On the minimal free resolution of a monomial ideal.

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August 2012

Abstract

Given a monomial ideal I in the polynomial ring $S = k[x_1, \ldots, x_n]$ over a field k, we construct a minimal free resolution for S/I. We give an introduction to Gröbner bases in order to prove Hilbert's Syzygy Theorem, which we use to resolve simple ideals, then by developing the study of free resolutions to that of cellular resolutions we construct Taylor's resolution and show by Hilbert's Syzygy Theorem that this is often non-minimal, giving some discussion to minimal resolutions.

1 Introduction

Let I be an ideal in a polynomial ring $S = k[x_1, \ldots, x_n]$, where k is any field. Our objective is to minimally resolve the quotient ring S/I, and in order to do so we first prove that such a finite free resolution exists for all ideals, the result of Hilbert's Syzygy Theorem. The first definition we provide to allow us to reach and prove this statement is that of a Gröbner basis, which we compute using Buchberger's algorithm. We apply Gröbner basis theory to syzygy modules, which are of interest to us as the relationship between a syzygy module and its generators can be encoded in a finite free resolution, and at this point we are in a position to state and prove Hilbert's Syzygy Theorem.

If our monomial ideal admits a Gröbner basis with relatively few elements and lies in a ring of only a few variables, then computing its syzygy module and its generators to obtain a free resolution is fairly straightforward, as we will show via application of Hilbert's Syzygy Theorem. However, we want to be able to construct a minimal free resolution for more complicated ideals, and in order to do this we look at cellular resolutions, which also give us a way to encode geometrically the information in a free resolution. We give particular attention to Taylor's resolution, as this is an example of a cellular resolution which we can compute for any monomial ideal. In some cases, Taylor's resolution is a minimal resolution and so at this point we have successfully resolved the quotient ring.

Our motivation for resolving the quotient ring lies in the fact that there is a wealth of algebraic and combinatorial information for S/I contained in its minimal free resolution. We do not include discussion of these properties, but simply highlight them once we have resolved the quotient ring to give an indication of the usefulness of what we have constructed.

2 Construction of a Gröbner Basis

Our fist step towards being able to minimally resolve S/I is to prove that a finite free resolution exists for the ideal we are studying. A crucial structure in being able to create such a resolution and prove its existence is that of a Gröbner basis, which we consider as a preferred generating set for an ideal, Cox, Little and O'Shea [CLO, Chapter 2].

Definition 2.1. Given a fixed monomial ordering and an ideal $I = \langle f_1, \ldots f_s \rangle$, a **Gröbner basis** $\mathcal{G} = \{g_1, \ldots, g_t\}$ for I, is a subset of I such that $\langle \text{LT}(g_1), \ldots, \text{LT}(g_t) \rangle = \langle \text{LT}(I) \rangle$, where $\text{LT}(g_i)$ is defined to be the *leading term* in g_i and $\langle \text{LT}(I) \rangle$ is the ideal generated by the leading terms of all the elements in I.

Our study of Gröbner bases will focus on how they can be used in the construction of finite free resolutions. They can also be applied in elimination theory to find the implicit equations that cut out a variety or the solutions of a set of polynomial equations. As a generating set they are also useful in determining whether a given polynomial lies in an ideal, [CLO, Chapter 3].

The first thing we must familiarise ourselves with before we can compute a Gröbner basis is the definition of a monomial ordering. A *monomial ordering* is defined to be a relation which is a linear ordering, preserves inequalities and is also a well ordering on $\mathbb{Z}^n_{\geq 0}$, [CLO, Chapter 2]. We will give examples of three commonly used monomial orderings, and of note in the following example is that the leading term in the polynomial is different in the case that different monomial orderings are used. This is not always the case, but it does mean that when computing a Gröbner basis, which is a potentially lengthy algorithmic procedure, it is crucial that we stick with the same monomial ordering throughout the computation.

Example 2.2. Let $m_1 = x^4y^3z^7, m_2 = x^4y^5z^2$ and $m_3 = x^2y^7z^5$ be monomials in the polynomial ring $S = \mathbb{Q}[x, y, z]$ such that $f = m_1 + m_2 + m_3$. To find the leading term in f, we must fix a monomial ordering to obtain an ordering of m_1, m_2 and m_3.

(i) **Lexicographic Order**
By considering the exponents of the monomials as vectors, $\mathbf{v}_{m_1} = (4, 3, 7), \mathbf{v}_{m_2} = (4, 5, 2)$ and $\mathbf{v}_{m_3} = (2, 7, 5)$, we are able to compute the vector differences:

$$\begin{aligned} \mathbf{v}_{m_2} - \mathbf{v}_{m_1} &= (0, 2, -5) \\ \mathbf{v}_{m_2} - \mathbf{v}_{m_3} &= (2, -2, -3) \\ \mathbf{v}_{m_1} - \mathbf{v}_{m_3} &= (2, -4, 2). \end{aligned}$$

The monomials' order is determined such that $m_i >_{lex} m_j$ if and only if the left most component in the vector difference is positive. Hence $m_2 >_{lex} m_1 >_{lex} m_3$ and m_2 is the leading term in f.

(ii) **Graded Lexicographic Order**
Graded orderings require first that we consider the *total degree* of the monomials:

$$\begin{aligned} |\mathbf{v}_{m_1}| &= 14 \\ |\mathbf{v}_{m_2}| &= 11 \\ |\mathbf{v}_{m_1}| &= 14. \end{aligned}$$

Clearly $m_1 >_{grlex} m_2$ and $m_3 >_{grlex} m_2$. To distinguish between monomials of the same total degree we use lexicographic order; it is now m_1 which is the leading term in f.

(iii) **Graded Reverse Lexicographic Order**
Again we are interested in the total degree of the monomials, but in this case those of equal degree are ordered by considering the right most component in the vector difference, with $m_i >_{grevlex} m_j$ if it is negative. So we consider $\mathbf{v}_{m_3} - \mathbf{v}_{m_1} = (-2, 4, -2)$ and conclude that $m_3 >_{grevlex} m_1 >_{grevlex} m_2$; the monomial m_3 is the leading term in f with respect to grevlex order.

The key definition that we must give before we examine the algorithm used to construct a Gröbner basis is that of an S-polynomial[1]. Let $\mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))$ be the least common multiple of the leading monomials[2] of f and g. Then for any pair of polynomials $f, g \in S$, the S-*polynomial* is

$$\mathrm{S}(f,g) = \frac{\mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))}{\mathrm{LT}(f)}.f - \frac{\mathrm{LCM}(\mathrm{LM}(f), \mathrm{LM}(g))}{\mathrm{LT}(g)}.g.$$

Gröbner bases were first proposed by Bruno Buchberger in his PhD thesis, [BB], in which he shows how a finite generating set for a polynomial ideal can be computed algorithmically using S-polynomials and a version of the division algorithm. The first step in Buchberger's algorithm is to define the set G of polynomials with which we are working. If $I = \langle f_1, \dots f_s \rangle$ is the ideal of S for which we want to compute the Gröbner basis, define $G = (f_1, \dots f_s)$. The next step is to compute the S-polynomials for every pair of polynomials in G with respect to a fixed monomial ordering. By developing the familiar division algorithm from the polynomial ring in one variable $S' = k[x]$ to define division over $S = k[x_1, \dots x_n]$ it is possible to divide each S-polynomial by all the polynomials in G, expressing them as $\mathrm{S}(f_i, f_j) = a_1 f_1 + \cdots + a_s f_s + r$ where $a_1, \dots, a_s, r \in S$, [CLO, Chapter 2]. If it is the case that every S-polynomial has zero remainder after undergoing the division algorithm, then G is a Gröbner basis for I. If this is not the case, then G is redefined to include those S-polynomials that have non-zero remainder, and the algorithm is repeated until all S-polynomials have zero remainder. This application of the division algorithm ensures that all elements in the ideal can be generated by elements in the basis; in this case we check that the S-polynomials, given elements of the ideal, can be generated by the basis we have defined. The *Ascending Chain Condition* ensures us that this algorithm must terminate, stating that for every chain of ideals $I_1 \subset I_2 \subset I_3 \subset \cdots$ there is some $N \in \mathbb{N}$ such that $I_N = I_{N+1} = I_{N+2} = \cdots$, [CLO, Chapter 2].

Although there are adjustments that can be made to make the algorithm more efficient, it can still be necessary to compute a great number of S-polynomials at each stage, and so Gröbner bases can be time consuming to compute manually. The *Macaulay 2* software developed by Grayson and Stillman [GS] can be used to quickly generate Gröbner bases. The basis generated by *Macaulay 2* is a *reduced Gröbner basis*, defined as a Gröbner basis in which the coefficient of the leading term is equal to one for all $g_i \in \mathcal{G}$ and moreover, that no monomial of g_i lies in the monomial ideal $\langle \mathrm{LT}(\mathcal{G} - \{g_i\}) \rangle$. An ideal may have infinitely many Gröbner bases, depending on the characteristic

[1] The 'S' in S-polynomials stands for syzygy, which we will explore in Section 3, it is not related to the polynomial ring S.

[2] The leading monomial is the leading term in the polynomial expression with the coefficient set to one.

of the field k. The reduced Gröbner basis is unique to a given ideal I, and therefore allows one to make comparisons between ideals, because two ideals are equal if and only if their reduced Gröbner bases are equal, [CLO, Chapter 2]. The following example demonstrates the steps of Buchberger's algorithm.

Example 2.3. Let $I = \langle x^4 + y + y^2z, x^3y^3z^2, xy^2z^3 + x^2z\rangle$ be an ideal in a polynomial ring $S = \mathbb{Q}[x, y, z]$. Set $G = (f_1, f_2, f_3)$ with $f_1 = x^4 + y + y^2z$, $f_2 = x^3y^3z^2$ and $f_3 = xy^2z^3 + x^2z$.

1. Select the leading terms with respect to a chosen monomial ordering. With respect to grevlex ordering, $\mathrm{LT}(f_1) = x^4, \mathrm{LT}(f_2) = x^3y^3z^2$ and $\mathrm{LT}(f_3) = xy^2z^3$.
2. Compute the S-polynomials for each pair of polynomials:

$$\begin{aligned}
\mathrm{S}(f_1, f_2) &= \frac{x^4y^3z^2}{x^4}(x^4 + y + y^2z) - \frac{x^4y^3z^2}{x^3y^3z^2}(x^3y^3z^2) \\
&= y^4z^2 + y^5z^3 \\
\mathrm{S}(f_2, f_3) &= \frac{x^3y^3z^3}{x^3y^3z^2}(x^3y^3z^2) - \frac{x^3y^3z^3}{xy^2z^3}(xy^2z^3 + x^2z) \\
&= -x^4yz \\
\mathrm{S}(f_3, f_1) &= \frac{x^4y^2z^3}{xy^2z^3}(xy^2z^3 + x^2z) - \frac{x^4y^2z^3}{x^4}(x^4 + y + y^2z) \\
&= x^5z - y^3z^3 - y^4z^4.
\end{aligned}$$

3. If the remainder of every S-polynomial on division by the elements of G is zero, then we have computed a Gröbner basis. We see that $G = (x^4 + y + y^2z, x^3y^3z^2, xy^2z^3 + x^2z)$ is not a Gröbner basis for I as there are no $a_1, \ldots, a_3 \in S$ such that

$$\mathrm{LT}(\mathrm{S}(f_1, f_2)) = -y^5z^3 = a_1\,\mathrm{LT}(f_1) + a_2\,\mathrm{LT}(f_2) + a_3\,\mathrm{LT}(f_3).$$

4. We must perform another step of the algorithm, redefining G to include the S-polynomials with non-zero remainder on division by G. When all remainders are zero we have arrived at a Gröbner basis. To generate a Gröbner basis using *Macaulay 2*, define the polynomial ring, the chosen monomial ordering and the ideal to produce the reduced Gröbner basis

$$\mathcal{G} = \{xyz, x^2z, y^2z^2 + yz, x^4 + y^2z + y\}.$$

3 Syzygy Modules

In the course of our proof of Hilbert's Syzygy Theorem in Section 4, we compute the Gröbner basis of a syzygy module. This section defines the first syzygy module and looks at how its Gröbner basis can be computed.

A *module* M, over a ring S, or an S-module, is an abelian group together with an action of S on M such that for all $s_i \in S, m_i \in M$:

(i) $s_1(m_1 + m_2) = s_1m_1 + s_1m_2$;

(ii) $(s_1 + s_2)m_1 = s_1m_1 + s_2m_1$;

(iii) $s_1(s_2m_1) = (s_1s_2)m_1$;

(iv) $1(m) = m$.

A *free module* is a module which has a basis. Ideals and quotient rings provide familiar examples of modules over $S = k[x_1, \dots, x_n]$, and we focus on these in later sections.

Free resolutions build sequences by considering the relationship between generators of a module, and this information is held in the syzygy module.

Definition 3.1. Let $F = (f_1 \dots, f_t)$ be the ordered t-tuple of elements such that $f_1, \dots f_t \in M$. The **first syzygy module** of F, $\mathrm{Syz}(f_1, \dots, f_t)$, is the set of relations $(a_1, \dots, a_t)^T \in S^t$ on F such that $a_1f_1 + \dots + a_tf_t = 0$.

As implied by its name, the first syzygy module is itself an S-module, and so we can compute the second syzygy module by considering the set of relations on the elements of the first syzygy module and so on. In order to see how we can form a Gröbner basis for a syzygy module, we consider Schreyer's Theorem [CLO2, Chapter 5].

Theorem 3.2. *[Schreyer's Theorem] Let $\mathcal{G} = (g_1, \dots, g_s)$ be a Gröbner basis of a module with respect to any monomial ordering. Define a column vector $\boldsymbol{a}_{ij} \in S^s$ for each* S*-polynomial* $\mathrm{S}(g_i, g_j)$ *by setting*

$$S(g_i, g_j) = \sum_{k=1}^{s} a_{ijk} g_k.$$

Let S^s be a free S-module of rank s with standard basis vectors $\boldsymbol{e}_1, \dots, \boldsymbol{e}_s$. For each $i, j \in \{1, \dots, s\}$ define an element $\boldsymbol{s}_{ij} \in S$ by setting

$$\boldsymbol{s}_{ij} := \frac{\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))}{\mathrm{LT}(g_i)}.\boldsymbol{e}_i - \frac{\mathrm{LCM}(\mathrm{LM}(g_i), \mathrm{LM}(g_j))}{\mathrm{LT}(g_j)}.\boldsymbol{e}_j - \boldsymbol{a}_{ij}.$$

The elements $\boldsymbol{s}_{ij}$ for all pairs of polynomials in $\mathcal{G}$ form a Gröbner basis for $Syz(g_1, \dots, g_s)$.

An understanding of Schreyer's Theorem will be central to proving Hilbert's Syzygy Theorem, showing as it does how we can manipulate the S-polynomials of a Gröbner basis in order to compute its syzygy module.

4 Hilbert's Syzygy Theorem

The main result of this section is the construction of a finite free resolution for any finitely generated S-module. Before we can define this object, we say that a sequence of S-modules

$$\cdots \longrightarrow M_{i+1} \xrightarrow{\psi_{i+1}} M_i \xrightarrow{\psi_i} M_{i-1} \longrightarrow \cdots$$

is *exact* at M_i if and only if $im(\psi_{i+1}) = ker(\psi_i)$. The sequence is exact if it is exact at every M_i in the sequence. Our interest lies in a particular kind of exact sequence.

Definition 4.1. A **finite free resolution** of an ideal I, with length ℓ is an exact sequence of the form

$$0 \longrightarrow F_\ell \xrightarrow{\psi_\ell} F_{\ell-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\psi_1} F_0 \longrightarrow I \longrightarrow 0 \tag{4.1}$$

where each F_i is a free module of rank r_i and where $F_\ell \neq 0$.

Our aim is to find a minimal free resolution for a quotient ring. The condition of exactness allows us to achieve this by finding a minimal resolution for an ideal. Consider the short exact sequence $0 \longrightarrow I \longrightarrow S \longrightarrow S/I \longrightarrow 0$. By 'splicing' this sequence with the finite free resolution defined in 4.1, we are able to consider the finite free resolution

$$0 \longrightarrow F_\ell \xrightarrow{\psi_\ell} F_{\ell-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\psi_1} F_0 \longrightarrow S \longrightarrow S/I \longrightarrow 0.$$

in which the free modules F_i are unchanged.

Having defined modules and free resolutions and looked at the computation of generating sets, we are now able to state and prove Hilbert's Syzygy Theorem[3].

Theorem 4.2. *[Hilbert's Syzygy Theorem] Let $S = k[x_1, \ldots, x_n]$. Every finitely generated S-module has a finite free resolution of length at most n.*

Proof. Let M be a finitely generated S-module. Choose a generating set $(f_1, \ldots, f_{r_0})$ for M. Calculate the first syzygy module on this set of generators using the method of Schreyer's Theorem, and compute a Gröbner basis $\mathcal{G}_0 = \{g_1, \ldots, g_{r_1}\}$ for $\mathrm{Syz}(f_1, \ldots, f_{r_0})$.

We now begin to construct our exact sequence. This choice of r_0 generators of M allows us to define a map $\psi : F_0 \to M$, where F_0 is a free S-module of rank r_0, in which each standard basis vector $\mathbf{e}_i$ for $i \in \{1, \ldots, r_0\}$ is mapped to $f_i \in M$. This map is surjective and we can therefore construct the exact sequence

$$F_0 \xrightarrow{\psi} M \longrightarrow 0.$$

Now if we consider the homomorphism ψ as a map sending $(g_1, \ldots, g_{r_o}) \in F_0$ to $\sum_{i=1}^{r_0} g_i f_i \in M$, we see that its kernel is the tuple $(g_1 \ldots, g_{r_o})$ such that $\sum_{i=1}^{r_0} g_i f_i = 0$. This is precisely the definition of the syzygy module of the generating set $(f_1 \ldots, f_{r_0})$, hence we can conclude that

$$\mathrm{Syz}(f_1, \ldots, f_{r_0}) = ker(\psi : F_0 \to M).$$

[3]For more detailed explanation of aspects of the theory of free resolutions used in this proof, see Chapter 6 of [CLO2].

Recall that the first syzygy module is itself an S-module, so choosing set of generators for the module $\mathrm{Syz}(f_1, \ldots, f_{r_0})$ allows us to obtain a sequence

$$F_1 \xrightarrow{\psi_1} F_0 \longrightarrow M \longrightarrow 0$$

which gives a presentation of M.

Define a monomial ordering on the elements of $\mathcal{G}_0$ such that if $\mathrm{LT}(g_i)$ and $\mathrm{LT}(g_j)$ contain the same standard basis vector $\mathbf{e}_k$, and $i < j$, then $\frac{\mathrm{LM}(g_i)}{\mathbf{e}_k} >_{lex} \frac{\mathrm{LM}(g_j)}{\mathbf{e}_k}$ where $>_{lex}$ is the lexicographic order on the variables of S, and order the elements of $\mathcal{G}$ accordingly to obtain the vector G_0. We now compute a Gröbner basis $\mathcal{G}_1$ for the module $\mathrm{Syz}(G_0)$ and obtain an exact sequence of the form

$$F_2 \xrightarrow{\psi_2} F_1 \xrightarrow{\psi_1} F_0 \longrightarrow M \longrightarrow 0.$$

This process is iterated, and as it can be shown that the ordering we have defined allows us to remove at least one variable from the leading terms of each consecutive syzygy module [CLO2, Lemma 2.2, Chapter 6], the length of the sequence can be at most the number of variables in the polynomial ring. Hence after $\ell \leq n$ steps we obtain the exact sequence

$$F_\ell \xrightarrow{\psi_\ell} F_{\ell-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\psi_1} F_0 \longrightarrow M \longrightarrow 0$$

where the leading terms in the Gröbner basis for F_ℓ do not contain any variables, and hence $F_{\ell+1} = 0$.

Finally we can extend this exact sequence to an exact sequence of length ℓ by considering ψ_ℓ as the inclusion mapping, leading to the free resolution

$$0 \longrightarrow F_\ell \xrightarrow{\psi_\ell} F_{\ell-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{\psi_1} F_0 \longrightarrow M \longrightarrow 0. \qquad \square$$

To complete this section, we will give an example of the construction of a finite free resolution for an ideal.

Example 4.3. Let $I = \langle x^2y, xyz^3, yz^2, xy^2 \rangle$ be a monomial ideal in $S = \mathbb{Q}[x, y, z]$. First, determine the Gröbner basis of I, $\mathcal{G}_0 = (yz^2, xy^2, x^2y)$.

We now calculate the set of relations on $\mathcal{G}_0$, the column vectors $\mathbf{a}_{ij}$ such that $a_{ij1}(yz^2) + a_{ij2}(xy^2) + a_{ij3}(x^2y) = 0$, and use them to construct the 3×3 monomial matrix

$$\psi_1 = \begin{bmatrix} xy & x^2 & 0 \\ -z^2 & 0 & x \\ 0 & -z^2 & -y \end{bmatrix}.$$

We next construct the monomial matrix ψ_2 which relates the elements of the Gröbner basis $\mathcal{G}_1 \subseteq F_1$ to F_2. As the sequence is exact we must have $\psi_1.\psi_2 = 0$. Hence we see that

$$\psi_2 = \begin{bmatrix} -x \\ y \\ -z^2 \end{bmatrix}.$$

This information is represented in the free resolution

$$0 \longrightarrow F_2 \xrightarrow{\begin{bmatrix} -x \\ y \\ -z^2 \end{bmatrix}} F_1 \xrightarrow{\begin{bmatrix} xy & x^2 & 0 \\ -z^2 & 0 & x \\ 0 & -z^2 & -y \end{bmatrix}} F_0 \longrightarrow I \longrightarrow 0$$

which has length $\ell = 2 \leq n = 3$, as guaranteed by Hilbert's Syzygy Theorem.

5 Cellular Resolutions

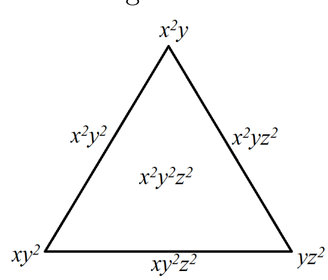
Having established that finite free resolutions exist for all finitely generated modules, we give some consideration to how they can be geometrically interpreted, and then examine how this method can be developed in order to construct free resolutions quickly for ideals where the method of applying Schreyer's Theorem to syzygy modules becomes overly lengthy. We concentrate on monomial ideals from this point, defined as any ideal generated by monomials.

Before we can construct this geometric object and define the resolution it provides us with, a few more general definitions are required. A *cell complex* is a collection of cells which can be built from a set of discrete points. By joining the points to each other by one dimensional lines and connecting these lines with two dimensional faces, and so on in higher dimensions, a complex of cells is constructed. In the case that these cells are simplices, then the cell complex is infact a simplicial complex. The cell complexes with which we are concerned have the property that the sequence of free modules which they encode are exact. Such complexes are said to be *acyclic*.

Definition 5.1. Let X be a finite acyclic cell complex. A **cellular resolution** of X is the finite resolution of free S-modules represented by the cellular monomial matrices which support X.

Our first example shows how we construct a cellular resolution for the ideal given in Example 4.3.

Example 5.2. Let $I = \langle x^2y, xyz^3, yz^2, xy^2 \rangle$ be a monomial ideal in $S = \mathbb{Q}[x, y, z]$. In order to construct the cellular resolution we first define the vertices in the simplex Δ to be the monomial elements in the Gröbner basis of I. The edge between each set of points is labelled by the least common multiple of the points, and the face of Δ which the three edges form is labelled by the least common multiple of the labels of the edges.



By labelling the vertices of a simplex with the monomials in an ideal, we can easily work out what the elements of the second and third syzygy module are, and we can continue this process in higher dimensions to find the next syzygy modules. So $S_1 = (x^2y^2, x^2yz^2, xy^2z^2)$ and $S_2 = (x^2y^2z^2)$. From this simple object we can determine the cellular resolution

$$0 \longrightarrow S_2^1 \xrightarrow{\phi_2} S_1^3 \xrightarrow{\phi_1} S_0^3 \longrightarrow I \longrightarrow 0 \tag{5.1}$$

where the grading on the free module is determined by the number of facets in each dimension. We can then construct the monomial matrices ϕ_1 and ϕ_2 in sequence 5.1 from the information encoded in this simplex. Determine the a_{ij} such that for the matrix $\phi_1 = \{a_{ij}\}_{3\times 3}$,

$$\phi_1 \times \begin{bmatrix} yz^2 \\ xy^2 \\ x^2y \end{bmatrix} = 0$$

Of course we find that $\phi_1 = \psi_1$ as defined in Example 4.3, and by the same procedure we find that $\phi_2 = \psi_2$. We have therefore constructed a cellular resolution for I, but without having to apply Schreyer's Theorem in order to compute syzygy modules, as Δ clearly shows the elements of each free module.

The remainder of this section will deal with a particular class of examples of cellular resolutions.

Definition 5.3. Let $I = \langle m_1, \dots, m_r\rangle$ be a monomial ideal in S with minimal generators. **Taylor's resolution**, is the free resolution which supports the full $(r-1)$-dimensional simplex whose vertices are labeled by the monomials which generate I.

These cellular resolutions may allow us to minimally resolve an ideal in some cases, but may lead to a non-minimal resolution in others. A *minimal resolution* is a cellular resolution in which no polytopes are represented by the same monomial, Bayer and Sturmfels, [BS, Remark 1.4]. Example 5.2 demonstrates a minimal Taylor's resolution for an ideal, however it is rare that this is the case; Taylor's resolution is normally non-minimal if the rank of S is much greater than the number of variables in the polynomial ring. Example 5.4 shows why this is the case.

Example 5.4. Let $I = \langle x^2yz, z^6, y^6, xy^3, x^3y^6\rangle$ be a monomial ideal in $S = \mathbb{Q}[x, y, z]$ with minimal generators. By definition, the Taylor's resolution for I is a four dimensional simplicial complex Δ_I. Consider the five minimal generators of I to be the vertices of Δ_I. To work out the grading on the next free module in the resolution, consider that all of the five points must be connected to each other by an edge, hence Δ_I has $\binom{5}{2} = 10$ edges. We continue with this procedure and see that Δ_I has ten two dimensional faces, five three dimensional faces and one four dimensional face. We begin to produce the Taylor's resolution

$$0 \longrightarrow S_4^1 \xrightarrow{\psi_4} S_3^5 \xrightarrow{\psi_3} S_2^{10} \xrightarrow{\psi_2} S_1^{10} \xrightarrow{\psi_1} S_0^5 \longrightarrow I \longrightarrow 0.$$

To determine the monomial matrices, we first determine the elements of each S-module. This is done by extending the labeling method presented in Example 5.2 to a four dimensional simplicial complex, and thus

$$\begin{aligned}
S_0 &= (x^2yz, z^6, y^6, xy^3, x^3y^6) \\
S_1 &= (x^3y^6z, x^3y^6, x^3y^6, x^3y^6z^6, x^2y^3z, x^2y^6z, x^2yz^6, xy^6, xy^3z^6, y^6z^6) \\
S_2 &= (xy^6z^6, x^3y^6z^6, x^2y^6z^6, x^3y^6z^6, x^2y^3z^6, x^3y^6z^6, x^3y^6z, x^3y^6z, x^2y^6z, x^3y^6) \\
S_3 &= (x^3y^6z, x^3y^6z^6, x^3y^6z^6, x^3y^6z^6, x^2y^6z^6) \\
S_4 &= (x^3y^6z^6).
\end{aligned}$$

Recall that elements of each module are labels of distinct faces, so if the same label appears multiple time in a module, then the resolution we are constructing is non minimal.

We can now determine the monomial matrices. Now that we are working with more elements in each S-module, we will consider how these matrices can be construted without necessarily considering syzygy modules. We know that ψ_1 is a 5×10 matrix, and we construct it by labelling each row with an element $a_j \in S_0$ for $j \in \{1, \ldots, 5\}$ and each column with an element of $b_k \in S_1$ for $k \in \{1, \ldots, 10\}$. The entries m_{jk} in the matrix are either

(i) zero, in the case that b_k is not a least common multiple of a_j and some other element of S_0;

(ii) non-zero, with m_{jk} such that $|a_j.m_{jk}| = b_k$.

Repeating this method allows us to obtain all the monomial matrices in Taylor's resolution, and to ensure the sequence is exact, check that $\psi_i.\psi_{i+1} = 0$ for all $i \in \{1, 2, 3\}$. Hence we have that

$$\psi_1 = \begin{bmatrix} z & 1 & 1 & z & 0 & 0 & 0 & 0 & 0 & 0 \\ -xy^5 & 0 & 0 & 0 & y^2 & y^5 & z^5 & 0 & 0 & 0 \\ 0 & -x^2y^3 & 0 & 0 & -xz & 0 & 0 & y^3 & z^6 & 0 \\ 0 & 0 & -x^3 & 0 & 0 & -x^2z & 0 & -x & 0 & z^6 \\ 0 & 0 & 0 & -x^3y^6 & 0 & 0 & -x^2y & 0 & -xy^3 & -y^6 \end{bmatrix}$$

$$\psi_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & -z^5 & -1 & -1 & 0 & 0 \\ 0 & 0 & 0 & -z^6 & 0 & 0 & 0 & z & 0 & -1 \\ 0 & z^6 & 0 & 0 & 0 & 0 & z & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -z^5 & 0 & 0 & -xy^3 & -y^3 & 0 \\ 0 & 0 & z^5 & 0 & 0 & 0 & -x & 0 & 1 & 0 \\ 0 & 0 & y^5 & 0 & y^2 & -xy^5 & 0 & 0 & 0 & 0 \\ -z^6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -xz & -x^2 \\ y^3 & 0 & 0 & -x^2y^3 & -x & 0 & 0 & 0 & 0 & 0 \\ -x & -x^3 & -x^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\psi_3 = \begin{bmatrix} 0 & x^2 & 0 & 0 & x \\ 0 & -1 & -1 & 0 & 0 \\ 0 & 0 & x & 0 & -1 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & xy^3 & y^3 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & -z^5 & 0 & 0 \\ -1 & 0 & 0 & -z^5 & 0 \\ x & 0 & 0 & 0 & -z^5 \\ -z & -z^6 & 0 & 0 & 0 \end{bmatrix}$$

$$\psi_4 = \begin{bmatrix} -z^5 \\ 1 \\ -1 \\ 1 \\ -x \end{bmatrix}.$$

These matrices, together with the elements of each S-module, provide a full cellular resolution for I. We can be sure that this resolution is non minimal as it has length $\ell = 4$ and by Hilbert's Syzygy Theorem, there exists a resolution of length $\ell \leq 3$. There must therefore be a subcomplex of the cellular complex supported on Taylor's resolution which is minimal.

6 Minimally Resolving S/I

The geometric interpretation of free resolutions allows us to construct Taylor's resolution for any monomial ideal. As this resolution has length determined by the number of minimal generators in the ideal, for ideals with large rank which exist in polynomial rings of only a few variables, Taylor's resolution is much longer than the finite free resolution with length $\ell \leq n$, the existence of which is guaranteed by Hilbert's Syzygy Theorem. Therefore, by considering a simple geometric interpretation of the minimal generating set of an ideal, the task of minimally resolving a quotient ring is straight forward in the cases that Taylor's resolution is minimal. In the more likely cases in which Talylor's resolution is non minimal, it is not just an overly long resolution which we encounter, but also one in which the monomial matrices become increasingly large as the resolution gets longer.

The next step in the study of free resolutions is to consider how an ideal can be minimally resolved. This process is different depending on the monomials in the given ideal; the method outlined by Bayer and Sturmels in [BS] works only for *generic ideals*, which they define to be monomial ideals in which no variable appears to the same degree in any two generators. In the case that the ideal is not generic, it is necessary to deform the ideal to a generic one before a minimal resolution can be constructed. Once such a minimal resolution has been constructed, it is possible to obtain algebraic information about the quotient ring, including Betti numbers, Euler characteristics and k-polynomials, as discussed in Chapter 6 of [MS].

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