb{N}^2 \longrightarrow \mathbb{Z}^2, (a, b) \mapsto (1a + 0b, 0a + 1b).$$

This extends to the homomorphism

$$\hat{\pi} : \mathbb{C}[x, y] \longrightarrow \mathbb{C}[s^{\pm 1}, t^{\pm 1}], x \mapsto s, y \mapsto t.$$

The kernel of this map is clearly the zero ideal, and so $I_{\mathcal{A}} = \langle 0 \rangle$. Hence

$$V = \mathrm{Spec}(\mathbb{C}[x, y]/\{0\}) = \mathbb{C}^2$$

is an affine toric variety.

Now let $\mathcal{B} = \{(2, 0), (1, 1), (0, 2)\}$. Similarly we obtain homomorphisms

$$\phi : \mathbb{N}^3 \longrightarrow \mathbb{Z}^2, (a, b, c) \mapsto (2a + b, b + 2c)$$

and

$$\hat{\phi} : \mathbb{C}[x, y, z] \longrightarrow \mathbb{C}[s^{\pm 1}, t^{\pm 1}], x \mapsto s^2, y \mapsto st, z \mapsto t^2.$$

The kernel of $\hat{\phi} = I_{\mathcal{B}} = \langle xz - y^2 \rangle$, and hence

$$\mathrm{Spec}(\mathbb{C}[x, y, z]/\langle xz - y^2 \rangle) = \{(u, v, w) \in \mathbb{C}^3 | uw = v^2\}$$

is an affine toric variety. In fact this is the cyclic quotient singularity of type $\frac{1}{2}(1, 1)$, which will be discussed in greater detail below.

The homomorphism $\hat{\pi}$ gives a grading of the $k[x_1, ..., x_n]$, with

$$deg(x_i) = \mathbf{a_i}.$$

The following proposition gives some important properties of toric ideals.

Proposition 4.4. *Let $I_{\mathcal{A}}$ be a toric ideal.*

- *$I_{\mathcal{A}}$ is a prime ideal in $k[\mathbf{x}]$.*
- *$I_{\mathcal{A}}$ is generated as a k-vector space by the infinite set of binomials*

$$\{\mathbf{x^u} - \mathbf{x^v} \in k[\mathbf{x}] : \pi(\mathbf{u}) = \pi(\mathbf{v})\}.$$

- *For every term order $\prec$, the reduced Groebner basis of $I_{\mathcal{A}}$ consists of a finite set of the above binomials.*

Proof. Let $I_{\mathcal{A}}$ be as above. $k[\mathbf{x}]/I_{\mathcal{A}} = k[\mathbf{x}]/ker(\hat{\pi}) \cong \pi(k[\mathbf{x}]) = k[\mathbf{t_1^a}, \mathbf{t^{a_2}}, ...\mathbf{t^{a_n}}]$. This is an integral domain, since the $\mathbf{a_i}$'s are linearly independent, and hence $I_{\mathcal{A}}$ is a prime ideal.

Note first that a binomial $\mathbf{x^u} - \mathbf{x^v}$ is in $I_{\mathcal{A}}$ if and only if

$$\hat{\pi}(\mathbf{x^u}) = \hat{\pi}(\mathbf{x^v}), \text{ i.e. } \pi(\mathbf{u}) = \pi(\mathbf{v}),$$

and hence

$$B := \{\mathbf{x^u} - \mathbf{x^v} \in k[\mathbf{x}] : \pi(\mathbf{u}) = \pi(\mathbf{v})\} \subseteq I_{\mathcal{A}}.$$

Let $\prec$ be some fixed term order on $k[\mathbf{x}]$, and suppose for a contradiction that there exists an $f \in I_{\mathcal{A}}$ that cannot be written as a k-linear combination of binomials in B. Choose such an f for which $LM_{\prec}(f) = \mathbf{x^u}$ is minimal. Since $f \in I_{\mathcal{A}} = ker(\hat{\pi})$, we conclude that $f(\mathbf{t^{a_1}}, ..., \mathbf{t^{a_n}}) = 0$. In particular, $\hat{\pi}(\mathbf{x^u}) = \mathbf{t}^{\pi(\mathbf{u})}$, the image of the leading monomial of f, must cancel in the expansion, and so there is a monomial $\mathbf{x^v}$ in f such that $\pi(\mathbf{u}) = \pi(\mathbf{v})$. Hence the polynomial

$$f' = f - \mathbf{x^u} + \mathbf{x^v}$$

is another polynomial in $I_{\mathcal{A}}$ that cannot be written as a k-linear combination of binomials in B, but of strictly lower order that f. This contradicts our minimality assumption, and the result follows.

By Hilbert's basis theorem and the above, we can find a finite subset C of B that generates $I_{\mathcal{A}}$. (This follows, since Hilbert's basis theorem gives us finitely many generators, all of which are a finite k-linear combination of elements of B.) We now apply Buchberger's algorithm to C. Clearly the S-polynomial of 2 binomials is also a binomial, and the normal form of a homogeneous binomial with respect to a set of homogeneous binomials is also a homogeneous binomial. Thus each Groebner basis of $I_{\mathcal{A}}$ compute from C is also a subset of B, as required.

□

Recall from above that the universal Groebner basis for an ideal I is the union of all the Groebner bases of I for all possible term orders. The *universal Groebner basis* for $I_{\mathcal{A}}$ is denoted $\mathcal{U}_{\mathcal{A}}$. Hence by the above $\mathcal{U}_{\mathcal{A}}$ will be a finite collection of $\mathcal{A}$-homogeneous binomials.

4.2. **Toric Varieties.** Recall that an affine algebraic set is a set of the form

$$\mathbb{V}(J) = \{\mathbf{p} \in \mathbb{C}^n : f(\mathbf{p}) = 0, \forall f \in J\},$$

where J is an ideal of S. This is just a set of common zeros to a collection of polynomial equations. By Hilbert's Basis Theorem, we can choose this collection to be finite. Irreducible algebraic sets are called *varieties*. It can be shown that finite unions of algebraic sets and arbitrary intersections are algebraic. Clearly $\emptyset, \mathbb{C}^n$ are algebraic, so the algebraic subsets of $\mathbb{C}^n$ form the closed sets of a topology on $\mathbb{C}^n$, called the *Zariski topology* on $\mathbb{C}^n$. This topological space is called *affine n-space*, and is denoted $\mathbb{A}^n_{\mathbb{C}}$.

Affine toric varieties are simply algebraic varieties of the toric ideals that were discussed above, but the general definition of toric varieties is a bit more complicated. There are several different ways to approach toric varieties, all of which look quite different. Two of these are discussed briefly below.

Definition 4.5. (Geometric Approach.) Let X be an algebraic variety. Then X is *toric* if and only if there exists a dense open algebraic torus $T \subseteq X$, such that the action

$$T \times T \to T \text{ extends to an action } T \times X \to X,$$

where $T := \text{Spec}(k[x_1^{\pm 1}, ..., x_n^{\pm 1}])$

Here, 'Spec' is a functor that takes a finitely generated k-algebra, and gives us an affine variety. For any finitely generated k-algebra $k[X]$, we know that $k[X] \cong k[t_1, ..., t_n]/I$, where I is an ideal in $k[t_1, ..., t_n]$. Then

$$\text{Spec}(k[X]) = \mathbb{V}(I).$$

For example $\mathbb{C}^2$ is a toric variety, containing a dense copy of the algebraic torus $(\mathbb{C}^*)^2$, and it's action on itself by multiplication clearly extends to $\mathbb{C}^2$.

Definition 4.6. (Local Approach.) Let X be a variety. Then X is *toric* if its combinatorial data can be encoded in a polyhedral fan.

Given a polyhedral fan $\Sigma \subset \mathbb{R}^n$, we will see that each cone σ defines an affine toric variety U_σ. This collection of affine toric varieties glue together along appropriate subvarieties U_τ, where τ is a cone in Σ of lower dimension. The resulting variety is a toric variety. So given a variety X, if we can find a fan Σ such that $X = X_\Sigma$, then X is toric.

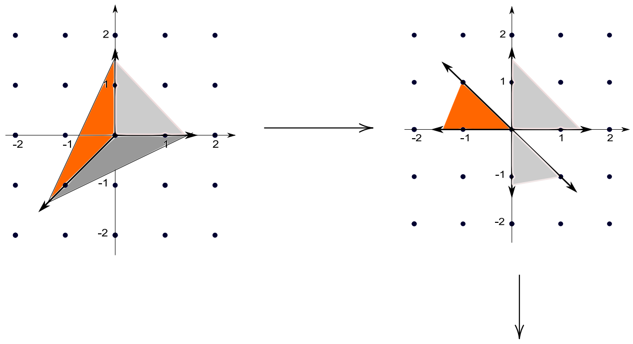
Hence for a toric variety X_Σ, we have a one to one correspondence between affine toric subvarieties U_σ, and cones σ in the fan Σ. A summary of how to find one from the other can be seen in the diagram below. Starting with a cone σ, we consider its *dual cone* $\sigma^\vee$ on the lattice $M := \mathbb{Z}^2$. Now there exists a finite subset $\{(m_i, n_i); 1 \leq i \leq r\} \subset \sigma \cap M$ such that the points of the set $\sigma^\vee \cap M$ can be expressed as a sum

$$\sum_{i=1}^{r} a_i(m_i, n_i),$$

where $a_i \in \mathbb{N}$. Then we can consider the coordinate ring

$$\mathbb{C}[\sigma \cap M] = \mathbb{C}[x^{m_i} y^{n_i}; 1 \leq i \leq r],$$

and finally take 'Spec' of it to get the corresponding affine variety U_σ.



$$\mathrm{Spec}(\mathbb{C}[U_{orange}]) \cong \mathbb{C}^2 \longleftarrow \mathbb{C}[U_{orange}] = \mathbb{C}[x^{-1}, x^{-1}y]$$

The following result ensures that 'gluing' U_σ and $U_{\sigma'}$ along U_τ, where τ is a face of each, makes sense.

Lemma 4.7. *Let τ, σ be cones in Σ, such that τ is a face of σ. Then U_τ is an open affine subset of U_σ.*

Proof. See [4], Lemma from Section 1.3. □

For example, the single cone generated by $(1, 0)$ and $(0, 1)$ in $\mathbb{R}^2_{\geq 0}$, together with all of its faces, is a polyhedral fan that encodes $\mathbb{C}^2$. This 'local approach' definition will be of the most use to us, as the proof of our theorem hinges on the fact that this polyhedral fan is the same as the Groebner fan. We have already seen in the previous subsection that $\mathbb{C}^2$ is an affine toric variety. In the following examples, we will consider some projective toric varieties.

Complex projective n-space can be thought of as the set of lines in $\mathbb{C}^{n+1}$ that pass through the origin. More formally,

$$\mathbb{P}^n_{\mathbb{C}} := \{[z_0 : z_1 : ... : z_n] : (z_0, z_1, ..., z_n) \neq \mathbf{0}\}/\mathbb{C}^*.$$

For each $n \in \mathbb{N}$, this is a toric variety. In the next two examples, we will consider the cases $n = 1, 2$.

Example 4.8. When $n = 1$ we have

$$\mathbb{P}^1_{\mathbb{C}} := \{[z_0 : z_1] : (z_0, z_1) \neq (0, 0)\}/\mathbb{C}^*.$$

It is determined by 2 co-ordinate charts

$$U_0 = [1 : z_1/z_0], \quad U_1 = [z_0/z_1 : 1].$$

These are isomorphic to $\mathbb{C}$, and are 'glued' together by the map

$$\phi_{0,1} : U_0 \setminus \{z_1/z_0 = 0\} \to U_1 \setminus \{z_0/z_1 = 0\} : [1 : x] \to [1/x : 1].$$

The charts have coordinate rings

$$\mathbb{C}[x], \quad \mathbb{C}[x^{-1}],$$

which can both be represented by a cone on the lattice $M := \mathbb{Z}$. This is because the lattice M can be thought of as the set of Laurent polynomials, where the point a corresponds to the monomial x^a. For example, the coordinate ring $\mathbb{C}[x]$ can be represented by the cone $\sigma_0^\vee$ with generators 1. All the monomials in $\mathbb{C}[x]$ correspond to point on the cone generated by 1. The cones $\sigma_0^\vee, \sigma_1^\vee$ are shown in the figure below, where $\sigma_1^\vee$ is the cone from the origin pointing in the negative direction.

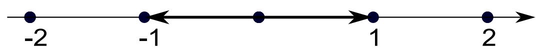


Figure 8. Charts of $\mathbb{P}^1_{\mathbb{C}}$.

The cones meet at the origin, which is a cone corresponding to the coordinate ring $\mathbb{C}[x, x^{-1}]$. So the copies of $\mathbb{C}^2$ parameterised by the cones $\sigma_0^\vee, \sigma_1^\vee$ are glued along $\mathrm{Spec}(\mathbb{C}[x, x^{-1}]) \cong \mathbb{C}^*$.

It is now convenient to consider the dual cones of the two cones above, drawn onto the dual lattice $N := Hom(M, \mathbb{Z})$. N is also isomorphic to $\mathbb{Z}$, since every linear map from $M = \mathbb{Z} \to \mathbb{Z}$ can be written as a 1×1 matrix with entry in $\mathbb{Z}$. The dual of a cone $\sigma^\vee$ is the set of vectors

$$\sigma := \{\mathbf{v} \in \mathbb{R}^2 : \mathbf{v}.\mathbf{u} \geq 0, \forall \mathbf{u} \in \sigma^\vee\}.$$

In this case the cones σ_1, σ_2 are the same as their duals, but in more complicated examples this is not the case, as we shall see later. Hence the cones in Figure 8 form a polyhedral fan in $\mathbb{R}$, with each cone corresponding to an affine chart of $\mathbb{P}^1_{\mathbb{C}}$. This is called the toric fan for $\mathbb{P}^1_{\mathbb{C}}$, hence $\mathbb{P}^1_{\mathbb{C}}$ is a toric variety.

Example 4.9. The complex projective plane is a very similar example, it can be thought of as the set of lines in $\mathbb{C}^3$ that pass through the origin. Its co-ordinate charts are

$$U_0 = [1 : z_1/z_0 : z_2/z_0], \quad U_1 = [z_0/z_1 : 1 : z_2/z_1] \quad U_2 = [z_0/z_2 : z_1/z_2 : 1].$$

These are all isomorphic to $\mathbb{C}^2$, and are 'glued' together by maps of the form

$$\phi_{0,1} : U_0 \setminus \{z_1/z_0 = 0\} \to U_1 \setminus \{z_0/z_1 = 0\} : [1 : x : y] \to [1/x : 1 : x/y]$$

The charts have coordinate rings

$$\mathbb{C}[x, y], \quad \mathbb{C}[x^{-1}, yx^{-1}], \quad \mathbb{C}[y^{-1}, xy^{-1}],$$

which can all be represented by a cone on the lattice $M := \mathbb{Z}^2$, since we are now considering Laurent monomials in 2 variables, where the point (a, b) corresponds to the monomial $x^a y^b$. For example, the coordinate ring $\mathbb{C}[x, y]$ can be represented by the cone $\sigma_0^\vee$ with generators $(1, 0), (0, 1)$. All the monomials in $\mathbb{C}[x, y]$ correspond to points within the cone generated by $(1, 0)$ and $(0, 1)$. The cones $\sigma_0^\vee, \sigma_1^\vee, \sigma_2^\vee$ are shown in the figure below.

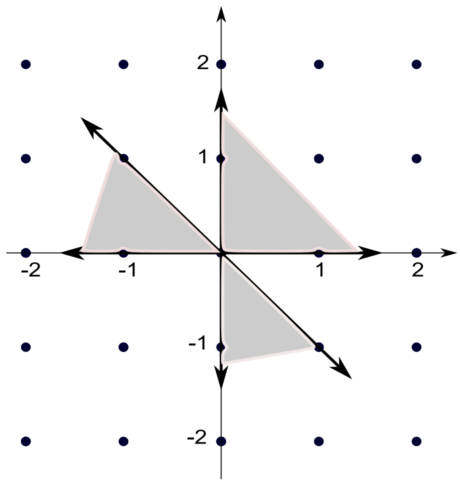


FIGURE 9. Charts of $\mathbb{P}^2_{\mathbb{C}}$.

As in the $n = 1$ case, we consider the dual cones, on the dual lattice $N := Hom(M, \mathbb{Z})$. N is also isomorphic to $\mathbb{Z}^2$, since every linear map from $M = \mathbb{Z}^2$ to $\mathbb{Z}$ can be written as a 2×1 matrix with entries in $\mathbb{Z}$.

The three dual cones are shown in the figure below left. They form a polytopal fan for the triangle shown below right, with each cone corresponding to an affine chart of $\mathbb{P}^2_{\mathbb{C}}$. This is called the toric fan for $\mathbb{P}^2_{\mathbb{C}}$, so $\mathbb{P}^2_{\mathbb{C}}$ is also a toric variety.

Example 4.10. A slightly more complicated example of a toric variety is weighted projective space. This is the projective variety of the form

$$\mathbb{P}_{\mathbb{C}}(a_1, ..., a_n) := \mathbb{C}^n \setminus \{0\}/\mathbb{C}^*,$$

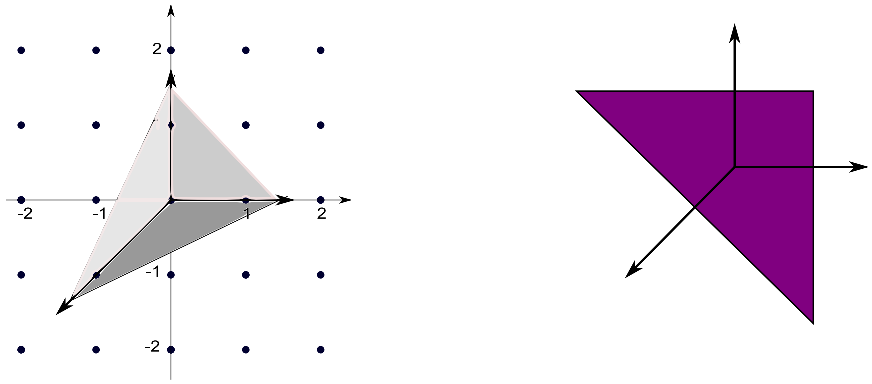
where $\mathbb{C}^*$ acts by

$$t \cdot (x_1, ..., x_n) = (t^{a_1}x_1, ..., t^{a_n}x_n).$$

A simple case of this is $\mathbb{P}_{\mathbb{C}}(1, 1, 2)$. It has coordinate charts

$$U_0 = [1 : z_1/z_0 : z_2/z_0^2], U_1 = [z_0 : 1 : z_2/z_1^2], U_2 = [z_0/\sqrt{z_2} : z_1/\sqrt{z_2} : 1].$$

This is because, for example, in order to move from $[z_0 : z_1 : z_2]$ to having a 1 in the first coordinate we must multiply by $1/z_0$, and this yields $[1 : z_1/z_0 : z_2/z_0^2]$ since the last coordinate has 'double weight'. To get a 1 in the 3rd coordinate,



(a) Toric fan

(b) State polytope with outer normal vectors.

FIGURE 10. The Toric Fan of $\mathbb{P}^2_{\mathbb{C}}$ and its State Polytope.

we can either divide by $1/\sqrt{z_2}$ or $1/(-\sqrt{z_2})$, and these give equivalent charts. These three charts are glued by maps of the form

$$\phi_{(0,1)} : U_0 \setminus \{z_1/z_0 = 0\} \to U_1 \setminus \{z_0/z_1 = 0\}, [1 : x : y] \mapsto [x^{-1} : 1 : yx^{-2}.]$$

The charts have coordinate rings

$$\mathbb{C}[U_0] = \mathbb{C}[x, y], \quad \mathbb{C}[x^{-1}, yx^{-2}], \quad \mathbb{C}[x^2y^{-1}, xy^{-1}, y^{-1}].$$

As before, these can be represented by cones on the lattice M. Note that the third coordinate ring has 3 variables, which are linearly dependant. This will be discussed in more detail in Example 4.12.

As before we now move to the dual lattice, and obtain a toric fan, and its state polytope. Hence $\mathbb{P}_{\mathbb{C}}(1, 1, 2)$ is also a toric variety.

4.3. **Cyclic Quotient Surface Singularities.** A *surface* is a variety of dimension 2. In this section we will restrict attention to toric surfaces. We have seen two examples of these already, namely $\mathbb{P}^2_{\mathbb{C}}$ and $\mathbb{P}_{\mathbb{C}}(1, 1, 2)$. By Definition 4.6 each toric variety is made up of affine toric varieties 'glued' together in a particular way. So if we can understand all the affine varieties, this will shed a lot of light on the toric variety that they make up. The example of ${\mathbb{P}_{\mathbb{C}}}^2$ is a very simple one: all the charts are isomorphic to $\mathbb{C}^2$, and are hence smooth. But what about other spaces? The following proposition gives a classification of all affine toric surfaces.

Proposition 4.11. *U_σ is either isomorphic to $\mathbb{C}^2$, or the cyclic quotient singularity of type $\frac{1}{r}(1, a)$, for some a, r relatively prime.*

Before going any further, we need to define the cyclic quotient singularity of type $\frac{1}{r}(1, a)$, and give some examples of it.

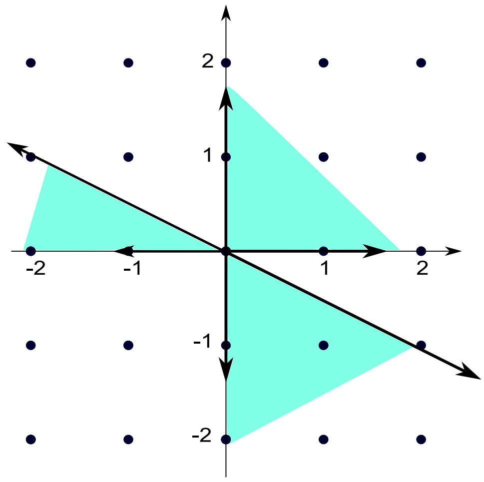
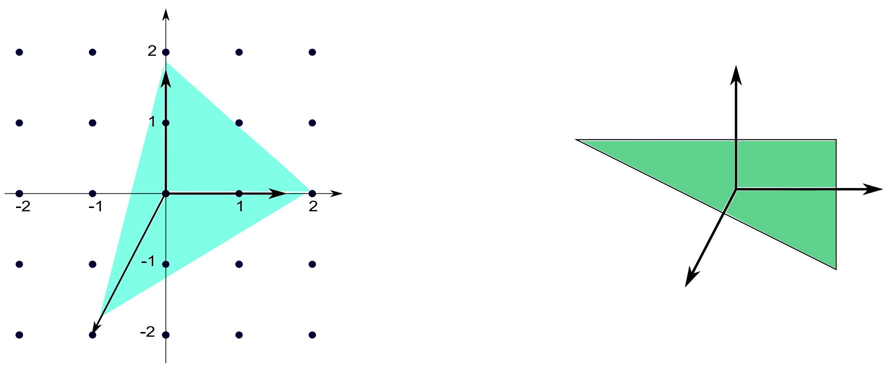


FIGURE 11. Charts of $\mathbb{P}_{\mathbb{C}}(1, 1, 2)$.



(a) Toric fan

(b) State polytope with outer normal vectors.

FIGURE 12. The Toric Fan of $\mathbb{P}_{\mathbb{C}}(1, 1, 2)$ and its State Polytope.

Let G be the finite cyclic group of order r, generated by the matrix

$$g := diag(\epsilon, \epsilon^a),$$

where ϵ is a primitive rth root of unity, and $gcd(a, r) = 1$. G acts on $\mathbb{C}^2$, and this determines an action on $\mathbb{C}[x, y]$, the coordinate ring of $\mathbb{C}^2$, with

$$g \cdot x = \epsilon x, \quad g \cdot y = \epsilon^a y.$$

Clearly a monomial $x^k y^l$ is invariant under the action if and only if $k + al \equiv 0 \bmod r$. Let $\overline{M}$ denote the lattice of Laurent monomials with dual lattice $\overline{N}$, and let M be the sublattice corresponding to the monomials that are invariant under the action of G. Hence, given the cone $\sigma^\vee$ generated by $(1,0)$ and $(0,1)$, the intersection of $\sigma^\vee$ with M corresponds to the coordinate ring

$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x,y]^G,$$

the coordinate ring of $\mathbb{C}^2/G$.

Example 4.12. (Cyclic Quotient Singularity of type $1/2(1,1)$). Here $r = 2$, $a = 1$, so G is the group generated by the matrix

$$g := diag(-1,-1),$$

with -1 the primitive 2nd root of unity. G acts on $\mathbb{C}[x,y]$ by

$$g \cdot x \mapsto -x, \quad g \cdot y \mapsto -y.$$

The invariant monomials $x^a y^b$ are those with $a + b \equiv 0 \bmod 2$, and these correspond to the points of $\mathbb{Z}^2_{\geq 0}$ with both odd or both even coordinates. These are given by $\sigma^\vee \cap M$. These are shown in the diagram below, where $\sigma^\vee$ is the shaded cone.

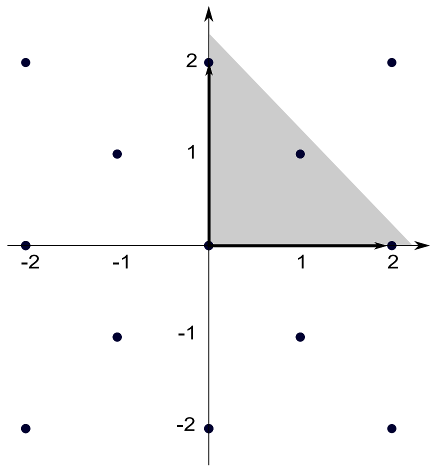


FIGURE 13. $\sigma^\vee \cap M$

Hence $\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x,y]^G = \mathbb{C}[x^2, xy, y^2]$, since all of the G invariant monomials are products of these.

This example has a strong link with the example of weighted projective space $\mathbb{P}_\mathbb{C}(1,1,2)$ that we saw earlier. In fact, $\sigma^\vee \cap M$ corresponds to the singular chart of $\mathbb{P}_\mathbb{C}(1,1,2)$, whose coordinate ring had 3 variables. The only difference is that in one example, we have used a standard lattice and deformed the cone, and in the other we have deformed the lattice and left the cone alone. The two fans shown below are both toric fans for $\mathbb{P}_{\mathbb{C}}(1,1,2)$, with the shaded cones corresponding to the singular cone. It is clear that in both cases their generators do not form a lattice basis, while the generators for the other (non-singular) cones do.

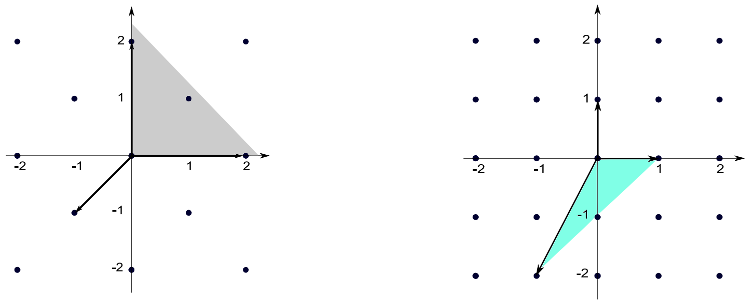


Figure 14. Alternative Presentations of the toric fan of $\mathbb{P}_{\mathbb{C}}(1,1,2)$

In the above example $\mathbb{C}[x,y]^G$ is a finitely generated $\mathbb{C}$-algebra, and hence defines an affine variety. Explicitly,

$$\mathbb{C}[x,y]^G = \mathbb{C}[x^2, xy, y^2] \cong \mathbb{C}[r,s,t]/\langle rt - s^2\rangle,$$

and this defines the affine variety

$$\mathbb{V}(\langle rt - s^2\rangle) = \{(k,l,m) \in \mathbb{C}^3 : km = l^2\}.$$

However, how can we be sure that this is true for every example? In fact, in Theorem 4.15 we will prove that this algebra is always finitely generated as a $\mathbb{C}$-algebra, so we can indeed take ‘Spec’ of it. We first establish some preliminary results.

So by the above proposition, we can deduce that for every finite group G acting on $\mathbb{C}^2$ we can define an affine variety $U = \mathrm{Spec}(\mathbb{C}[x,y]^G)$ which corresponds to the set of G-orbits $\mathbb{C}^2/G$.

We use the *Jung-Hirzebruch fraction* of $\frac{r}{r-a}$ to find the generators of $\mathbb{C}[x,y]^G$. For any fraction a/b, we can find an expression of the form

$$\frac{a}{b} = c_1 - \cfrac{1}{c_2 - \cfrac{1}{c_3 - \cfrac{1}{\dots - \frac{1}{c_t}}}}.$$

This is called the Jung-Hirzebruch fraction for a/b and is usually written as $[c_1, c_2, \dots, c_t]$ for convenience.

Lemma 4.13. *Let $[c_1, c_2, ...c_k]$ be a Jung Hirzebruch fraction, and let $p_i/q_i = [c_1, ..., c_i]$ be the ith convergent, for each $i \leq k$. Then*

$$\begin{pmatrix} -q_{i-1} & p_{i-1} \\ -q_i & p_i \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & c_i \end{pmatrix} \cdots \begin{pmatrix} 0 & 1 \\ -1 & c_1 \end{pmatrix}.$$

Proof. Proof is by induction on i. The 0th convergent is not defined, and so the $i = 1$ case doesn't make sense. We begin with the $i = 2$ case.

$$[c_1, c_2] = c_1 - \frac{1}{c_2} = \frac{c_1c_2 - 1}{c_2},$$

and

$$\begin{pmatrix} 0 & 1 \\ -1 & c_2 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & c_1 \end{pmatrix} = \begin{pmatrix} -1 & c_1 \\ -c_2 & -1 + b_1b_2 \end{pmatrix} = \begin{pmatrix} -q_1 & p_1 \\ -q_2 & p_2 \end{pmatrix}$$

as required. Now suppose the result holds for all Jung-Hirzebruch fractions of length $i = j - 1$. Then

$$[c_1, ..., c_{j-1}] = \frac{c_{j-1}(p_{j-2}) - p_{j-3}}{c_{j-1}(q_{j-2}) - q_{j-3}}.$$

Also

$$[c_1, ..., c_j] = c_1 - \frac{1}{[c_2, ..., c_j]},$$

where the denominator is a Jung-Hirzebruch fraction of length $j - 1$. Hence

$$\begin{aligned} [c_1, ..., c_j] &= c_1 - \frac{c_j(q_{[c_2,...,c_{j-1}]}) - q_{[c_2,...,c_{j-2}]}}{c_j(p_{[c_2,...,c_{j-1}]}) - p_{[c_2,...,c_{j-2}]}} \\ &= \frac{c_1c_jp_{[c_2,...,c_{j-1}]} - c_1q_{[c_2,...,c_{j-2}]} - c_jp_{[c_2,...,c_{j-1}]} + p_{[c_2,...,c_{j-2}]}}{c_j(p_{[c_2,...,c_{j-1}]}) - p_{[c_2,...,c_{j-2}]}}. \end{aligned}$$

By the inductive argument,

$$\frac{p_{j-1}}{q_{j-1}} = c_1 - \frac{q_{[c_2,...,c_{j-1}]}}{p_{[c_2,...,c_{j-1}]}} = \frac{b_1p_{[c_2,...,c_{j-1}]} - q_{[c_2,...,c_{j-1}]}}{p_{[c_2,...,c_{j-1}]}},$$

and hence we deduce that

$$[c_1, ..., c_j] = \frac{c_jp_{j-1} - p_{j-2}}{c_jq_{j-1} - q_{j-2}} = \frac{p_j}{q_j}.$$

Now

$$\begin{pmatrix} 0 & 1 \\ -1 & c_j \end{pmatrix}\begin{pmatrix} -q_{j-2} & p_{j-1} \\ -q_{j-1} & p_{j-1} \end{pmatrix} = \begin{pmatrix} -q_{j-1} & p_{j-1} \\ q_{j-2} - c_jq_{j-1} & c_jp_{j-1} - p_{j-2} \end{pmatrix},$$

and so by the above the result follows. □

The embedding is given by the following proposition.

Proposition 4.14. *Writing $r/(r-a) = [c_1, ..., c_k]$ as above, and setting $u_0 = (r, 0)$ and $u_1 = (r-a, 1)$, we can define*

$$u_{i+1} := c_i.u_i - u_{i-1}.$$

Then the set $\{u_0, ..., u_{k+1}\}$ form the convex hull of points in $\sigma^\vee \cap M \setminus \{0\}$, and

$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[\chi^{u_0}, ..., \chi^{u_{k+1}}] \cong \mathbb{C}[x_1, ..., x_{k+1}]/I.$$

Proof. The vectors $u_0 = (r, 0), u_1 = (r-a, 1)$ correspond to the monomials $x^r, x^{r-a}y \in M$. These vectors form a $\mathbb{Z}$-basis for M, since every G-invariant monomial $x^c y^d$ satisfies $c + ad = kr$ for some $k \in \mathbb{Z}$. So $c = kr - ad = d(r-a) + (k-d)r$, i.e. $(c, d) = d(r-a, 1) + (k-d)(r, 0)$. Therefore we deduce that these vectors form the convex hull for the cone that they generate. We now apply a change of basis matrix

$$\begin{pmatrix} r-a & 1 \\ c_1(r-a) - r & c_1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & c_1 \end{pmatrix} \begin{pmatrix} r & 0 \\ r-a & 1 \end{pmatrix},$$

giving rise to the pair $u_1 = (r-a, 1)$ and $u_2 = (c_1(r-a)-r, c_1)$ which also form a $\mathbb{Z}$ basis for M. Then $\{u_0, u_1, u_2\}$ form a convex hull of the lattice points in the cone spanned by $\{u_0, u_2\}$. By induction we deduce that the $\{u_0, ..., u_{k+1}\}$ form the convex hull of lattice points in the cone generated by $\{u_0, u_{k+1}\}$. By 4.13 it follows that the change of basis from $\{u_0, u_1\}$ to $\{u_k, u_{k+1}\}$ is given by

$$\begin{pmatrix} r - ap_{k-1} - rq_{k-1} & p_{k-1} \\ 0 & r \end{pmatrix} = \begin{pmatrix} -q_{k-1} & p_{k-1} \\ (a-r) & r \end{pmatrix} \begin{pmatrix} r & 0 \\ r-a & 1 \end{pmatrix}.$$

Hence $u_{k+1} = (0, r)$. In particular the whole cone is generated by $\{u_0, u_{k+1}\}$, so the convex hull of all the lattice points in the cone $\sigma^\vee \setminus \{0\}$. These lattice points generate the semigroup $S_\sigma = \sigma^\vee \cap M$, since adjacent pairs base M, which gives

$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[\chi^{u_0}, ..., \chi^{u_{k+1}}].$$

□

It is also worth noting that, since every pair $\{u_i, u_{i+1}\}$ is a lattice basis for M, then the volume of the parallelogram generated by u_i and u_{i+1} must equal r, the *size* of the lattice M. Now the final pair u_k, u_{k+1} base M, so the parallelogram spanned by $((r-a)p_{k-1} - rq_{k-1}, p_{k-1})$ and $(0, r)$ must have area r. This implies that $r \cdot (r-a)p_{k-1} - rq_{k-1} = r$, i.e. $(r-a)p_{k-1} - rq_{k-1} = 1$ which in turn implies that $p_{k-1} \cdot (r-a) \equiv 1 \mod r$. Hence $u_k = (1, \beta)$ where $(r-a)\beta \equiv \mod r$.

We are now in a position to prove that $\mathbb{C}[x, y]^G$ is a finitely generated $\mathbb{C}$-algebra.

Theorem 4.15. *For $G \subset GL(2, \mathbb{C})$ a cyclic group, as above, the algebra $\mathbb{C}[x, y]^G$ is finitely generated.*

Proof. $\mathbb{C}[x,y]^G$ is the $\mathbb{C}$-algebra generated by the monomials of $\mathbb{C}[x,y]$ that are invariant under the action of G. This is precisely the monomials in the set $\sigma \cap M$, since M is the set of invariant Laurent monomials, and σ is the cone of monomials in $\mathbb{C}[x,y]$. So

$$\mathbb{C}[x,y]^G = \mathbb{C}[\sigma \cap M],$$

and the result follows from Proposition 4.14. □

Now by the following proposition, we see that we have found a closed embedding of $\mathbb{C}^2/G$ into $\mathbb{C}^{k+1}$.

Proposition 4.16. *Let $G \subset GL(2,\mathbb{C})$ be a finite group. Then the inclusion $\mathbb{C}[x,y]^G \hookrightarrow \mathbb{C}[x,y]$ induces a surjective homomorphism*

$$\pi : {\mathbb{A}_{\mathbb{C}}}^2 \to Spec(\mathbb{C}[x,y]^G)$$

with $\pi(p) = \pi(p') \Leftrightarrow G \cdot p = G \cdot p'$. Hence $Spec(\mathbb{C}[x,y]^G) = \mathbb{C}^2/G$, which parameterises all G-orbits in $\mathbb{C}^2$.

Proof. See [**?**], Proposition 2.4. □

Example 4.17. (Cyclic Quotient Singularity of type $\frac{1}{17}(1,7)$.) Let G be the cyclic subgroup of $GL(2,\mathbb{C})$ generated by the matrix

$$g := diag(\epsilon, \epsilon^7),$$

where ϵ is a primitive 17th root of unity.

$$\frac{r}{r-a} = \frac{17}{10} = 2 - \frac{1}{4 - \frac{1}{2 - \frac{1}{2}}} = [2,4,2,2].$$

$u_0 = (17,0)$, $u_1 = (10,1)$ and from these we can deduce that

- $u_2 = 2(10,1) - (17,0) = (3,2)$
- $u_3 = 4(3,2) - (10,1) = (2,7)$
- $u_4 = 2(2,7) - (3,2) = (1,12)$
- $u_2 = 2(1,12) - (2,7) = (0,17)$.

These points generate the lattice of G-invariant monomials, as shown below. By the above proposition, we can deduce that

$$\mathbb{C}[\sigma^\vee \cap M] = \mathbb{C}[x^{17}, x^{10}y, x^3y^2, x^2y^7, xy^{12}, y^{17}] = \mathbb{C}[z_1, ...z_5]/I,$$

where I is generated by the equations

$$rank \begin{pmatrix} z_1 & z_2 & z_3^3 & z_3^2 z_4 & z_3^2 z_5 \\ z_2 & z_3 & z_4 & z_5 & z_6 \end{pmatrix} \leq 1.$$

Hence we obtain the embedding

$$\mathbb{V}(I) \hookrightarrow \mathbb{C}^5.$$

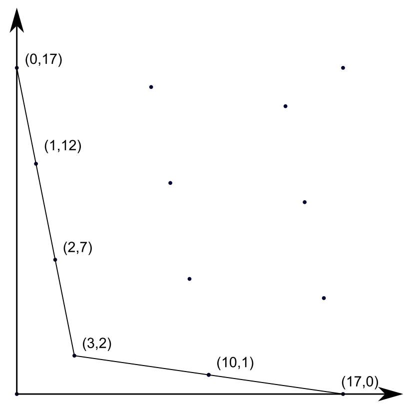


FIGURE 15. $\sigma^\vee \cap M$

We will now prove the theorem stated at the beginning of this subsection, namely that the variety U_σ associated with the cone σ on the lattice N, is either $\mathbb{C}^2$, or the cyclic quotient singularity of type $\frac{1}{r}(1,a)$ for some a, r relatively prime.

Proof of 4.11. Fix a lattice $N \cong \mathbb{Z}^2$ in $\mathbb{C}^2$, and a cone σ. Choose primitive generators $\mathbf{v_1}, \mathbf{v_2}$ of σ. We can then choose coordinates such that $\mathbf{v_2} = (0,1)$ and $\mathbf{v_1} = (r,s)$ for $r, s \geq 0$. Thinking of the coordinates of points in $\mathbb{C}^2$ as a column vector, we can apply the lattice automorphism

$$\theta \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ c & 1 \end{pmatrix} \begin{pmatrix} r \\ s \end{pmatrix} = \begin{pmatrix} r \\ cr + s \end{pmatrix},$$

which allows us to add any multiple of r onto s, so we choose $s = -a$ for $0 \leq a < r$. If $a = 0$, then $\mathbf{v_1} = (r, 0)$, and hence $r = 0$ since $\mathbf{v_1}$ is primitive. This implies that σ is the non-negative quadrant, and so $U_\sigma \cong \mathbb{C}^2$. Otherwise we have that $0 < a < r$, and since $\mathbf{v_1}$ is primitive, a and r are coprime. Now we can apply the matrix

$$\frac{1}{r} \begin{pmatrix} 1 & 0 \\ a & r \end{pmatrix}.$$

This alters both the lattice N and the cone σ such that the resulting lattice is generated by $\frac{1}{r}(1,a)$ and $(0,1)$, and the resulting cone has generators $(1,0)$ and $(0,1)$. This data determines the cyclic quotient singularity of type $\frac{1}{r}(1,a)$ as required. $\square$

4.4. **Resolution of Singularities.** In the above examples we showed how U_σ could be embedded into $\mathbb{C}^n$, where $U_\sigma \cong \mathbb{C}^2/G$, and by Proposition 4.11

every affine chart U_σ of a toric surface X is isomorphic to $\mathbb{C}^2/G$ for some cyclic group

$$G = \langle diag(\epsilon, \epsilon^a)\rangle,$$

where ϵ is a primitive rth root of unity, and $gcd(a, r) = 1$. However, when G is non-trivial, U_σ is singular. This is seen when we consider the dual cone σ in the dual space. It's generators do not form a lattice basis for N, the dual lattice of M, and this implies that they do not generate the cone of $\mathbb{C}^2$.

We can resolve the singularities of U_σ by subdividing the cone corresponding to it, so that the generators of each subcone do form a lattice basis for N. We can compute this by using the Jung-Hirzebruch continued fraction of r/a in the following way.

Proposition 4.18. *Write $r/a = [b_1, ..., b_s]$, and set $v_0 = (0, 1)$, $v_1 = \frac{1}{r}(1, a)$. Then define*

$$v_{i+1} = b_i v_i - v_{i-1} \text{ for } i = 1, ...s.$$

Then the set $\{v_0, ..., v_{s+1}\}$ form the convex hull of points in $\sigma \cap N \setminus \{0\}$, and the fan Σ obtained by subdividing σ along the rays $\rho_i = \mathbb{R}_{\geq 0} v_i$ determines the minimal resolution

$$\phi : U_\Sigma \to U_\sigma \cong \mathbb{C}^2/G.$$

Proof. In the proof of Proposition 4.14 we calculated the points on the convex hull of $\sigma^\vee \cap M \setminus \{0\}$ by beginning with the two points nearest the y-axis and using the continued fraction expansion to find all the other points. We now have the 2 points closest to the x-axis, and we want to find a continued fraction r/α which will give these as the final two points. By the remark following the proof, this α satisfies $\alpha < r$ and $\alpha \cdot a \equiv 1 \bmod r$. But given fractions $r/\alpha, r/\beta$ such that $\alpha \cdot \beta \equiv 1 \bmod r$ implies that if $r/\alpha = [b_1, ..., b_n]$, then $r/\beta = [b_n, ..., b_1]$ So we will find the same points by starting with the points $(0, r)$ and $(1, a)$ and working clockwise with the continued fraction expansion of r/a. So in a similar way to 4.14 we see that $\{v_0, ...v_{k+1}\}$ form the convex hull of points in $\sigma \cap N \setminus \{0\}$. By construction, each v_i lies in N and consecutive pairs generate the lattice, so the toric variety X_Σ equivalent to the fan obtained by drawing a line ρ_i through each v_i is smooth. This is the minimal resolution of U_σ.

Each $v_i = \frac{1}{r}(\alpha_i, \beta_i)$ for some $\alpha_i, \beta_i \in \mathbb{C}$. Let σ_i be the 2-dimensional cone spanned by the vectors v_i and v_{i+1}. Then we can cover the minimal resolution by affine charts by setting $U_{\sigma_i} = \text{Spec}(\mathbb{C}[\xi_i, \eta_i])$, where $\xi_i = x^{\beta_i}/y^{\alpha_i}$ and $\eta_i = y^{\alpha_{i+1}}/x^{\beta_{i+1}}$. □

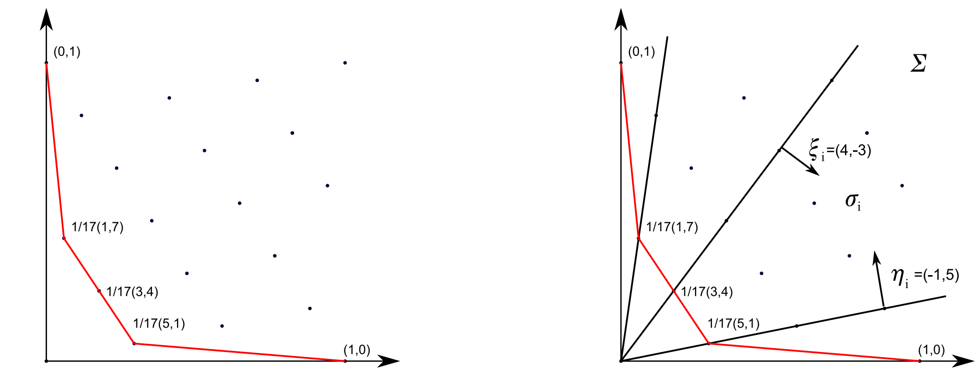
Example 4.19. Let us consider the cyclic quotient singularity of type $\frac{1}{17}(1, 7)$ again.

$$\frac{r}{a} = \frac{17}{7} = 3 - \frac{1}{2 - \frac{1}{4}} = [3, 2, 4].$$

We have that $v_0 = (0, 1)$ and $v_1 = \frac{1}{17}(1, 7)$, and we obtain that

- $v_2 = \frac{3}{17}((1,7) - (0,17)) = \frac{1}{17}(3,4)$
- $v_3 = \frac{2}{17}((3,4) - (1,7)) = \frac{1}{17}(5,1)$
- $v_4 = \frac{4}{17}((5,1) - (3,4)) = (1,0)$.

These points form the convex hull of points in $\sigma \cap N \setminus \{0\}$, as shown on the graph below left. The line through these points is sometimes called the Newton Boundary.



(a) $\sigma \cap N$ with Newton Boundary. (b) The Resolved Cone.

FIGURE 16. Resolution of the Singularity $\frac{1}{17}(1,7)$.

With the above proposition, we have a method for subdividing a singular cone in a fan into smaller non singular cones. We now show how to cover this 'resolved cone' with affine charts. The figure on the right shows the resolved cone for the above example.

The 'extra lines' on our cone all have primitive generators of the form $\frac{1}{r}(\alpha_i, \beta_i)$, and hence the cone σ_i corresponds to the chart with coordinate ring $\mathbb{C}[\xi_i, \eta_i]$, where

$$\xi_i := x^{\beta_i}/y^{\alpha_i}, \eta_i := y^{\alpha_{i+1}}/x^{\beta^{i+1}}.$$

This choice of co-ordinate ring arises by looking at each σ_i, and choosing the Laurent monomials that correspond to the inward pointing vectors. For example, the coordinate ring for the cone σ_i shown above is

$$\mathbb{C}[x^4/y^3, y^5/x].$$

This $\mathbb{C}$-algebra is isomorphic to $\mathbb{C}[z_0, z_2]$, and hence $\mathrm{Spec}(\mathbb{C}[x^4/y^3, y^5/x]) = \mathbb{C}^2$, so this cone does indeed correspond to the affine variety $\mathbb{C}^2$.

5. The Main Theorem.

Having given an introduction to Groebner fans and toric varieties, we now proceed to state prove the theorem that links this all together. We begin by a consideration of finite subgroups in $GL(2,\mathbb{C})$.

5.1. Finite Subgroups of $GL(2,\mathbb{C})$.

Let G be a finite group of order r.

Definition 5.1. A *representation* of G over $\mathbb{C}$ is a homomorphism

$$\rho : G \to GL(n,\mathbb{C})$$

for some $n \in \mathbb{N}$. The *degree* of ρ is defined to be n.

Example 5.2. For example, let $G = \mathbb{Z}_r$ for some non-zero $r \in \mathbb{Z}$. Then the map

$$\rho : \mathbb{Z}_r \to GL(2,\mathbb{C}) : \epsilon \mapsto \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix}$$

is a representation of G over $\mathbb{C}$ of degree 2. This is an isomorphism, and so we can think of the group either as an abstract cyclic group, or as a subgroup of $GL(2,\mathbb{C})$.

Definition 5.3. Two representation ρ and ρ' are said to be *equivalent* if there exists an invertible matrix T such that, for all $g \in G$,

$$\rho(g) = T^{-1}\rho'(g)T.$$

We have already considered the group

$$H = \left\langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix} \right\rangle$$

in a previous section, and it is clear by the above example that for any finite cyclic group G we can find a representation ρ such that $\rho(G) = H$ for some $a, r \in \mathbb{N}$, where $gcd(a,r) = 1$. But what about other finite abelian groups G? Can we also find a representation for these groups with image of this form? Since ρ is a homomorphism, we can deduce that the image of ρ is also a finite abelian group in $GL(2,\mathbb{C})$. The following theorem will help us.

Theorem 5.4. *Every finite abelian group G of $GL(2,\mathbb{C})$ is diagonalisable.*

Proof. See [7], Chapter 9, Proposition 9.11. □

This implies that for every presentation ρ of G with image H, we can find a matrix T such that for all $A_g \in H$, $T^{-1}A_gT$ is a diagonal matrix. So for every presentation $\rho : G \to GL(2,\mathbb{C})$ there exists an equivalent representation ρ' with a diagonal image.

Definition 5.5. An element $g \in GL(n, \mathbb{C})$ is called a *quasi-reflection* if the matrix $g - I$ has rank 1, where I is the $n \times n$ identity matrix. This is equivalent to saying that the action of $\mathbb{C}$ on $\mathbb{C}^n$ by left multiplication by g does not fix any hyperplane of $\mathbb{C}^n$. A group $G \subset GL(2, \mathbb{C})$ is called *small* if it does not contain any quasi-reflections.

It can be shown that an action of a group G generated by quasi-reflections on $\mathbb{C}^2$ doesn't alter its geometry in any way, i.e. that $\mathbb{C}^2/G \cong \mathbb{C}^2$. Thus we can safely ignore them, and for any group G, we simply consider the group G/H, where H is the normal subgroup generated by the quasi-reflections in G.

Theorem 5.6. *Every finite small abelian subgroup of $GL(2, \mathbb{C})$ is cyclic.*

Proof. Let $G \subset GL(2, \mathbb{C})$ be a small finite abelian group of order r. Then by Theorem 5.4 we can diagonalise it, and so, up to isomorphism every $g \in G$ takes the form $diag(\lambda, \mu)$ for some $\lambda, \mu \in \mathbb{C}$. $|G| = r$, and so λ and μ must be rth roots of unity, and so

$$g = \begin{pmatrix} \epsilon^m & 0 \\ 0 & \epsilon^n \end{pmatrix},$$

for some $0 \leq m, n < r$. If $m = 0$, then g would be a quasi-reflection, which contradicts the 'smallness' assumption, so $m > 0$. In fact, since $g^k \in G$ for all $k \in \mathbb{Z}$, and by the same argument, no power $(\epsilon^m)^k$ is trivial for $0 < k < r$, so ϵ^m is a primitive rth root of unity. Hence we can take an appropriate power l of g such that

$$g' = g^l = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix},$$

for some $0 \leq a < r$. Similarly we can deduce that ϵ^a is a primitive rth root of unity, and hence a and r are relatively prime. Now $\langle g' \rangle$ is a cyclic subgroup of order r, and since $|G| = r$, we deduce that $G = \langle g' \rangle$, and hence G is cyclic as required. ☐

Thus, given a finite, small abelian subgroup G in $GL(2, \mathbb{C})$, we may choose coordinates so that it is isomorphic to the subgroup generated by the diagonal matrix $diag(\epsilon, \epsilon^a)$ for ϵ a primitive rth root of unity, and $gcd(a, r) = 1$.

5.2. **Statement and Proof of the Main Theorem.**

Theorem 5.7. *Let G be a small, finite, abelian group and let $I_G \subset \mathbb{C}[x, y]$ be the G-homogeneous ideal generated by the binomials of the form*

$$x^a y^b - x^c y^d \in \mathbb{C}[x, y] : (a - c, b - d) \in M,$$

where M is the lattice of G-invariant Laurent monomials. Then the Groebner fan for the I_G determines a toric variety which is isomorphic to the minimal resolution of $\mathbb{C}^2/G$.

Let H be some multiplicative group contained in $\mathbb{C}^*$. Then we can consider the action of H on $\mathbb{C}^2$ by

$$h \cdot (r, s) = (h^m r, h^n s),$$

where $m, n \in \mathbb{N}$. This action can be represented by the group of matrices

$$\left\{diag(h^m, h^n); h \in H\right\}$$

acting on $\mathbb{C}^2$ by matrix multiplication. This action also defines a dual action on $\mathbb{C}[x, y]$, the coordinate ring of $\mathbb{C}^2$ with

$$h \cdot x \mapsto h^m x, h \cdot y \mapsto h^n y.$$

So for every monomial $x^a y^b$, we have that

$$h \cdot x^a y^b = h^{ma+nb} x^a y^b.$$

Such an action of H on $\mathbb{C}^2$ is equivalent to a grading of $\mathbb{C}[x, y]$. For example, if we take $H = \mathbb{C}^*$, we obtain a $\mathbb{Z}$-grading $\bigoplus \mathbb{C}[x, y]_{\mathbb{Z}}$, with $x^a y^b \in \mathbb{C}[x, y]_{ma+nb}$. If H is a cyclic group of order r, then it's action on $\mathbb{C}[x, y]$ is equivalent to a grading of $\mathbb{C}[x, y]$ into r pieces.

Definition 5.8. Let $G \subset GL(2, \mathbb{C})$ be a finite cyclic group with generator

$$g := diag(\epsilon, \epsilon^a),$$

where ϵ is a primitive rth root of unity. A polynomial $f \in \mathbb{C}[x, y]$ is G *semi-invariant* if all the terms of f lie in the same graded piece of $\mathbb{C}[x, y]$ with respect to the grading defined above by the action of G.

Definition 5.9. Let $G \subset GL(2, \mathbb{C})$ be as above. Then an ideal $I \subseteq \mathbb{C}[x, y]$ is *G-homogeneous* if it is generated by polynomials that are semi-invariant under the action of G.

Before proving Theorem 5.7, we first provide a geometric interpretation of the ideal I_G.

Lemma 5.10. *Let G be a finite cyclic group in $GL(2, \mathbb{C})$ generated by $g := diag(\epsilon, \epsilon^a)$. Let $I_G \subseteq \mathbb{C}[x, y]$ be the G-homogeneous ideal defined in the statement of Theorem 5.7. Then*

$$Spec(\mathbb{C}[x, y]/I_G) = \mathbb{V}(I_G)$$

is the free G-orbit of the point $(1, 1)$.

Proof. Let Z be the G-orbit of $(1, 1)$. Then

$$Z = \{(\epsilon^i, \epsilon^{ai \bmod r}) : 0 \leq i < r\}.$$

Clearly the number of distinct points in the G-orbit Z is $|G| = r$. We want to show that $\mathbb{V}(I_G) = Z$. Let $h = x^c y^d - x^e y^f \in I_G$. Then $(c-e, d-f) \in M$, which implies that $c + ad \equiv e + af \bmod r$. Hence for all $(\epsilon^i, \epsilon^{ai \bmod r}) \in Z$,

$$h(\epsilon^i, \epsilon^{ai \bmod r}) = 0.$$

Since h was an arbitrary generator of I_G, we deduce that $Z \subseteq \mathbb{V}(I_G)$. It is now enough to show that $|\mathbb{V}(I_G)| \leq r$.

$\mathbb{V}(I_G)$ has coordinate ring $R = \mathbb{C}[x,y]/I_G$. $\mathbb{C}[x,y]$ has $\mathbb{C}$-vector space basis consisting of the infinite set of monomials $\{x^\alpha y^\beta : \alpha, \beta \in \mathbb{N}\}$. We know that the points $(r,0), (0,r) \in M$, so $x^r - 1, y^r - 1 \in I_G$, and so R is generated by a subset of $\{x^\alpha y^\beta ; 0 \leq \alpha, \beta < r\}$, and is hence a k-dimensional $\mathbb{C}$-vector space for some finite k. Also, the point $(r-a, 1) \in M$, so each y in R can be replaced by x^{r-a}. Hence the set $1, x, ..., x^{r-1}$ spans R, and $k \leq r$. Since

$$\mathbb{C}[x,y]/I_G \cong \mathbb{C}[\mathbb{V}(I_G)],$$

where $\mathbb{C}[\mathbb{V}(I_G)]$ is the ring of regular functions on $\mathbb{V}(I_G)$, we can conclude that $|\mathbb{V}(I_G)| = k \leq r$. But $|Z| = r$, and $Z \subseteq \mathbb{V}(I_G)$, so $Z = \mathbb{V}(I_G)$ as required.

□

Proof of 5.7. Let G be a small finite abelian group. Then we can find a representation $\rho : G \to GL(2,\mathbb{C})$ with image equal to the group

$$\left\langle \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon^a \end{pmatrix} \right\rangle,$$

where ϵ is a primitive rth root of unity, and $gcd(a,r) = 1$.

By the above,

$$I_G = \left\langle x^a y^b - x^c y^d : (a-c, b-d) \in M \right\rangle.$$

Without loss of generality, we may assume $a \geq c$. Then $x^a y^b - x^c y^d = x^c(x^{a-c} y^b - y^d)$, and $(a-c, b) - (0, d) \in M$, so $x^{a-c} y^b - y^d \in I_G$. Similarly, if $d \geq b$, we have $x^{a-c} - yd - b \in M$, or when $d < b$ $x^{a-c} y^{b-d} - 1 \in M$. $x^a y^b - x^c y^d$ is obtainable from whichever lower degree generator we have found, hence a smaller generating set of I_G is

$$A := A_1 \cup A_2 = \left\{x^s - y^t : (s, r-t) \in M\right\} \cup \left\{x^u y^v - 1 : (u,v) \in M\right\}.$$

For all monomials $x^a y^b \in LT(I_G)$, and term orders $\prec$, we can find a binomial in the above set whose leading term divides $x^a y^b$, so the universal Groebner basis $\mathcal{G}$ is a subset of A.

Let $(s, r-t) \in M$. Then $x^s y^{r-t}$ is invariant under the action of G, which sends $x \to \epsilon^1 x$, and $y \to \epsilon^a y$. This implies that

$$s \equiv at \bmod r.$$

Now points in $N = \mathbb{Z}^2 + \frac{1}{r}(1, a)\mathbb{Z}$ are of the form $\frac{1}{r}(n_1, n_2)$, where $n_1 = k + rm_1, n_2 = ak + rm_2$, for some $k, m_1, m_2 \in \mathbb{Z}$. Clearly

$$n_2 \equiv an_1 \bmod r,$$

so we have a one to one correspondence between binomials $x^s - y^t \in A$ and points $\frac{1}{r}(t, s) \in N$.

The points on the Newton boundary N' form a lattice basis of N, and all points of N contained in the positive quadrant of $\mathbb{R}^2$ are obtained by taking the sum of positive multiples of them. We now establish an equivalence between members of the Groebner basis contained in A_1, and points on the Newton boundary N'. Suppose $\frac{1}{r}(t, s)$ is not contained in N'. Then there exist $\frac{1}{r}(t_i, s_i) \in N', a_i \in \mathbb{Z}_{\geq 0}$ such that

$$\frac{1}{r}(t, s) = \Sigma a_i \frac{1}{r}(t_i, s_i).$$

By an inductive argument, we may assume we have $\frac{1}{r}(t_1, s_1), \frac{1}{r}(t_2, s_2) \in N$ such that

$$\frac{1}{r}(t_1, s_1) + \frac{1}{r}(t_2, s_2) = \frac{1}{r}(t_i, s_i).$$

By the above correspondence, we have binomials $x^{s_1} - y^{t_1}, x^{s_2} - y^{t_2} \in A_1$ such that

$$x^{s_1} - y^{t_1} + x^{s_2} - y^{t_2} = x^s - y^t.$$

Hence $x^s - y^t$ is obtained from other members of A_1 of lower degree in both monomials, and so is not in the reduced Groebner basis. Suppose now that $x^s - y^t \in A_1 \setminus \mathcal{G}$. Then we can find elements of $\mathcal{G}$ that generate $x^s - y^t$. By an inductive argument we can assume $\exists x^{s_1} - y^{t_1}, x^{s_2} - y^{t_2} \in I_G$ such that $s_1 + s_2 = s, t_1 + t_2 = t$. Hence we deduce that

$$\frac{1}{r}(t, s) = \frac{1}{r}(t_1, s_1) + \frac{1}{r}(t_2, s_2), t_1, t_2, s_1, s_2 \geq 0, \frac{1}{r}(t_1, s_1), \frac{1}{r}(t_2, s_2) \in N.$$

So $\frac{1}{r}(t, s)$ are not in N'. Hence we have a one to one correspondence between points on the Newton boundary and binomials in $A_1 \cup \mathcal{G}$.

Let $N' = \{\mathbf{p_i} = \frac{1}{r}(a_i, b_i) : 1 \leq i \leq s\}$ for some $s \in \mathbb{N}$. Let $\mathbf{w} = (a_i, b_i)$. Then by the above $x^{b_i} - y^{a_i} \in \mathcal{G}$ and $in_{\mathbf{w}}(I)$ has a binomial generator $x^{b_i} - y^{a_i}$. However if you take $\mathbf{w}' = (a_i + \delta, b_i)$ where $\delta > 0$ is small, then $in_{\mathbf{w}'}(I)$ has the monomial generator x^{b_i} instead. Similarly for $\mathbf{w}'' = (a_i, b_i + \delta)$ has monomial generator y^{a_i}. So we have two different equivalence classes and a line l_i on the Groebner fan between these two cones, passing through the point $\mathbf{p_i}$. Hence for every $\mathbf{p_i} \in N'$ we have a line l_i on the Groebner fan. These are the only lines on the Groebner fan, since elements in $A_2 \cap \mathcal{G}$ don't give a line, since their leading term is the same for every monomial order. Hence the Groebner fan determines the toric variety isomorphic to the minimal resolution of $\mathbb{C}^2/G$ as required. □

6. The G-Hilbert Scheme

In the preceding section, we established an correspondence between the minimal resolution of $\mathbb{C}^2/G$, and the Groebner fan of the ideal I_G. This implies that the minimal resolution of $\mathbb{C}^2/G$ is understood by I_G, i.e. that all we need to obtain it is the ideal I_G. This ideal is given by the group G, and hence we deduce that we can obtain the minimum resolution of $\mathbb{C}^2/G$ using only information about the group G, via the ideal I_G. The natural question to ask now is, can we can find a description of the minimal resolution of $\mathbb{C}^2/G$ more directly in terms of G? In this section we consider the *G-Hilbert Scheme*, which enables us to construct the minimal resolution more directly, by considering only information about G, namely certain G-homogeneous ideals contained in the polynomial ring $\mathbb{C}[x, y]$. We begin by giving an informal description of the G-Hilbert Scheme.

A *moduli space* is a space in which each point parameterises a given algebro-geometric object, and, moreover, all objects in a given class are parameterised by some point. A simple example is ${\mathbb{P}_{\mathbb{C}}}^2$, which is a moduli space with every point in ${\mathbb{P}_{\mathbb{C}}}^2$ parameterising a straight line through the origin in $\mathbb{C}^3$.

We now introduce a class of algebro-geometric objects called *G-clusters* which are relevant to our moduli problem. For G a finite cyclic group in $GL(2, \mathbb{C})$ of order r, a G-cluster I is an ideal in $\mathbb{C}[x, y]$ such that $\mathbb{C}[x, y]/I$ is an r-dimensional $\mathbb{C}$-vector space with basis elements all in different eigenspaces of the G-action. A *monomial G-cluster* is a G-cluster with monomial generators.

For any abelian group G, we can say that G acts on itself by multiplication. Such an action has only one orbit, and one orbit stabiliser, namely 1_G. For a given field k, the *regular representation* of G is the vector space over k, where all the group elements are taken as basis vectors. From the above it is clear that for any G-cluster I, the coordinate ring $\mathbb{C}[x, y]/I$ is (isomorphic to) the regular representation of G over $\mathbb{C}$.

Definition 6.1. The *G-Hilbert Scheme* is the moduli space of G-clusters, i.e. the points of the G-Hilbert Scheme all correspond to G-clusters.

However, the G-Hilbert scheme is not just a set. In general it carries a complicated algebro-geometric structure, but in this special case of G a finite abelian group in $GL(2, \mathbb{C})$, we will show that it is a toric variety.

Example 6.2. Let us first consider the example of

$$G := \left\{ \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right\} \cong \mathbb{Z}_2.$$

This is the minimal cyclic quotient singularity of type $\frac{1}{2}(1, 1)$ as we saw in Example 4.12, and it has minimal resolution given by the cones spanned by $(2, 0), (1, 1)$ and $(1, 1), (0, 2)$.

The monomial G-clusters for G are the ideals

$$I_1 := \langle x, y^2 \rangle, I_2 := \langle x^2, y \rangle,$$

where $\mathbb{C}[x,y]/I_1$ has vector space basis $\{1, y\}$, and $\mathbb{C}[x,y]/I_2$ has vector space basis $\{1, x\}$. In fact, any G-cluster is just a deformation of either I_1 or I_2, with

$$I_1 := \langle x - a_1 y, y^2 - b_1, xy - a_1 b_1 \rangle, I_2 := \langle x^2 - a_2, y - b_2 x, xy - a_2 b_2 \rangle,$$

where $(a_i, b_i) \in \mathbb{C}^2$. So for each I_i, and each $(a, b) \in \mathbb{C}^2$, we define a G-cluster. For each I_i then, we can parameterise all the G-clusters of this form by a copy of $\mathbb{C}^2$. So in this example, all the G-clusters are parameterised by two copies of $\mathbb{C}^2$, where the origins parameterise the monomial G-clusters.

In particular, the ideal $I_G = \langle x^2 - 1, y^2 - 1, xy - 1 \rangle$, as defined in the previous section always corresponds the ideal defined by $(a_i, b_i) = (1, 1)$. This is easy to see, since we have

$$I_1 := \langle x - y, y^2 - 1, xy - 1 \rangle, I_2 := \langle x^2 - 1, y - x, xy - 1 \rangle,$$

and all the generators of I_1 and I_2 are contained in I_G and vice versa. This tells us that the copies of $\mathbb{C}^2$ which parameterise the G-clusters must glue together in some natural way.

Recall that the minimal resolution of $\mathbb{C}^2/G$ was given by a collection of cones, each corresponding to a copy of $\mathbb{C}^2$ glued together along the edges of the cones. In the following example we will show that the gluing in the G-Hilbert Scheme is precisely the same as the gluing in the minimal resolution of $\mathbb{C}^2/G$.

Example 6.3. Consider the cyclic quotient singularity of type $\frac{1}{3}(1, 2)$. The minimal resolution is shown below on the deformed lattice, and on the straight lattice, as for $\mathbb{P}_{\mathbb{C}}(1, 1, 2)$ in Figure 4.12.

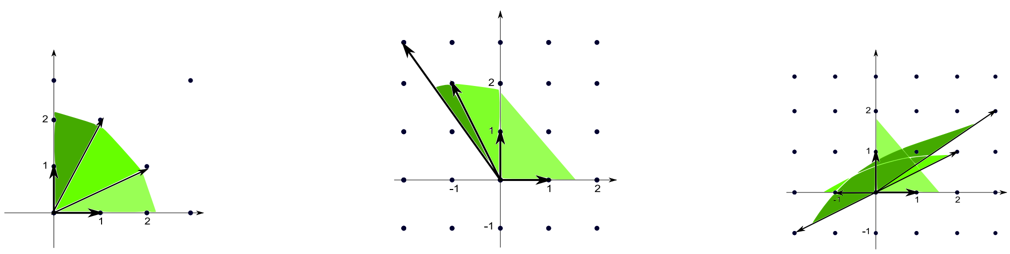


FIGURE 17. Minimal Resolution for $\frac{1}{3}(1, 2)$, and the Dual Picture.

Dualising the second picture gives the dual cones on the far right.

These cones correspond to the coordinate rings

$$\mathbb{C}[x, y], \mathbb{C}[x^2 y, x^{-1}], \mathbb{C}[x^3 y^2, x^{-2} y^{-1}].$$

Three G-clusters for this group are

$$I_1 := \langle x - a_1y^2, y^3 - b_1, xy - a_1b_1\rangle, I_2 := \langle x^2 - a_2y, y^2 - b_12x, xy - a_2b_2\rangle,$$
$$I_3 := \langle x^3 - a_3, y - b_3x^2, xy - a_3b_3\rangle.$$

There is an obvious correspondence between the cones of the minimal resolution and these ideals. These parameterise all the possible G-clusters, but sometimes points in different copies of $\mathbb{C}^2$ parameterise the same G-cluster. These are the points at which the copies of $\mathbb{C}^2$ are 'glued.'

For example $I_1 = I_2$ if and only if $a_1 = b_2^{-1}$ and $a_1b_1 = a_2b_2$, and these imply that $b_1 = a_2b_2^2$. This implies that $\mathbb{C}[a_2, b_2] = \mathbb{C}[a_1^2b_1, a_1^{-1}]$. So we see that the G-cluster $I_i(a_i, b_i)$ and $I_j(a_j, b_j)$ are the same exactly when the two coordinate rings corresponding to them are the same. Hence the copies of $\mathbb{C}^2$ parameterising the G-clusters glue in the same way as the copies of $\mathbb{C}^2$ glue in the minimal resolution.

The following theorem generalises these ideas for all possible G-clusters for the action of a finite cyclic group G on $\mathbb{C}^2$.

Theorem 6.4. *For every finite cyclic group in $GL(2,\mathbb{C})$ and every G-cluster I, the generator of the ideal I can be chosen as the system of 3 equations $\{x^a = \alpha y^c, y^b = \beta x^d, x^{a-d}y^{c-b} = \alpha\beta\}$ where $\alpha, \beta \in \mathbb{C}$ and both x^a and y^c (resp. y^b and x^d) belong to the same eigenspace of the G-action.*

Proof. Let I_Z be a G-cluster, where $|G| = r$. As we saw earlier we can think represent the action of G on $\mathbb{C}^2$ by multiplication by the matrix $g := diag(\epsilon, \epsilon^a)$. By our definition of a G-cluster we have that the $\mathbb{C}$ algebra $\mathbb{C}[x, y]/I_Z$ is a $\mathbb{C}$-vector space with dimension r. There are r different eigenspaces of the G action, and the ith eigenspace is spanned by the set of monomials with the property that

$$g \cdot x^\alpha y^\beta = \epsilon^i x^\alpha y^\beta.$$

By definition, each eigenspace of $\mathbb{C}[x, y]/I_Z$ is spanned by one monomial $\mathbf{m}$. If $\mathbf{m} = \mathbf{m_0m_1}$, where $\mathbf{m_0}$ is a G-invariant monomial, then m_1 is also a basis of the same eigenspace. Hence we can find a basis for $\mathbb{C}[x, y]/I_Z$ where each basis element is a *basic monomial*, i.e. it is not a multiple of any G-invariant monomial.

Under this action of G the first invariant power of x is x^r. Now suppose that x^{r-1} is not contained in I_Z. Then the monomials $1, x, ..., x^{r-1}$ are all not in I_Z and are basic monomials in $\mathbb{C}[x, y]/I_Z$, all lying in different eigenspaces of the action. Since x^r is in the 0th eigenspace of the G action which has basis 1, we have the relation $x^r = \alpha y^0$ for some $\alpha \in \mathbb{C}$. Also the monomial y must be spanned by the monomials x^i, and since it is contained in the dth eigenspace, for some $d \leq r$ we have the relation $y = \beta x^d$. Hence

$$x^r y = \alpha y = \alpha\beta x^d \text{ and so } x^{r-d}y = \alpha\beta,$$

as required. Now suppose that $x^{r-1} \in I_Z$. Then we can say that there is some $a < r$ with $1, x, ..., x^{a-1}$ basic monomials and $x^a \in I_Z$. Now the eigenspace of x^a must be spanned by a basic monomial $\mathbf{m}$ in $\mathbb{C}[x,y]/I_Z$. Suppose $\mathbf{m} = x\mathbf{m}'$. Then $\mathbf{m}'$ must also be a basic monomial, in the same eigenspace as x^l. So $x^l = a\mathbf{m}'$ for some $a \in \mathbb{C}$. But this implies $x^a = x\mathbf{m}'$, which is a contradiction, since $x^a \in I_Z$ and $x\mathbf{m}'$ is not. So $\mathbf{m} = y^c$ for some $c \in \mathbb{N}$, and so we obtain $x^a = \alpha y^c$. Applying a similar argument to y we obtain basic monomials $y, ..., y^{b-1}$ with $y^b = \beta x^d$ for some $d \in \mathbb{N}$. Similarly to the above we obtain the third equation

$$x^{a-d}y^{b-c} = \alpha\beta.$$

It is clear that a, b must be such that there are precisely r monomials not contained in the set ideal $\langle x^a, y^b x^{a-d}y^{b-c}\rangle$, and these form the basic monomial basis for $\mathbb{C}[x,y]/I_Z$, so there is no further relation between these monomials. Hence

$$I_Z = \langle x^a = \alpha y^c, y^b = \beta x^d, x^{a-d}y^{b-c} = \alpha\beta\rangle.$$

□

Example 6.5. Let G be the cyclic quotient singularity of type $\frac{1}{17}(1,7)$. The minimal resolution is given in Figure 4.19. By the above theorem, the G-clusters are

$$I_1 := \langle x - a_1y^5, y^{17} - b_1, xy^{12} - a_1b_1\rangle$$
$$I_2 := \langle x^4 - a_2y^3, y^5 - b_2x, x^3y^2 - a_2b_2\rangle$$
$$I_3 := \langle x^7 - a_3y, y^3 - b_3x^4, x^3y^2 - a_3b_3\rangle$$
$$I_4 := \langle x^{17} - a_4, y - b_4x^7, x^{10}y - a_4b_4\rangle.$$

Each of these classes of ideals is derived from the generators of one of the cones in the minimal resolution. It is clear that points (a,b) on the Newton Boundary give x^b and y^a in the same eigenspace of the action of G, i.e. $x^b - y^a$ is semi-invariant under the action of G. Now as we have seen above Figure 4.19 also gives the Groebner fan for the ideal I with universal Groebner basis

$$\{x^{17} - 1, x^7 - y, x^4 - y^3, x - y^5, y^{17} - 1, x^3y^2 - 1\}.$$

The fan has 4 cones, and hence we have 4 equivalence classes of weight vectors and 4 corresponding initial ideals.

- $\langle y^{17}, x\rangle$
- $\langle x^3y^2, y^5, x^4\rangle$
- $\langle x^3y^2, y^3, x^7\rangle$
- $\langle x^{17}, y\rangle$

Now if we set $a_i = b_i = 0$, I_i is the initial ideal corresponding to one of the cones in the Groebner fan, and hence we see that the G-Hilbert Scheme is the same as the toric resolution of G.

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