The McKay Correspondence

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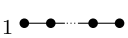
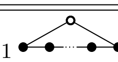
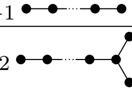
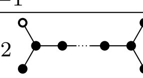
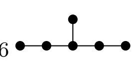
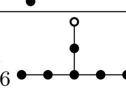
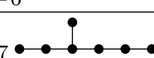
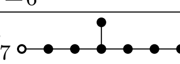
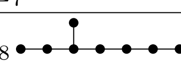
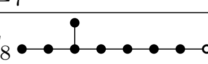
1 Introduction

The McKay correspondence, named after mathematician J. McKay, is a correspondence linking the finite subgroups of SL(2,$\mathbb{C}$) and the Dynkin diagrams of type $A_r, D_r, E_{6,7,8}$. It had been known that the so called Kleinian (or Du Val) singularities have resolution graphs of this type, but it was first noticed by McKay that in certain constructions using the irreducible representations of these groups, the same graphs appear.

In this project, I will be considering the simpler cases of the correspondence, that is, Dynkin diagrams of type A_r, D_r only. This is in the interest of time. It would have been possible to give a more complete account of any section of this project, but my interest lay more in giving an overview of the complete picture. As such, I will construct the cyclic and binary dihedral groups as subgroups of SL(2,$\mathbb{C}$) first. Then I will construct the ring of invariants corresponding to each of these families of groups. From these rings, we can create the Kleinian singularities. From here we employ the method of blowing-up in order to resolve singularities, and hence we obtain our resolution graphs corresponding to the Dynkin diagrams.

The final sections of this project deal with the reverse correspondence. From the subgroups, we construct the McKay graphs ,which correspond to extended versions of the Dynkin diagrams, by considering tensor products of irreducible representations with a 'natural' representation. We also give a construction of the irreducible representations themselves. The last chapter is devoted to constructing the singularities from the graphs, by first converting them into quivers and then considering the category of representations of the quivers and traces of oriented cycles.

G	Kleinian Singularity	Resolution Graph	McKay Graph
$C_n, n \geq 2$	$\mathbb{V}(xy - z^n)$	A_{n-1} 	$\tilde{A}_{n-1}$ 
$\mathcal{B}D_{4n}$	$\mathbb{V}(x^{n+1} + xy^2 + z^2)$	D_{n+2} 	$\tilde{D}_{n+2}$ 
$\mathcal{B}T$	$\mathbb{V}(x^4 + y^3 + z^2)$	E_6 	$\tilde{E}_6$ 
$\mathcal{B}O$	$\mathbb{V}(x^3y + y^3 + z^2)$	E_7 	$\tilde{E}_7$ 
$\mathcal{B}I$	$\mathbb{V}(x^5 + y^3 + z^2)$	E_8 	$\tilde{E}_8$ 

2 Finite Subgroups of SL(2,$\mathbb{C}$)

2.1 Fractional-linear transformations

Definition 2.1. The *special linear group* of $\mathbb{C}^2$, denoted SL(2,$\mathbb{C}$), is the group of 2×2 complex valued matrices with determinant 1. Such matrices act naturally on $\mathbb{C}^2$ by matrix multiplication of vectors.

The *projective complex line*, $\mathbb{P}^1$ is the set of lines through the origin in the complex plane $\mathbb{C}^2$. Points in $\mathbb{P}^1$ are represented by *homogeneous coordinates*, $[z_0 : z_1]$, which have the property that $[z_0 : z_1] = [\lambda z_0 : \lambda z_1]$, $\forall \lambda \in \mathbb{C}\backslash\{0\}$.

Aut($\mathbb{P}^1$) is the group of automorphisms of $\mathbb{P}^1$, that is, linear transformations from $\mathbb{P}^1$ to itself.

Proposition 2.2. SL(2,$\mathbb{C}$) has an action on $\mathbb{P}^1$ defined by:

$$\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot [z_0 : z_1] = [az_0 + bz_1 : cz_0 + dz_1].$$

Moreover, any linear transformation of $\mathbb{P}^1$ can be expressed this way, and so any element $g \in$ Aut($\mathbb{P}^1$) can be represented by some matrix in SL(2,$\mathbb{C}$).

Proof. We first show that the map defined in the statement is well-defined. For any $\lambda \in \mathbb{C}\backslash\{0\}$ we have:

$$\begin{aligned} \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \cdot [\lambda z_0 : \lambda z_1] &= [a(\lambda z_0) + b(\lambda z_1) : c(\lambda z_0) + d(\lambda z_1)] \\ &= [\lambda(az_0 + bz_1) : \lambda(cz_0 + dz_1)] \\ &= [az_0 + bz_1 : cz_0 + dz_1]. \end{aligned}$$

So our choice of representatives for the point $[z_0 : z_1]$ doesn't matter. We further need to check that no element of $\mathbb{P}^1$ is mapped to $[0 : 0]$. Since $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}(2, \mathbb{C})$ we have that $ad - bc = 1$. Consider first the case that $z_0 \neq 0$. Then

$$[az_0 + bz_1 : cz_0 + dz_1] = [0 : 0] \implies a = c = 0 \implies ad - bc = 0.$$

The only other point is $[0 : 1]$, in which case

$$[b : d] = [0 : 0] \implies b = d = 0 \implies ad - bc = 0$$

and both of these cases are contradictory. Hence our map is well-defined. Furthermore, for any $[z_0 : z_1] \in \mathbb{P}^1$ we have that $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot [z_0 : z_1] = [z_0 : z_1]$ and if $M, N \in \mathrm{SL}(2, \mathbb{C})$ then it is routine to check that

$$M \cdot (N \cdot [z_0 : z_1]) = (MN) \cdot [z_0 : z_1].$$

So the map we have defined is indeed an action.

Recall that a linear projective transformation $\varphi \in \mathrm{Aut}(\mathbb{P}^1)$, is of the form:

$$\varphi([z_0 : z_1]) = T([z_0 : z_1]),$$

for some $T \in \mathrm{GL}(2, \mathbb{C})$. Moreover, for any $\lambda \in \mathbb{C}\backslash\{0\}$, the linear bijections λT and T define the same projective transformation. Hence we can scale T so that its determinant is one, meaning that there exists $T' \in \mathrm{SL}(2, \mathbb{C})$ which defines φ. □

When $z_0 \neq 0$ we identify $[z_0 : z_1]$ with a point $z = \frac{z_1}{z_0} \in \mathbb{C}$ known as the affine coordinate and we identify the point $[0 : 1]$ with ∞. This gives an isomorphism $\mathbb{P}^1 \cong \mathbb{C} \cup \{\infty\}$. In terms of affine coordinates, the action defined in Proposition 2.2 is the transformation $z \mapsto \frac{dz+c}{bz+a}$, known as a *fractional-linear* transformation. This transformation sends ∞ to $\frac{d}{b}$ and $\frac{-a}{b}$ to ∞.

2.2 Finite subgroups of Aut($\mathbb{P}^1$)

Lemma 2.3. Let G be a finite subgroup of Aut($\mathbb{P}^1$). The action of any nontrivial element $g \in G$ on $\mathbb{P}^1$ has two distinct fixed points.

Proof. Any $g \in G$ can be represented by an element in SL(2,$\mathbb{C}$) by Proposition 2.2. Any 2×2 matrix has two eigenvalues, and there are two possibilities:

- The eigenvalues are equal. Then the eigenspace of the repeated eigenvalue has dimension 2, meaning that every line through the origin in $\mathbb{C}^2$ is fixed by our matrix. Hence in this case $g = 1 \in \text{Aut}(\mathbb{P}^1)$, the identity transformation.
- The eigenvalues are distinct. The corresponding eigenspace for each eigenvalue must have dimension 1, i.e. the eigenspace is a line in $\mathbb{C}^2$ through the origin. Thus each eigenvalue corresponds to a distinct fixed point of $\mathbb{P}^1$ under the action of g.

We conclude that the action of any non-trivial element $g \in G$ has two distinct fixed points in $\mathbb{P}^1$. □

We define the set

$$Z := \big\{(g, x) \in G\backslash\{1\} \times \mathbb{P}^1 : g(x) = x\big\}.$$

We wish to count the number of elements of Z in two different ways to obtain some restrictions on G. For this, first write $\mathcal{P}$ for the projection of Z to its second component, that is $\mathcal{P}$ is the set of points in $\mathbb{P}^1$ which are fixed by some non-trivial element of G.

Lemma 2.4. $\mathcal{P}$ is invariant under the action of G.

Proof. If $x \in \mathcal{P}$ and $g \in G$, let $h \in G\backslash\{1\}$ be such that $h(x) = x$. The element ghg^{-1} is non-trivial in G and fixes $g(x)$, so $g(x) \in \mathcal{P}$. □

We decompose $\mathcal{P}$ into orbits under the G-action, write

$$\mathcal{P} = \mathcal{O}_1 \cup ... \cup \mathcal{O}_k$$

as the decomposition. Write $e_i(x)$ to be the order of the stabiliser subgroup of any $x \in \mathcal{O}_i$. Since the stabiliser subgroups of elements in the same orbit are conjugate, we have $e_i(x) = e_i(y)$ for any $x, y \in \mathcal{O}_i$, so from now on we simply write e_i to mean the order of the stabiliser subgroup of any element of $\mathcal{O}_i$.

Proposition 2.5. The integers $k, e_1, ..., e_k$ satisfy the relationship:

$$k - 2 + \frac{2}{|G|} = \sum_{i=1}^{k} \frac{1}{e_i}. \qquad (*)$$

Proof. The Orbit-Stabiliser theorem tells us that $|G| = |\mathcal{O}_i|e_i$ for any $i \in \{1, 2, ..., k\}$. Now:

$$|Z| \overset{\text{Lemma 2.3}}{=} 2(|G| - 1) = \sum_{x \in \mathcal{P}} (e_i(x) - 1)$$
$$= \sum_{i=1}^{k} \sum_{x \in \mathcal{O}_i} (e_i(x) - 1)$$
$$= \sum_{i=1}^{k} |\mathcal{O}_i|(e_i - 1)$$
$$= k|G| - \sum_{i=1}^{k} |\mathcal{O}_i|.$$

Rearranging this equation and dividing throughout by $|G|$ we end up with the desired expression $(*)$. □

The expression $(*)$ only has finitely many solutions! Using the fact that $2 \leq e_i \leq |G|$ and also that G is finite, we have that either $k = 2$ or $k = 3$.

Case 1: $k = 2$

In this case our expression becomes

$$\frac{2}{|G|} = \frac{1}{e_1} + \frac{1}{e_2}$$

and the only solution is $e_1 = e_2 = |G|$. The Orbit-Stabiliser Theorem then tells us that $|\mathcal{O}_1| = |\mathcal{O}_2| = 1$ and so there are two fixed points under the action of G , say z_1 and z_2. Let $h \in \text{Aut}(\mathbb{P}^1)$ be the transformation which maps $z_1 \mapsto 0$ and $z_2 \mapsto \infty$. Then let $\bar{G} = hGh^{-1}$ so that $\bar{G}$ fixes 0 and ∞. Let $g \in \bar{G}$ and write $g(z) = \frac{dz+c}{bz+a}$ for any $z \in \mathbb{P}^1$. Then:

$$g(0) = \frac{c}{a} = 0 \implies c = 0$$

and recalling that $\infty = [0 : 1]$

$$g(\infty) = \frac{d}{b} = \infty \implies [b : d] = [0 : 1] \implies b = 0.$$

Then we can write $g(z) = \omega z$ for some $\omega \in \mathbb{C}$. Since the order of g is finite we must have that ω is a primitive n^{th} root of unity for some $n \in \mathbb{N}$. This argument applies for an arbitrary element of $\bar{G}$ and so we must have that actually $\bar{G}$ is a cyclic group. Since G is conjugate to a cyclic group, it also must be cyclic. ♢

Case 2: $k = 3$

Now our expression is

$$1 + \frac{2}{|G|} = \frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3}.$$

which has four solutions;

(2.1) $e_1 = e_2 = 2, e_3 = \frac{|G|}{2} \quad |G|$ even,

(2.2) $e_1 = 2, e_2 = e_3 = 3, |G| = 12$,

(2.3) $e_1 = 2, e_2 = 3, e_3 = 4, |G| = 24$,

(2.4) $e_1 = 2, e_2 = 3, e_3 = 5, |G| = 60$.

Subcase: $e_1 = e_2 = 2, e_3 = \frac{|G|}{2}$**.** As in the solution for $k = 2$, by replacing G with an appropriate conjugate group we may assume that $\mathcal{O}_3$ consists of the two points 0 and ∞. Since $\{0, \infty\}$ is an orbit, there is an element $h \in G$ such that $h(0) = \infty$ and $h(\infty) = 0$. Let $H = \mathrm{Stab}_G(0)$. Since $e_3 = |H| = \frac{|G|}{2}$, we have that the index of H in G is 2, meaning that H is a normal subgroup. Then:

$$H(\infty) = hHh^{-1}(\infty) = \{\infty\},$$

so actually every element of H fixes the point ∞ too. Thus by our worked solution for the case $k = 2$, we conclude that H is the cyclic group of order $\frac{|G|}{2}$, say $H = \langle \alpha | \alpha^{\frac{|G|}{2}} = 1 \rangle$. Now write $h : z \mapsto \frac{dz+c}{bz+a}$, and we can deduce that:

$$h(0) = \frac{c}{a} = \infty \implies [a : c] = [0 : 1] \implies a = 0$$

and

$$h(\infty) = \frac{d}{b} = 0 \implies d = 0$$

so we have that $h : z \mapsto \frac{\lambda}{z}, \lambda \neq 0$. Clearly h is of order 2, and also $h \notin H$. Furthermore $h\alpha h^{-1} = \alpha^{-1}$ so:

$$G = \langle \alpha, h \mid \alpha^{\frac{|G|}{2}} = h^2 = 1, h\alpha h^{-1} = \alpha^{-1} \rangle,$$

is isomorphic to $D_{|G|}$ the *dihedral group* of order $|G|$. $\diamondsuit$

Solution (2.2) corresponds to the *tetrahedral group*, that is the group of rotational symmetries of the tetrahedron. This group is isomorphic to A_4, the *alternating group* of permutations of 4 objects.

Solution (2.3) corresponds to the *octahedral group*, the group of rotational symmetries of the octahedron (or cube). This group is isomorphic to S_4, the *symmetric group* of permutations of 5 objects.

Solution (2.4) corresponds to the *icosahedral group*, the group of rotational symmetries of the icosahedron (or dodecahedron). This group is isomorphic to A_5, the smallest non-abelian simple group!

2.3 Lifting subgroups of Aut($\mathbb{P}^1$) to subgroups of SL(2,$\mathbb{C}$)

Now that we have fully classified all of the finite subgroups of Aut($\mathbb{P}^1$), how can we relate them to the finite subgroups of SL(2,$\mathbb{C}$)? We consider the surjective action homomorphism, α, for the action we defined in Proposition 2.1. This homomorphism is not injective:

Lemma 2.6. The kernel of α is $\{\pm I\}$

Proof. Since the identity transformation on $\mathbb{P}^1$ must fix 0 and ∞, any element $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \text{Ker}(\alpha)$ satisfies $b = c = 0$ and so must map $z \mapsto \frac{d}{a}z$. Since we want this transformation to be the identity, we must have that $d = a$, implying that Ker(α) consists only of scalar multiples of the identity matrix. The only such multiples which have determinant 1 are I and $-I$. □

The action homomorphism thus provides a *double cover* of Aut($\mathbb{P}^1$) and so any finite $G <$ Aut($\mathbb{P}^1$) can be lifted to a finite $\bar{G} <$ SL(2,$\mathbb{C}$), with twice the number of elements. This means our finite subgroups of SL(2,$\mathbb{C}$) are:

- C_n, the cyclic groups of order n. The cyclic subgroups of Aut($\mathbb{P}^1$) are lifted to cyclic subgroups of even order, but taking appropriate subgroups we may also obtain all of the cyclic subgroups of odd order. We have the presentation:

$$C_n = \langle a \rangle, \quad a = \left(\begin{smallmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{smallmatrix}\right), \omega = e^{\frac{2\pi i}{n}}$$

- The *binary dihedral* group, $\mathcal{B}D_{4n}$ of order $4n$. This group has presentation:

$$\langle a, b \rangle, \quad a = \left(\begin{smallmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{smallmatrix}\right), b = \left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right), \omega = e^{\frac{\pi i}{n}}$$

- The *binary polyhedral* groups, which are
  - the *binary tetrahedral* group, $\mathcal{B}T$, with presentation:

$$\langle a, b, c \mid a^2 = b^3 = c^3 = abc \rangle,$$

  - the *binary octahedral* group, $\mathcal{B}O$, with presentation:

$$\langle a, b, c \mid a^2 = b^3 = c^4 = abc \rangle,$$

  - the *binary icosahedral* group, $\mathcal{B}I$, with presentation:

$$\langle a, b, c \mid a^2 = b^3 = c^5 = abc \rangle,$$

This completes the classification of finite subgroups of SL(2,$\mathbb{C}$).

3 Rings of Invariants

3.1 $\mathbb{C}$-algebras

We recall Definition A.1, that of a $\mathbb{C}$-algebra. Now that we have classified the finite subgroups of SL(2,$\mathbb{C}$), we are interested in creating $\mathbb{C}$-algebras for each of them. From there we can construct algebraic varieties, which will allow us to recover our Dynkin diagrams. The action that we defined in §1 can actually be extended to an action of our groups on the $\mathbb{C}$-algebra $\mathbb{C}[x,y]$. Throughout, we let G be a finite subgroup of SL(2,$\mathbb{C}$).

Proposition 3.1. We define an action of G on $\mathbb{C}[x,y]$ as follows:

$$\forall f \in \mathbb{C}[x,y], g \in G, (g \cdot f)(v) := f(g^{-1} \cdot v)\,\forall v \in \mathbb{C}^2$$

This is indeed an action.

Proof. For I the identity matrix, we have:

$$(\mathrm{I} \cdot f)(v) = f(\mathrm{I} \cdot v) = f(v)$$

so $\mathrm{I}\cdot f = f$ for any $f \in \mathbb{C}[x,y]$. Let $g, h \in \mathrm{SL}(2,\mathbb{C})$. Then:

$$\begin{aligned} (g \cdot (h \cdot f))(v) &= (h \cdot f)(g^{-1} \cdot v) \\ &= f(h^{-1} \cdot (g^{-1} \cdot v)) \\ &= f((h^{-1}g^{-1}) \cdot v) \\ &= f((gh)^{-1} \cdot v) \\ &= ((gh) \cdot f)(v) \end{aligned}$$

so this map is indeed an action. □

Definition 3.2. We define the *ring of invariants* of G to be the $\mathbb{C}$-subalgebra of $\mathbb{C}[x,y]$ for which each element is invariant under the action of G:

$$\mathbb{C}[x,y]^G := \big\{ f \in \mathbb{C}[x,y] \mid g \cdot f = f\ \forall g \in G \big\}.$$

3.2 Calculating $\mathbb{C}$-vector space generators

Proposition 3.3. Define a $\mathbb{C}$-linear map called the *Reynold's operator* $\varrho_G : \mathbb{C}[x,y] \to \mathbb{C}[x,y]^G$ by:

$$\varrho_G(f) = \frac{1}{|G|} \sum_{g \in G} g \cdot f.$$

The Reynold's operator is a projection, in particular, it is surjective.

Proof. Actually $\varrho_G|_{\mathbb{C}[x,y]^G}$ is the identity map, because for any $f \in \mathbb{C}[x,y]^G$ and $g \in G$ we have $g \cdot f = f$ by definition. This tells us that ϱ_G is a projection map ($\varrho_G \circ \varrho_G = \varrho_G$). □

The Reynold's operator allows us to compute explicitly a $\mathbb{C}$-vector space basis for $\mathbb{C}[x,y]^G$ by evaluating at the $\mathbb{C}$-vector space basis of $\mathbb{C}[x,y]$, namely, polynomials of the form $x^r y^s$ where $r, s \in \mathbb{N}$.

Case 1: $G = C_n$, the cyclic group of order n

G is generated by the matrix $g = \left(\begin{smallmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{smallmatrix}\right)$ where $\omega = e^{\frac{2\pi i}{n}}$. Then $g \cdot x = \omega x$ and $g \cdot y = \omega^{-1} y$, so

$$\begin{aligned} \varrho_G(x^r y^s) &= \frac{1}{n} \sum_{k=1}^{n} g^k \cdot x^r y^s \\ &= \frac{1}{n} \sum_{k=1}^{n} \omega^{k(r-s)} x^r y^s. \end{aligned}$$

If $r \neq s \bmod n$ then we are summing over all n^{th} roots of unity, which gives 0. If $r = s \bmod n$, then our sum is $x^r y^s$ and so the $\mathbb{C}$-basis for $\mathbb{C}[x,y]^G$ is $\{x^r y^s \mid r = s \bmod n\}$. We write:

$$\mathbb{C}[x,y]^G = \operatorname{span}_{\mathbb{C}}\{x^r y^s \mid r = s \bmod n\}$$

Case 2: $G = \mathcal{B}D_{4n}$, the binary dihedral group of order 4n

G is generated by the matrices $g = \left(\begin{smallmatrix} \omega^{-1} & 0 \\ 0 & \omega \end{smallmatrix}\right)$ where $\omega = e^{\frac{\pi i}{n}}$ and $h = \left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right)$. We see that $h \cdot x = iy$ and that $h \cdot y = ix$. Any element of the group can be written as $g^p h^q$ for $1 \leq p \leq 2n, 0 \leq q \leq 1$. We compute

$$\begin{aligned} \varrho(x^r y^s) &= \frac{1}{4n} \sum_{k=1}^{2n} (g^k \cdot x^r y^s + g^k h \cdot x^r y^s) \\ &= \frac{1}{4n} \sum_{k=1}^{2n} \omega^{k(r-s)} (x^r y^s + i^{r+s} x^s y^r) \\ &= \begin{cases} 0 & r \neq s \bmod 2n \\ \frac{1}{2}(x^r y^s + x^s y^r) & r = s \bmod 2n, r \text{ even} \\ \frac{1}{2}(x^r y^s - x^s y^r) & r = s \bmod 2n, r \text{ odd} \end{cases}. \end{aligned}$$

Hence in this case we may write

$$\mathbb{C}[x,y]^G = \operatorname{span}_{\mathbb{C}}\{x^r y^s + (-1)^r x^s y^r \mid r = s \bmod 2n\}.$$

3.3 Calculating $\mathbb{C}$-algebra generators

Recall from MA40188 the following definition;

Definition 3.4. A *finitely-generated* $\mathbb{C}$-algebra, R, is a $\mathbb{C}$-algebra for which there exists a surjective $\mathbb{C}$-algebra homomorphism $\phi : \mathbb{C}[x_1, ...x_k] \to R$ for some $k \in \mathbb{N}$.

Remark. What this means in practise is that there are finitely many elements of R, say $f_1, ..., f_k$, such that any other element can be expressed as a polynomial in $f_1, ..., f_k$. In this case, we write $R = \mathbb{C}[f_1, ..., f_k]$.

Proposition 3.5. Let $G = C_n$ the cyclic group of order n. Then $\mathbb{C}[x,y]^G = \mathbb{C}[x^n, y^n, xy]$

Proof. It suffices to show that any basis element $x^r y^s \in \mathbb{C}[x^n, y^n, xy]$. Let $r = an+r'$ and $s = bn+s'$ where $a, b, r', s' \in \mathbb{N}$ and $0 \le r', s' < n$. From the calculations in §3.2, we have that $s' = r'$. Thus we can write

$$x^r y^s = x^{(an+r')} y^{(bn+r')} = (x^n)^a (y^n)^b (xy)^{r'} \in \mathbb{C}[x^n, y^n, xy].$$

□

Proposition 3.6. Let $G = \mathcal{B}D_{4n}$ the binary dihedral group of order $4n$. Then $\mathbb{C}[x,y]^G = \mathbb{C}[x^2y^2, x^{2n}+y^{2n}, xy(x^{2n}-y^{2n})]$

Proof. It is sufficient to show that any basis element $x^r y^s + (-1)^r x^s y^r \in \mathbb{C}[x^2y^2, x^{2n}+y^{2n}, xy(x^{2n} - y^{2n})]$. Write $r = a(2n)+t, s = b(2n)+t$ where $0 \le t < 2n$.
Assume r is even and $b \le a$. Then

$$x^r y^s + (-1)^r x^s y^r = (xy)^t (xy)^{(2n)b} (x^{(2n)(a-b)} + y^{(2n)(a-b)}). \tag{†}$$

Claim: $x^{(2n)c} + y^{(2n)c} \in \mathbb{C}[x^2y^2, x^{2n}+y^{2n}, xy(x^{2n}-y^{2n})] \quad \forall c \in \mathbb{N}$.
Proof of claim: By induction. We need to check two base cases, $c = 1$ and $c = 2$.The base case of $c = 1$ follows by assumption. The base case of $c = 2$ follows from the calculation

$$(x^{(2n)2} + y^{(2n)2}) = (x^{2n}+y^{2n})^2 - 2(x^2y^2)^n.$$

Now assume the claim holds for all $c < C$ and consider $x^{(2n)C} + y^{(2n)C}$. We can write

$$x^{(2n)C} + y^{(2n)C} = (x^{(2n)} + y^{(2n)})^C - \sum_{k=1}^{C-1} \binom{C}{k} x^{(2n)(C-k)} y^{(2n)k}$$

but we can group the k and $C-k$ terms in the sum together to get

$$\begin{aligned}\sum_{k=1}^{C-1} \binom{C}{k} x^{(2n)(C-k)} y^{(2n)k} &= \sum_{k=1}^{\frac{C-1}{2}} \binom{C}{k} \left(x^{(2n)(C-k)} y^{(2n)k} + x^{(2n)k} y^{(2n)(C-k)}\right) \\ &= \sum_{k=1}^{\frac{C-1}{2}} \binom{C}{k} \left(x^2y^2\right)^{nk} \left(x^{(2n)(C-2k)} + y^{(2n)(C-2k)}\right)\end{aligned}$$

when $C-1$ is even and then the claim follows from the inductive hypothesis. The case where $C-1$ is odd is similar but with an additional term for $k = \frac{C}{2}$ which is $2\binom{C}{k}(x^2y^2)^{nk}$.
Then the expression (†) belongs to $\mathbb{C}[x^2y^2, x^{2n}+y^{2n}, xy(x^{2n} - y^{2n})]$. The case where $a \le b$ is identical.
Now assume that r is odd and that $b \le a$. We then have the expression

$$x^r y^s + (-1)^r x^s y^r = (xy)^{t-1} (xy)^{(2n)b} (xy) (x^{(2n)(a-b)} - y^{(2n)(a-b)}). \tag{††}$$

Claim: $xy(x^{(2n)c} - y^{(2n)c}) \in \mathbb{C}[x^2y^2, x^{2n}+y^{2n}, xy(x^{2n}-y^{2n})] \quad \forall c \in \mathbb{N}$.
Proof of claim: By induction. Again the base case of $c = 1$ follows immediately so assume the claim holds for all $c < C$. By the previous claim, $x^{(2n)(C-1)} + y^{(2n)(C-1)} \in \mathbb{C}[x^2y^2, x^{2n}+y^{2n}, xy(x^{2n}-y^{2n})]$. We compute

$$xy(x^{2n}-y^{2n})(x^{(2n)(C-1)} + y^{(2n)(C-1)}) = xy(x^{(2n)C} - y^{(2n)C}) - (xy)^{2n}\left(xy(x^{2n(C-2)} - y^{2n(C-2)})\right)$$

and rearranging yields

$$xy(x^{(2n)C} - y^{(2n)C}) = xy(x^{2n} - y^{2n})(x^{(2n)(C-1)} + y^{(2n)(C-1)}) + (xy)^{2n}\big(xy(x^{2n(C-2)} - y^{2n(C-2)})\big).$$

The right hand side is an element of $\mathbb{C}[x^2y^2, x^{2n} + y^{2n}, xy(x^{2n} - y^{2n})]$ by the inductive hypothesis. Therefore, in the expression (††), every factor is an element of $\mathbb{C}[x^2y^2, x^{2n} + y^{2n}, xy(x^{2n} - y^{2n})]$ by the claim and the fact that t is odd (which follows from the assumption that r is odd). The case where $b \leq a$ is the same. □

Theorem 3.7. For any finite subgroup of SL(2,$\mathbb{C}$), G, the $\mathbb{C}$-algebra $\mathbb{C}[x,y]^G$ is finitely generated by 3 elements.

3.4 The Ideal of Relations

For any finite $G <$ SL(2,$\mathbb{C}$), we can construct a surjective $\mathbb{C}$-algebra homomorphism $\varphi_G : \mathbb{C}[u,v,w] \to \mathbb{C}[x,y]^G, F \mapsto F(f_1, f_2, f_3)$, where $\mathbb{C}[x,y]^G = \mathbb{C}[f_1, f_2, f_3]$ by Theorem 3.7. The kernel of φ_G is an ideal in $\mathbb{C}[u,v,w]$ and so $\mathbb{V}(\mathrm{Ker}(\varphi_G)) \subset \mathbb{C}^3$ is an affine algebraic set with coordinate ring $\mathbb{C}[x,y]^G$.

Definition 3.8. We let the affine algebraic set $\mathbb{V}(\mathrm{Ker}(\varphi_G)) \subset \mathbb{C}^3$ be denoted $\mathbb{C}^2/G$.

We wish to show that $\mathbb{C}^2/G$ is an irreducible hypersurface and according to Theorem A.5, we need only show that it has dimension 2. Since the ring of invariants is a $\mathbb{C}$-subalgebra of the ring of polynomials in 2 variables, we let $\iota : \mathbb{C}[x,y]^G \hookrightarrow \mathbb{C}[x,y]$ be the inclusion $\mathbb{C}$-algebra homomorphism. We make use of Theorem A.4 from MA40188 applied to the algebraic sets $\mathbb{C}^2/G$ and $\mathbb{C}^2$ (note that $\mathbb{C}[x,y]$ is the coordinate ring for $\mathbb{C}^2$) to obtain a polynomial map

$$\pi : \mathbb{C}^2 \to \mathbb{C}^2/G.$$

Now we want to show that this map has finite fibres. It would suffice to show that $\mathbb{C}^2/G$ is the set of G-orbits of $\mathbb{C}^2$ as the notation suggests, since the group G is finite. This is not an obvious fact from the definition and requires more commutative algebra than we have time to establish. As some motivation, we prove the following result:

Proposition 3.9. Consider the set of G-orbits, $\mathbb{C}^2/G$ and assume that this set is an affine algebraic set. The coordinate ring for this affine algebraic set is the ring of invariants $\mathbb{C}[x,y]^G$.

Proof. Let $f \in \mathbb{C}[\mathbb{C}^2/G]$ so f is a polynomial which is constant on any G-orbit. Let $p \in \mathbb{C}^2$,

$$\begin{aligned} f(p) = f(g^{-1}p) = (g \cdot f)(p) \quad \forall g \in G \\ \implies f \in \mathbb{C}[x,y]^G. \end{aligned}$$

Likewise, for any $h \in \mathbb{C}[x,y]^G$ and for any point $p \in \mathbb{C}^2$,

$$\begin{aligned} h(p) = (g^{-1} \cdot h)(p) = h(gp) \quad \forall g \in G \\ \implies h \in \mathbb{C}[\mathbb{C}^2/G] \end{aligned}$$

as h is constant on each G-orbit. □

So the polynomial map π has finite fibres, which implies that $\dim(\mathbb{C}^2/G) = 2$ and hence that $\mathbb{C}^2/G = \langle f \rangle$ for some irreducible $f \in \mathbb{C}[u,v,w]$. We call this f the *defining equation* for the hypersurface $\mathbb{C}^2/G$.

Proposition 3.10. Let G be the cyclic group of order n. Then $\mathbb{C}^2/G = \mathbb{V}(xy - z^n)$.

Proof. We seek an irreducible polynomial $f \in \mathbb{C}[u, v, w]$ such that $f(x^n, y^n, xy) = 0 \in \mathbb{C}[x, y]^G$. Consider $f = uv - w^n \in \text{Ker}(\varphi_G)$. This polynomial is irreducible since if we assumed it wasn't, any factors would have to have degree one or zero in u but there is only one term in f which involves u. □

Proposition 3.11. Let G be the binary dihedral group of order $4n$. Then $\mathbb{C}^2/G = \mathbb{V}(x^{n+1} + xy^2 + z^2)$.

Proof. We now seek an irreducible polynomial $f \in \mathbb{C}[u, v, w]$ such that $f(x^2y^2, xy(x^{2n} - y^{2n}), x^{2n} + y^{2n}) = 0$. We note that $\mathbb{C}[x^2y^2, x^{2n} + y^{2n}, xy(x^{2n} - y^{2n})] = \mathbb{C}[ax^2y^2, b(x^{2n} + y^{2n}), cxy(x^{2n} - y^{2n})]$ for any non-zero scalars $a, b, c \in \mathbb{C}$. We compute

$$\begin{aligned}&(ax^2y^2)^{n+1} + (ax^2y^2)(b(x^{2n} + y^{2n}))^2 + (c(xy)(x^{2n} - y^{2n}))^2 \\ &= (a^{n+1} + 2ab^2 - 2c^2)(xy)^{2(n+1)} + (ab^2 + c^2)(x^2y^2)(x^{4n} + y^{4n}). \qquad (*)\end{aligned}$$

If we pick a, b, c such that $c^2 = -1, ab^2 = 1, a^{n+1} = -4$ we get that (*) is equal to zero, which is precisely what we wanted to show. □

In the interest of completeness, the following is a list of the defining equations for each of the finite subgroups of SL(2,$\mathbb{C}$):

- $G = C_n, \mathbb{C}^2/G = \mathbb{V}(xy - z^n)$
- $G = \mathcal{B}D_{4n}, \mathbb{C}^2/G = \mathbb{V}(x^{n+1} + xy^2 + z^2)$
- $G = \mathcal{B}T, \mathbb{C}^2/G = \mathbb{V}(x^4 + y^3 + z^2)$
- $G = \mathcal{B}O, \mathbb{C}^2/G = \mathbb{V}(x^3y + y^3 + z^2)$
- $G = \mathcal{B}I, \mathbb{C}^2/G = \mathbb{V}(x^5 + y^3 + z^2)$

4 Resolution of Singularities

Proposition 4.1. For any finite subgroup of SL(2,$\mathbb{C}$), except the trivial subgroup (which we write as the cyclic group with one element C_1), the affine hypersurface $\mathbb{C}^2/G$ is singular. It has only one singularity, which is at the origin.

Proof. We simply need to find $\mathbb{V}(f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$. In each case, this can be computed explicitly. If $G = C_n$, then we are looking for solutions to the simultaneous equations:

$$xy - z^n = 0, y = 0, x = 0, nz^{n-1} = 0$$

If $n = 1$ then there is no solution to the above system, as the last equation becomes $1 = 0$. If $n \neq 1$ then the only solution to the system is $(x, y, z) = (0, 0, 0)$. A similar argument works for $G = \mathcal{B}T, \mathcal{B}I$.

If $G = \mathcal{B}D_{4n}$, we want simultaneous solutions to:

$$x^{n-1} + xy^2 + z^2 = 0, (n-1)x^{n-2} + y^2 = 0, 2xy = 0, 2z = 0.$$

The last equation immediately tells us that $z = 0$, where the penultimate equation gives us that either $x = 0$ or $y = 0$. In either case, substituting the result into the second equation yields that the only solution is $(x, y, z) = (0, 0, 0)$. A similar argument works for $\mathcal{B}O$. □

When we have singularities in affine varieties, we wish to be able to look at the way that the variety behaves at the singularity. We develop some tools in the next section to deal with this.

4.1 Blow-up

Definition 4.2. Consider the Cartesian product,

$$\mathbb{C}^n \times \mathbb{P}^{n-1} = \Big\{\big((x_1, ..., x_n), [y_1 : ... : y_n]\big) \mid (x_1, ..., x_n) \in \mathbb{C}^n, [y_1 : ... : y_n] \in \mathbb{P}^{n-1}\Big\}.$$

The *blow-up* of $\mathbb{C}^n$ at the origin O is

$$B = \mathbb{V}(x_i y_j - x_j y_i \mid i, j = 1, ..., n) \subset \mathbb{C}^n \times \mathbb{P}^{n-1}.$$

In this space we call $O \times \mathbb{P}^{n-1}$ the *exceptional divisor* of B and denote it by E.

The composition of the inclusion $B \hookrightarrow \mathbb{C}^n \times \mathbb{P}^{n-1}$ and the projection $\mathbb{C}^n \times \mathbb{P}^{n-1} \twoheadrightarrow \mathbb{C}^n$ defines a morphism

$$\psi : B \to \mathbb{C}^n.$$

The space B and morphism ψ satisfy the properties that we want in order to "resolve" singularities. We summarise these properties in the following result.

Lemma 4.3.
1. $\psi^{-1}(O) \cong \mathbb{P}^{n-1}$ and each point in $\psi^{-1}(O)$ corresponds to a unique line through the origin in $\mathbb{C}^n$, so $E = \psi^{-1}(O)$.
2. If $O \neq p \in \mathbb{C}^n$, then $\psi^{-1}(p)$ is a point.
3. B is an irreducible algebraic set.

Proof. 1. The first claim follows from the fact that

$$\psi^{-1}(O) = \Big\{ \big((0, ..., 0), [y_1 : ... : y_n]\big) \mid [y_1 : ... : y_n] \in \mathbb{P}^{n-1} \Big\}.$$

Let L be a line through the origin in $\mathbb{C}^n$, given by the parameterisation:

$$L = \big\{ (a_1 t, ..., a_n t) \mid t \in \mathbb{C}^1 \big\},$$

for some fixed $a_i \in \mathbb{C}$ for $1 \leq i \leq n$ not all 0. Let $L' = \psi^{-1}(L \backslash \{O\}) \subset B \backslash \psi^{-1}(O)$, so then:

$$\begin{aligned} L' &= \Big\{ \big((a_1 t, ..., a_n t), [a_1 t : ... : a_n t]\big) \mid t \in \mathbb{C}^1 \backslash \{0\} \Big\} \\ &= \Big\{ \big((a_1 t, ..., a_n t), [a_1 : ... : a_n]\big) \mid t \in \mathbb{C}^1 \backslash \{0\} \Big\} \end{aligned}$$

Then the closure of L' in B, $\overline{L'} = \Big\{ \big((a_1 t, ..., a_n t), [a_1 : ... : a_n]\big) \mid t \in \mathbb{C}^1 \Big\}$, meets $\psi^{-1}(O)$ at the point $O \times [a_1 : ... : a_n] \in B$. So the map which sends L to the point $[a_1 : ... : a_n]$ defines a one-to-one correspondence between lines through the origin in $\mathbb{C}^n$ and points in $\psi^{-1}(O)$.

2. We show that ψ defines an isomorphism between $B \backslash \psi^{-1}(O)$ and $\mathbb{C}^n \backslash \{O\}$. Let $p = (a_1, ..., a_n) \in \mathbb{C}^n$ where some $a_i \neq 0$. If $p \times [y_1 : ... : y_n] \in \psi^{-1}(p)$, then by definition of B we must have that $y_j = \frac{a_j}{a_i} y_i, \forall j = 1, ..., n$. This defines a unique point $[y_1 : ... : y_n] = [a_1 : ... : a_n] \in \mathbb{P}^{n-1}$. Hence, $\psi^{-1}(p) = (a_1, ..., a_n) \times [a_1 : ... : a_n]$ and we have achieved the isomorphism we wanted to.

3. We can write $B = (B \backslash \psi^{-1}(O)) \cup (\psi^{-1}(O))$. From (2), we have that $B \backslash \psi^{-1}(O) \cong \mathbb{C}^n \backslash \{O\}$, which is irreducible. From (1), we know that any point of $\psi^{-1}(O)$ belongs to the closure of some subset of $B \backslash \psi^{-1}(O)$, meaning that $B \backslash \psi^{-1}(O)$ is dense in B, i.e., $B = \overline{B \backslash \psi^{-1}(O)}$. However the closure of an irreducible set is irreducible, so B must be irreducible.

□

Definition 4.4. Let $X \subset \mathbb{C}^n$ be an algebraic variety passing through the origin. The *blow-up* of X at the origin is $\widetilde{X} = \overline{\psi^{-1}(X \backslash \{O\})}$.

Remark. This space is contained in B by definition. In order to blow-up a different point, $p \in X$, we simply apply a linear change of coordinates so that p is sent to O and then apply our blow up. If the resulting variety is smooth we say that we have achieved a *resolution of singularities.* If not, we may apply the process again repeatedly until we do end up with a smooth variety.

Example. Consider the affine hypersurface $X = \mathbb{V}(y^2 - x^2(x+1)) \subset \mathbb{C}^2$. This hypersurface is singular only at the origin and we resolve its singularity by blowing up. Let the projective coordinates of $\mathbb{P}^1$ be given by $[t : v]$. The blow-up of X is $\widetilde{X} = \mathbb{V}(xv - yt, y^2 - x^2(x+1)) \subset \mathbb{C}^2 \times \mathbb{P}^1$. We can cover $\mathbb{C}^2 \times \mathbb{P}^1$ by two affine sets $U_t = \{((x, y), [t : v]) \in \mathbb{C}^2 \times \mathbb{P}^1 \mid t \neq 0\}$ and U_v defined similarly. We first consider $\widetilde{X} \cap U_t =: \widetilde{X}_t$. In this space we may associate the projective coordinate $[t : v]$ with an affine coordinate $\frac{v}{t}$ so that we can talk about points in $\widetilde{X}_t$ as points in $\mathbb{C}^3$ with coordinates $(x, y, \frac{v}{t})$. Now $\widetilde{X}_t = \mathbb{V}(x\frac{v}{t} - y, y^2 - x^2(x+1)) \subset U_t \cong \mathbb{C}^3$. Substituting the equation $y = x\frac{v}{t}$ into $y^2 = x^2(x+1)$, we get that $x^2((\frac{v}{t})^2 - x - 1) = 0$.

We must have that either $x^2 = 0$ or that $(\frac{v}{t})^2 = x + 1$. The first case means that $y = x = 0$ and can have any solution for $\frac{v}{t}$, which corresponds to $E \cap U_t$. The other case corresponds to

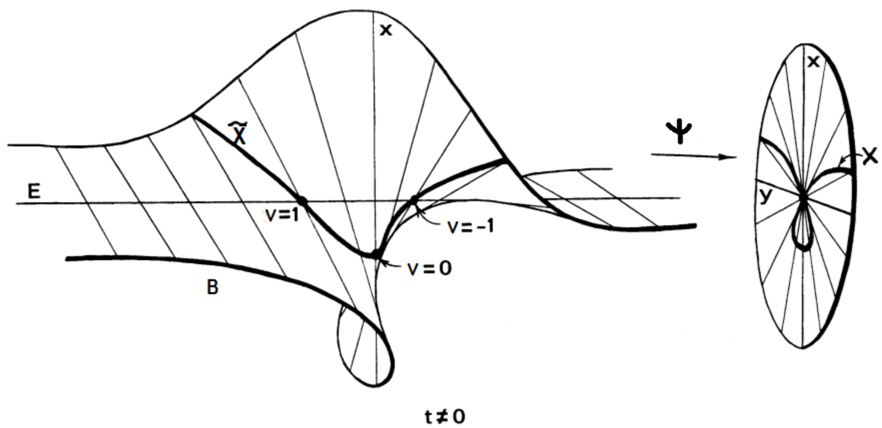


Figure 1: Blow-up of $\mathbb{V}(y^2 - x^2(x+1))$ at the origin, [1]p.29

$\widetilde{X}_t$ and intersects E when $(x, y) = (0, 0)$, leaving the equation $\frac{v}{t} = \pm 1$. Hence we have two points $(x, y, \frac{v}{t}) = (0, 0, \pm 1)$ which correspond to the lines through the origin in which our variety 'approaches' its singularity. $\diamondsuit$

Armed with this powerful technique, we are ready to resolve the singularities of our varieties $\mathbb{C}^2/G$, for the cyclic and binary dihedral cases. First we need a brief discussion of notation so that our results are consistent. We will use the coordinates $\big((x, y, z), [a : b : c]\big) \in \mathbb{C}^3 \times \mathbb{P}^2$. Then

$$B = \mathbb{V}(xb - ya, xc - za, yc - zb)$$

and we will write $\widetilde{X}$ for the blow up at the origin of the variety $X \subset \mathbb{C}^3$. We let

$$U_a = \big\{\big((x, y, z), [a : b : c]\big) \in \mathbb{C}^3 \times \mathbb{P}^2 \mid a \neq 0\big\}$$

and define U_b, U_c similarly. Write $\widetilde{X}_a = X \cap U_a$ and similarly for $\widetilde{X}_b$ and $\widetilde{X}_c$. We call these sets the affine charts of X in the respective coordinates a, b and c.

Definition 4.5. Let X be an affine variety in $\mathbb{C}^3$ and $\widetilde{X}$ be its resolution. The *resolution graph* for $\widetilde{X}$ is a graph where each vertex corresponds to an irreducible component of $\widetilde{X} \cap E$ and two vertices are joined by an edge if the corresponding irreducible components intersect.

4.2 Resolving singularities for the cyclic group

Theorem 4.6. Let $G = C_n$, the cyclic group of order n where $n \geq 2$. The resolution graph for $\mathbb{C}^2/G$ is the affine Dynkin diagram A_{n-1}, •—•⋯•—•.

This subsection will be a proof of Theorem 4.6 by induction. We split the proof up so that it is more easily digestible.

Proof of base case $n = 2$. The defining equation for $\mathbb{C}^2/G$ is $f = xy - z^2$ so the blowup of $X = \mathbb{C}^2/G$ at the origin is the variety $\widetilde{X} = \mathbb{V}(xb - ya, xc - za, yc - zb, xy - z^2)$. We consider first the affine chart in the coordinate a;

$$\widetilde{X}_a = \mathbb{V}(x\tfrac{b}{a} - y, x\tfrac{c}{a} - z, yc - zb, xy - z^2).$$

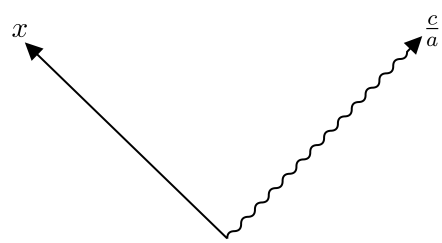
Substituting the first two equations into the last one, we end up with the equation:

$$x^2\left(\frac{b}{a} - (\frac{c}{a})^2\right) = 0.$$

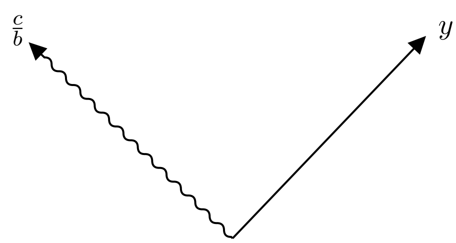
This has two solutions, $x^2 = 0$ and $\frac{b}{a} = (\frac{c}{a})^2$. The first solution forces $x = y = z = 0$ and so corresponds to $E \cap U_a$. Using the second solution, and the first two equations in $\widetilde{X}_a$, we can construct an invertible polynomial morphism

$$(x, y, z, \frac{b}{a}, \frac{c}{a}) = (x, x(\frac{c}{a})^2, x\frac{c}{a}, (\frac{c}{a})^2, \frac{c}{a}) \mapsto (x, \frac{c}{a}) \in \mathbb{C}^2.$$

Since $\mathbb{C}^2$ is itself a smooth affine variety, and we have just shown that $\widetilde{X}_a$ is isomorphic to $\mathbb{C}^2$, we must have that $\widetilde{X}_a$ is smooth, and that it can be represented as a copy of $\mathbb{C}^2$ with coordinates x and $\frac{c}{a}$ as shown below:



In this diagram the wavy line corresponds to $\widetilde{X}_a \cap E$.
If we consider the affine chart in the coordinate b, we go through identical calculations with the roles of x and $\frac{c}{a}$ replaced by y and $\frac{c}{b}$ respectively. Importantly we get another isomorphism to $\mathbb{C}^2$ so $\widetilde{X}_b$ is smooth and can be represented as a copy of $\mathbb{C}^2$ with coordinates y and $\frac{c}{b}$:



The only affine chart left to check is $\widetilde{X}_c$. We have that:

$$\widetilde{X}_c = \mathbb{V}(xb - ya, x - z\frac{a}{c}, y - z\frac{b}{c}, xy - z^2).$$

Substituting the second and third equations into the last gives us the equation:

$$(z\frac{a}{c})(z\frac{b}{c}) - z^2 = z^2(\frac{a}{c}\frac{b}{c} - 1) = 0.$$

As before, this equation has two solutions. The first is $z^2 = 0$, which forces $x = y = z = 0$ so that this solution corresponds to $E \cap U_c$. The second solution is $\frac{a}{c}\frac{b}{c} = 1$. Notice that in this second solution, we have that $a \neq 0$ and $b \neq 0$, so that any solution of this type is actually in $\widetilde{X}_a \cup \widetilde{X}_b$. Hence there is no new information gained from looking in this chart.

Now that we have found that our variety is smooth in its affine charts, it must be smooth globally and we are interested in 'gluing' our charts together to get a picture of the whole variety.
Claim: $\widetilde{X}_a \backslash \{\frac{c}{a} = 0\} \cong \widetilde{X}_b \backslash \{\frac{c}{b} = 0\}$.
Proof of claim: In $\widetilde{X}_a \backslash \{\frac{c}{a} = 0\}$, we have

$$\frac{a}{c} = (\frac{c}{a})^{-1} = \frac{c/a}{(c/a)^2} = \frac{c/a}{b/a} = \frac{c}{b}.$$

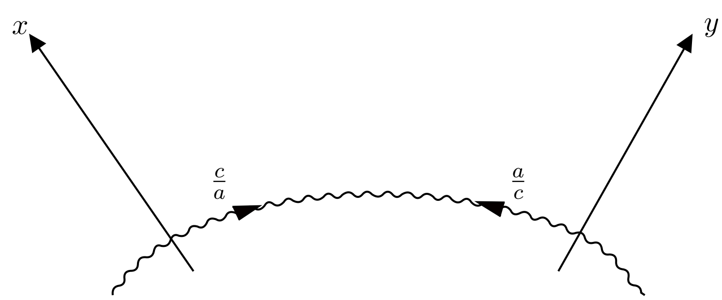
Therefore we can define our isomorphism by the morphism

$$(x, \frac{c}{a}) \mapsto (x(\frac{c}{a})^2, (\frac{c}{a})^{-1}) = (y, \frac{c}{b}),$$

with inverse morphism

$$(y, \frac{c}{b}) \mapsto (y(\frac{c}{b})^2, (\frac{c}{b})^{-1}) = (x, \frac{c}{a}).$$

The point $(x, y, z, \frac{b}{a}, \frac{c}{a}) = (0,0,0,0,0) \in \widetilde{X}_a$ corresponds to the point $((x, y, z), [a : b : c]) = ((0,0,0), [1 : 0 : 0]) \in \widetilde{X}$ which is the only point in $E \cap \widetilde{X}_a \backslash \widetilde{X}_b$. This is the point at infinity of the projective line corresponding to the $\frac{a}{c}$ axis of $\widetilde{X}_b$ in the following picture:



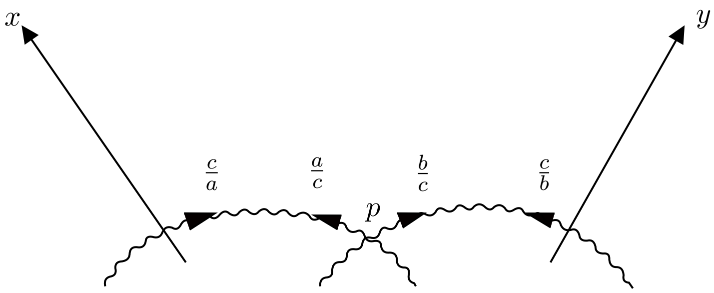
which represents $\widetilde{X}$, where the wavy line now corresponds to a projective line inside the exceptional divisor. □

Proof of base case n = 3. The defining equation for $\mathbb{C}^2/G$ is $f = xy - z^3$ so the blowup of $\mathbb{C}^2/G$ at the origin is the variety $\widetilde{X} = \mathbb{V}(xb - ya, xc - za, yc - zb, xy - z^3)$. Much as in the C_2 case, the affine charts $\widetilde{X}_a$ and $\widetilde{X}_b$ are isomorphic to $\mathbb{C}^2$, with the same coordinates as before. The interesting chart this time, is $\widetilde{X}_c$.

First, we divide throughout by c to get that $\widetilde{X}_c = \mathbb{V}(xb - ya, x - z\frac{a}{c}, y - z\frac{b}{c}, xy - z^3)$. Substituting the second and third equations into the last one we get

$$(z\tfrac{a}{c})(z\tfrac{b}{c}) - z^3 = z^2(\tfrac{a}{c}\tfrac{b}{c} - z) = 0.$$

This equation has solutions $z^2 = 0$ and $\frac{a}{c}\frac{b}{c} = z$. The first solution forces $x = y = z = 0$ and so corresponds to $U_c \cap E$. The second solution corresponds to $\widetilde{X}_c$. Notice that if either $a \neq 0$ or $b \neq 0$ then the point $\big((x, y, z), [a : b : c]\big) \in \widetilde{X}_a \cup \widetilde{X}_b$. Hence the only point of $\widetilde{X}_c$ which lies outside of the affine charts we have already considered is the point $p = \big((0,0,0), [0 : 0 : 1]\big)$. Notice that in our affine charts $\widetilde{X}_a, \widetilde{X}_b$, the axis which corresponds to the intersection with the exceptional divisor meets this point in the projective plane as the point at infinity. Hence we get the following picture for $\widetilde{X}$:



where again the wavy lines correspond to the intersection with the exceptional divisor. Notice that this time we have two projective lines and that they intersect, so our resolution graph is . □

Proof of Theorem 4.6. Now we have our two base cases, we are ready to make our inductive step. Assume that the resolution graph of $\mathbb{C}^2/G$ is the Dynkin diagram A_{n-1} for $G = C_n, n \leq N$. We now look at $\mathbb{C}^2/G$ for $G = C_{N+1}$. The defining equation for this hypersurface is $f = xy - z^{N+1}$ so the blowup of $\mathbb{C}^2/G$ at the origin is the variety $\widetilde{X} = \mathbb{V}(xb - ya, xc - za, yc - zb, xy - z^{N+1})$.
In the chart $\widetilde{X}_a$, substituting the first two equations into the last one we obtain the equation:

$$x(x\tfrac{b}{a}) - (x\tfrac{c}{a})^{N+1} = x^2(\tfrac{b}{a} - x^{N-1}(\tfrac{c}{a})^{N+1}) = 0.$$

The first solution to this equation, $x^2 = 0$, corresponds to $U_a \cap E$. The second solution is $\frac{b}{a} = x^{N-1}(\frac{c}{a})^{N+1}$. Using this solution, we can construct a polynomial isomorphism from $\widetilde{X}_a$ to $\mathbb{C}^2$:

$$(x, y, z, \tfrac{b}{a}, \tfrac{c}{a}) = (x, x^N(\tfrac{c}{a})^{N+1}, x\tfrac{c}{a}, x^{N-1}(\tfrac{c}{a})^{N+1}, \tfrac{c}{a}) \mapsto (x, \tfrac{c}{a}) \in \mathbb{C}^2.$$

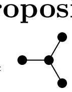
This isomorphism implies that $\widetilde{X}_a \cong \mathbb{C}^2$ is smooth with coordinates x and $\frac{c}{a}$. We can use a similar argument for the affine chart $\widetilde{X}_b$ to get that $\widetilde{X}_b \cong \mathbb{C}^2$, with coordinates y and $\frac{c}{b}$.
Now for the affine chart $\widetilde{X}_c$. Here we substitute the second and third equations into the last one to get the equation:

$$(z\tfrac{a}{c})(z\tfrac{b}{c}) - z^{N+1} = z^2(\tfrac{a}{c}\tfrac{b}{c} - z^{N-1}) = 0.$$

The first solution to this equation, $z^2 = 0$, corresponds to $U_c \cap E$. The second solution is $\frac{a}{c}\frac{b}{c} - z^{N-1} = 0$. The set of solutions to this equation is $\mathbb{V}(xy - z^{N-1})$, which is a singular variety in $\mathbb{C}^3$, it is in fact $\mathbb{C}^2/G$ for $G = C_{N-1}$(!). Since this affine chart is singular, and in fact has a singularity at the origin, we blow-up again in order to resolve this singularity. However, since we know that the defining equation of this variety is the same for the defining equation of $\mathbb{C}^2/C_{N-1}$, we know by induction that the resolution graph of this repeated blow-up will be the resolution graph of $\mathbb{C}^2/C_{N-1}$. Since in our first blow-up we had two projective lines which met at our singular point, we end up with a resolution graph of , the Dynkin diagram of type A_N. □

4.3 Resolving singularities for $\mathcal{B}D_8$

In the interest of time, the only other case that we will work through will be the most basic interesting case of the binary dihedral group.

Proposition 4.7. Let $G = \mathcal{B}D_8$. Then the resolution graph for $\mathbb{C}^2/G$ is the affine Dynkin diagram D_4  .

Proof. In this case the defining equation is $f = x^3 + xy^2 + z^2$ so our blow up is given by:

$$\widetilde{X} = \mathbb{V}(xb - ya, xc - za, yc - zb, x^3 + xy^2 + z^2).$$

As before, we check each affine chart in turn. In $\widetilde{X}_a$, substituting the first two equations into the last one yields the equation:

$$x^3 + x(x\tfrac{b}{a})^2 + (x\tfrac{c}{a})^2 = x^2\Big(x\big(1 + (\tfrac{b}{a})^2\big) + (\tfrac{c}{a})^2\Big) = 0.$$

The first solution to this equation, $x^2 = 0$, again corresponds to $U_a \cap E$. We solve the system of equations:

$$f(x, \tfrac{b}{a}, \tfrac{c}{a}) = x\big(1 + (\tfrac{b}{a})^2\big) + (\tfrac{c}{a})^2 = 0, \tag{1}$$
$$\frac{\partial f}{\partial x} = 1 + (\tfrac{b}{a})^2 = 0, \tag{2}$$
$$\frac{\partial f}{\partial \frac{b}{a}} = 2x\tfrac{b}{a} = 0, \tag{3}$$
$$\frac{\partial f}{\partial \frac{c}{a}} = 2\tfrac{a}{c} = 0. \tag{4}$$

Equation (4) tells us that $\frac{a}{c} = 0$, and equation (3) tells us that either $x = 0$ or $\frac{b}{a} = 0$. However, for this to be consistent with equation (2), we must have $x = 0$, because $(\frac{b}{a})^2 = -1$. We find that the solutions $(x, \frac{b}{a}, \frac{c}{a}) = (0, \pm i, 0)$ also satisfy equation (1), and so $\mathbb{V}(f)$ has two singularities, one at each of these points. This means that we will need to blow up this chart again.
Let us now look at $\widetilde{X}_b$. In this chart, substituting the first and third equations into the last, we end up with the equation:

$$(y\tfrac{a}{b})^3 + (y\tfrac{a}{b})y^2 + (y\tfrac{c}{b})^2 = y^2\Big(y\big((\tfrac{a}{b})^3 + \tfrac{a}{b}\big) + (\tfrac{c}{b})^2\Big) = 0.$$

The irreducible component $y^2 = 0$ corresponds to $U_b \cap E$. We check now for singularities in the other irreducible part. The system of equations:

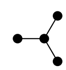
$$g(y, \tfrac{a}{b}, \tfrac{c}{b}) = y(\tfrac{a}{b})^3 + y\tfrac{a}{b} + (\tfrac{c}{b})^2 = 0 \tag{5}$$
$$\frac{\partial g}{\partial y} = (\tfrac{a}{b})^3 + \tfrac{a}{b} = 0 \tag{6}$$
$$\frac{\partial g}{\partial \frac{a}{b}} = 3y(\tfrac{a}{b})^2 + y = 0 \tag{7}$$
$$\frac{\partial g}{\partial \frac{c}{b}} = 2\tfrac{c}{b} = 0 \tag{8}$$

has solutions at 3 points, with coordinates $y = \frac{c}{b} = 0$, $(\frac{a}{b})^3 + \frac{a}{b} = 0$. Hence $\frac{a}{b} \in \{0, \pm i\}$, but these second two solutions belong to $\widetilde{X}_a$ as well. They are in fact the solutions we already found, so we have only found three singularities of $\tilde{X}$, the ones to be found in $\widetilde{X}_b$.
Lastly, we look at $\widetilde{X}_c$. In this chart, substituting the second and third equations into the last one, leaves us with the equation:

$$(z\tfrac{a}{c})^3 + (z\tfrac{a}{c})(z\tfrac{b}{c})^2 + z^2 = z^2\big(z(\tfrac{a}{c})^3 + \tfrac{a}{c}(\tfrac{b}{c})^2 + 1\big) = 0.$$

As before, the first irreducible component of this equation corresponds to $U_c \cap E$. We check for singularities in the second irreducible part. The system of equations:

$$h(z, \tfrac{a}{c}, \tfrac{b}{c}) = z(\tfrac{a}{c})^3 + \tfrac{a}{c}(\tfrac{b}{c})^2 + 1 = 0 \tag{9}$$
$$\frac{\partial h}{\partial z} = (\tfrac{a}{c})^3 = 0 \tag{10}$$
$$\frac{\partial h}{\partial \frac{a}{c}} = 3z(\tfrac{a}{c})^2 + (\tfrac{b}{c})^2 = 0 \tag{11}$$
$$\frac{\partial h}{\partial \frac{b}{c}} = 2\tfrac{a}{c}\tfrac{b}{c} = 0 \tag{12}$$

has no solutions. Hence this affine chart is smooth.
So our blow-up is singular and we need to reiterate the process in order to resolve our singularities. We make that appeal that under the change of coordinates $y \mapsto y, (\frac{a}{b})^3 + \frac{a}{b} \mapsto x, \frac{c}{b} \mapsto iz$, our defining equation for the singular part of $\widetilde{X}_b$ is $xy - z^2$, which is in fact the defining equation for our A_1 hypersurface. Thus, making a linear change of coordinates in order to position our singularities at the origin if required, we end up with the exceptional divisor of our new blow-up containing only a projective line. Note that all of these singularities lie on a projective line, so our resolution graph will contain a node connected to three others, where the other nodes will come from our second blow-ups. Hence the resolution graph of this blow-up is  . □

A similar process can be used to find that the resolution graph for the binary dihedral group of order $4n$ is the Dynkin diagram of type D_{n+2} and that the resolution graphs for the binary tetrahedral, octohedral and icosohedral groups are the Dynkin diagrams of type $E_{6,7,8}$ respectively.

5 McKay Graphs

Now that we have established the correspondence in one direction, we wish to establish it in the reverse direction. That is, can we construct the Dynkin diagrams directly from the groups and then reconstruct the singularities from the diagrams? We make use throughout this section of the content of the module MA40054 Representation Theory of Finite Groups.

5.1 Constructing the McKay Graph

Definition 5.1. The direct sum of vector spaces $V_1, ..., V_n$, written $V_1 \oplus ... \oplus V_n$ or $\bigoplus_{i=1}^n V_i$ is the vector space of n-tuples $(v_1, ..., v_n)$ where $v_i \in V_i$ and addition and scalar multiplication are defined component-wise. If $V_1, ..., V_n$ are G-modules, we make $\bigoplus_{i=1}^n V_i$ a G-module via the linear action:

$$g \cdot (v_1, ..., v_n) = (g \cdot v_1, ..., g \cdot v_n) \quad \forall g \in G, (v_1, ..., v_n) \in \bigoplus_{i=1}^{n} V_i.$$

If ρ_i is the representation for the G-module V_i, we also write $\bigoplus_{i=1}^n \rho_i$ to mean the representation for the G-module $\bigoplus_{i=1}^n V_i$.

Theorem 5.2 (Maschke's Theorem)**.** Let G be a finite group and V a reducible G-module. Then there exist irreducible submodules $U_1, ..., U_k$ of V such that

$$V = \bigoplus_{n=1}^{k} U_n$$

Definition 5.3. Let V, W be G-modules, with vector space bases $v_1, ..., v_n$ and $w_1, ..., w_m$ over $\mathbb{C}$ respectively. The *tensor product* $V \otimes W$ of V and W is the G-module with vector space basis $\{v_i \otimes w_j \mid i = 1, ..., n; j = 1, ..., m\}$ and with linear G-action defined by:

$$g \cdot (v_i \otimes w_j) = (g \cdot v_i) \otimes (g \cdot w_j), \quad \forall g \in G.$$

If ρ_V, ρ_W are representations for V, W respectively, we write $\rho_V \otimes \rho_W$ for the representation of $V \otimes W$.

Since the groups we are interested in are finite subgroups of SL(2,$\mathbb{C}$), the natural linear action of each group makes $\mathbb{C}^2$ into a 2-dimensional G-module, henceforth denoted by V. Then for the irreducible G-module of G, given by $V_0, ...V_k$ we want to calculate:

$$V \otimes V_i = \bigoplus_{j=0}^{k} V_j^{\oplus a_{ij}}$$

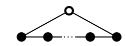
which is possible by Maschke's Theorem. Once we have calculated the a_{ij}, we construct a graph by introducing a vertex for each irreducible representation, and adding an edge from the vertex representing V_i to the vertex representing V_j if $a_{ij} = a_{ji}$.

Definition 5.4. The graph as constructed above is called the *McKay graph* of the group G.

For $k \in \{0, ..., n-1\}$ we henceforth write $\rho_k : C_n \to \mathrm{GL}(V)$ to denote the representation such that $\rho(\alpha)(v) = e^{\frac{2\pi ik}{n}} v$, as in the construction in Proposition A.4.

Proposition 5.5. Let $G = C_n$. Then for $k \in \{0, ..., n-1\}$, we have that:

$$V \otimes V_k = V_{k-1} \oplus V_{k+1} \mod n-1.$$

In particular, the McKay graph for C_n is the extended Dynkin diagram of type $\tilde{A}_{n-1}$, . Here the white vertex corresponds to the trivial representation and is always present so there are n nodes in total.

Proof. We can write V as $\mathbb{C}^2$ with standard basis elements e_1, e_2 and the action

$$\alpha \cdot e_1 = \omega e_1$$
$$\alpha \cdot e_2 = \omega^{-1} e_2$$

where $\omega = e^{\frac{2\pi i}{n}}$. Then V_k is the 1-dimensional G-module with vector space basis v_k and linear action of G defined by $\alpha \cdot v_k = \omega^k v_k$. Thus $V \otimes V_k$ is a G-module with vector space basis $\{e_1 \otimes v_k, e_2 \otimes v_k\}$ and linear G-action defined by:

$$\alpha \cdot (e_1 \otimes v_k) = \omega^{k+1}(e_1 \otimes v_k)$$
$$\alpha \cdot (e_2 \otimes v_k) = \omega^{k-1}(e_2 \otimes v_k).$$

The G-module $V_{k+1} \oplus V_{k-1}$ has vector space basis $\{(v_{k+1}, 0), (0, v_{k-1})\}$ and linear G-action given by

$$\alpha \cdot (v_{k+1}, 0) = \omega^{k+1}(v_{k+1}, 0)$$
$$\alpha \cdot (0, v_{k-1}) = \omega^{k-1}(0, v_{k-1}).$$

Hence we see that we have the stated decomposition. □

5.2 Characters

For our discussion of the binary dihedral group, we will employ the more sophisticated theory of characters in order to streamline calculations.

Definition 5.6. Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ a representation of G over $\mathbb{C}$. The *character* of ρ is the function:

$$\chi_\rho : G \to \mathbb{C}, \quad g \mapsto \mathrm{tr}(\rho(g)).$$

where $\mathrm{tr}(\rho(g))$ is the trace of the linear map $\rho(g)$. This is the trace of the matrix representing $\rho(g)$ with respect to a basis of V.

Remark. The definition of character demands a choice of basis for V, but since the trace is invariant under a change of basis, so is the character and it is therefore well defined.

Definition 5.7. Define an inner product of characters by

$$\langle \chi_1, \chi_2 \rangle = \frac{1}{|G|} \sum_{g \in G} \chi_1(g) \chi_2(g^{-1}).$$

This is indeed an inner product.

We recall the following two results from MA40054;

Lemma 5.8. A representation ρ with character χ_ρ is irreducible iff $\langle \chi_\rho, \chi_\rho \rangle = 1$.

Proposition 5.9. Let $V_1, ..., V_k$ be a complete list of irreducible G-modules, that is, they are pairwise non-isomorphic and any irreducible G-module is isomorphic to some V_i. Let $d_i = \dim(V_i)$. Then

$$|G| = \sum_{i=1}^{k} d_i^2$$

Proposition 5.10. There are precisely 4 one dimensional and $(n-1)$ two dimensional irreducible representations of $\mathcal{B}D_{4n}$.

We have the following candidate irreducible representations:

1. Degree 1:
   - $\rho_0 : \mathcal{B}D_{4n} \to \mathrm{GL}(V); \rho_0(\alpha) = \mathrm{Id}, \rho_0(\beta) = \mathrm{Id}$
   - $\rho_{+-} : \mathcal{B}D_{4n} \to \mathrm{GL}(V); \rho_{+-}(\alpha) = \mathrm{Id}, \rho_{+-}(\beta) = \text{-Id}$
   - $\rho_{-+} : \mathcal{B}D_{4n} \to \mathrm{GL}(V); \rho_{-+}(\alpha) = \text{-Id}, \rho_{-+}(\beta) = \begin{cases} \mathrm{Id} & n \text{ even} \\ i\mathrm{Id} & n \text{ odd} \end{cases}$
   - $\rho_{--} : \mathcal{B}D_{4n} \to \mathrm{GL}(V); \rho_{--}(\alpha) = \text{-Id}, \rho_{--}(\beta) = \begin{cases} \text{-Id} & n \text{ even} \\ -i\mathrm{Id} & n \text{ odd} \end{cases}$

2. Degree 2: For $k \in \{1, ..., n-1\}$,
   - $\rho_k : \mathcal{B}D_{4n} \to \mathrm{GL}(V); \rho_k(\alpha)(v) = \omega^k v, \rho_k(\alpha)(\beta \cdot v) = \omega^{-k} v$

   where $\omega = e^{\frac{i\pi}{n}}$.

Proof. For an explicit construction of these representations, see Theorem A.5. With the above candidate representations in hand, we simply check that the inner product of their characters is 1 and that the sum of squares of their dimensions is $4n$. The maps defined are indeed representations and they have the character table

$\mathcal{B}D_{4n}$	$\alpha^j \quad 0 \le j < 2n$	$\alpha^j \beta \quad 0 \le j < 2n$
ρ_0	1	1
ρ_{+-}	1	-1
ρ_{-+}	$(-1)^j$	$(-1)^j(m)$
ρ_{--}	$(-1)^j$	$(-1)^j(-m)$
$\rho_k, 1 \le k \le n-1$	$\omega^{kj} + \omega^{-kj}$	0

where $m = 1$ if n is even and $m = i$ if n is odd. There are 4 representations of dimension 1 and $n-1$ representations of dimension two so we have the sum $4 \cdot 1 + (n-1) \cdot 4 = 4n$ which is the order of the group. We also have that one-dimensional representations are irreducible, so all we have to check is that the two-dimensional representations have the correct inner products. Let χ_k be the character for the representation $\rho_k, 1 \le k \le n-1$. We compute

$$\begin{aligned}
\langle \chi_k, \chi_k \rangle &= \frac{1}{4n} \sum_{j=1}^{2n} \left(\chi_k(\alpha^j)\chi_k(\alpha^{-j}) + \chi_k(\alpha^j \beta)\chi_k(\alpha^{j+n}\beta) \right) \\
&= \frac{1}{4n} \sum_{j=1}^{2n} \left((\omega^{kj} + \omega^{-kj})(\omega^{-kj} + \omega^{kj}) \right) \\
&= \frac{1}{4n} \sum_{j=1}^{2n} (\omega^{2kj} + \omega^{-2kj} + 2) \\
&= 1
\end{aligned}$$

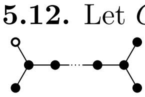
by the property that the sum of all powers of roots of unity is 0. Hence we have shown that we have a complete list of irreducible G-modules for the binary dihedral group. □

We can determine the characters of direct sums and tensor products of representations using the following lemma:

Lemma 5.11. Let ρ_1, ρ_2 be representations of G a finite group. Then for any $g \in G$;

$$\begin{aligned}
\chi_{\rho_1 \oplus \rho_2}(g) &= \chi_{\rho_1}(g) + \chi_{\rho_2}(g) \\
\chi_{\rho_1 \otimes \rho_2}(g) &= \chi_{\rho_1}(g)\chi_{\rho_2}(g).
\end{aligned}$$

Thus, when we have the character table for a certain group, it is possible to compute the decomposition into irreducibles for an arbitrary finite dimensional representation, and also to calculate the character for a tensor product of representations.

Proposition 5.12. Let $G = \mathcal{B}D_{4n}$. Then the McKay graph of G is the extended Dynkin diagram of type $\tilde{D}_{n+2}$,  where the white vertex corresponds to the trivial representation.

Proof. We first compute the characters of V, the G-module with linear G-action given by the natural action of $\mathcal{B}D_{4n}$ on $\mathbb{C}^2$. This action has matrix representation give by:

$$\rho : \mathcal{B}D_{4n} \to \mathrm{GL}(2, \mathbb{C}), \rho(\alpha) = \left(\begin{smallmatrix} \omega & 0 \\ 0 & \omega^{-1} \end{smallmatrix}\right), \rho(\beta) = \left(\begin{smallmatrix} 0 & -i \\ -i & 0 \end{smallmatrix}\right), \omega = e^{\frac{\pi i}{n}}.$$

Then we have the character table:

$\mathcal{B}D_{4n}$	$\alpha^j \quad 0 \le j < 2n$	$\alpha^j\beta \quad 0 \le j < 2n$
ρ	$\omega^j + \omega^{-j}$	0
ρ_0	1	1
ρ_{+-}	1	-1
ρ_{-+}	$(-1)^j$	$(-1)^j(m)$
ρ_{--}	$(-1)^j$	$(-1)^j(-m)$
$\rho_k, 1 \le k \le n-1$	$\omega^{kj} + \omega^{-kj}$	0

where $m = 1$ if n is even and $m = i$ if n is odd. This table is sufficient as any $g \in \mathcal{B}D_{4n}$ can be uniquely written in the form $g = \alpha^j\beta^k$ for some $j \in \{0, ..., n-1\}$ and $k \in \{0, 1\}$. Lemma 5.11 gives:

$$\chi_{\rho\otimes\rho_k}(\alpha^j) = (\omega^{(k+1)j} + \omega^{-(k+1)j}) + (\omega^{(k-1)j} + \omega^{-(k-1)j})$$
$$\chi_{\rho\otimes\rho_k}(\alpha^j\beta) = 0$$

Hence, when $k \in \{2, ..., n-2\}$ we have that $\chi_{\rho\otimes\rho_k} = \chi_{\rho_{k+1}\oplus\rho_{k-1}}$. If $k = 1$, we then have that:

$$\chi_{\rho\otimes\rho_1}(\alpha^j) = (\omega^{2j} + \omega^{-2j}) + 2 = \chi_{\rho_2}(\alpha^j) + \chi_{\rho_0}(\alpha^j) + \chi_{\rho_{+-}}(\alpha^j)$$
$$\chi_{\rho\otimes\rho_1}(\alpha^j\beta) = 0 = \chi_{\rho_2}(\alpha^j\beta) + \chi_{\rho_0}(\alpha^j\beta) + \chi_{\rho_{+-}}(\alpha^j\beta)$$

so we must have that $\chi_{\rho\otimes\rho_1} = \chi_{\rho_2} + \chi_{\rho_0} + \chi_{\rho_{+-}}$. A similar calculation yields $\chi_{\rho\otimes\rho_{n-1}} = \chi_{\rho_{n-2}} + \chi_{\rho_{-+}} + \chi_{\rho_{--}}$.

Now, clearly $\chi_{\rho\otimes\rho_0} = \chi_{\rho\otimes\rho_{+-}} = \chi_\rho = \chi_{\rho_1}$, so the only products we have left to check are ρ_{-+} and ρ_{--}. We appeal to the symmetry of their characters so that we need only check one, as their product will be the same. Now:

$$\chi_{\rho\otimes\rho_{-+}}(\alpha^j) = (-1)^j(\omega^j + \omega^{-j})$$
$$\chi_{\rho\otimes\rho_{-+}}(\alpha^j\beta) = 0.$$

Since $\omega = e^{\frac{\pi i}{n}}$, then $-1 = \omega^n = \omega^{-n}$ so we may write

$$\chi_{\rho\otimes\rho_{-+}}(\alpha^j) = \omega^{-nj}\omega^j + \omega^{nj}\omega^{-j} = \omega^{(n-1)j} + \omega^{-(n-1)j} = \chi_{\rho_{n-1}}(\alpha^j)$$

and we end up with the McKay graph being the Dynkin diagram of type $\tilde{D}_{n+2}$ as in Figure 2. $\square$

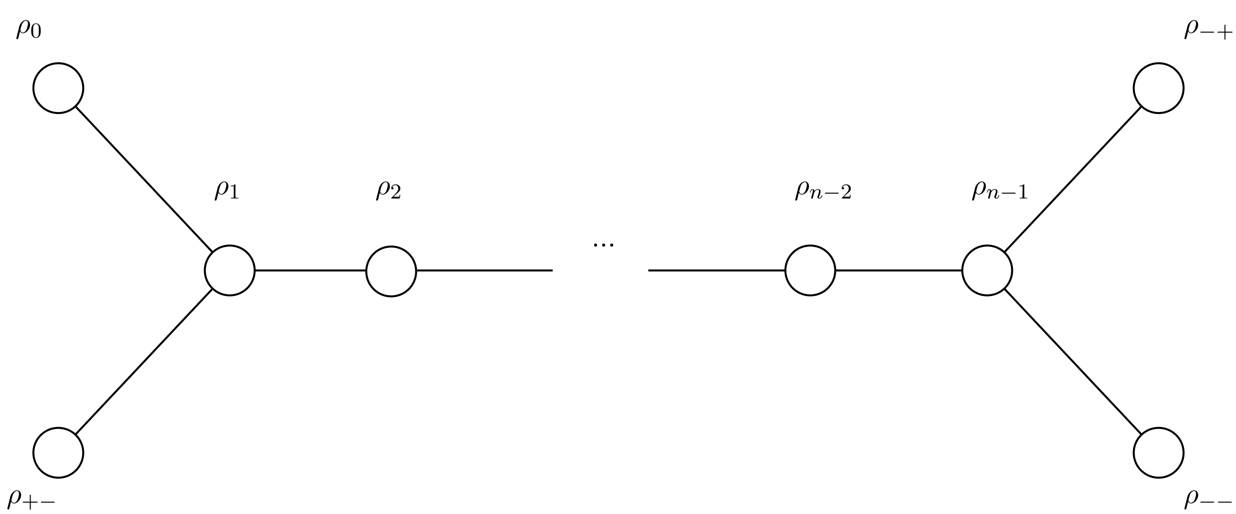


Figure 2: $\tilde{D}_{n+2}$; The Mckay graph for $\mathcal{B}D_{4n}$ with vertices labelled by the corresponding representations.

Theorem 5.13. The McKay graph for a finite subgroup of SL(2,$\mathbb{C}$) is the extended resolution graph for the Kleinian singularity corresponding to the same group.

6 Quivers

We have been able to recover the Dynkin diagrams from the subgroups very directly but we wish to also be able to recover the Kleinian singularities from the McKay graphs. For this, we need a discussion on quivers which are covered in more detail in many of the projects of my contemporaries.

6.1 Quivers and quiver representations

Definition 6.1. A *quiver*, Q, is a tuple (Q_0, Q_1, h, t) where Q_0 is a set of vertices, Q_1 is a set of arrows and $h : Q_1 \to Q_0, t : Q_1 \to Q_0$ are known as the head and tail functions respectively.
A representation of a quiver is an association of a vector space V_q to each vertex $q \in Q_0$ and a linear map ϕ to each arrow $a \in Q_1$ where $\phi : V_{t(a)} \to V_{h(a)}$.
For a vector $\delta = (\delta_1, \delta_2, ...) \in \mathbb{N}_0^{|Q_0|}$, the space $\text{Rep}(Q, \delta)$ consists of those representations where the vector space associated with vertex i is $\mathbb{C}^{\delta_i}$. Hence

$$\text{Rep}(Q, \delta) = \bigoplus_{a \in Q_1} \text{Hom}(\mathbb{C}^{\delta_{t(a)}}, \mathbb{C}^{\delta_{h(a)}}).$$

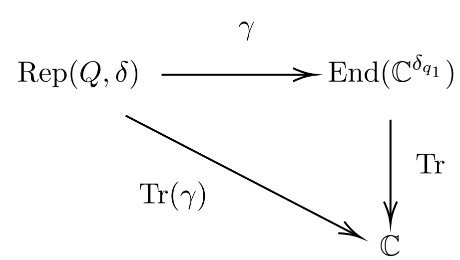
Definition 6.2. Let G be a finite subgroup of SL(2,$\mathbb{C}$) and let ρ be its natural representation, $\rho_0, ...\rho_k$ be a complete list of the irreducible representations. Define integers a_{ij} for every $i, j \in \{0, ..., k\}$ by

$$\rho \otimes \rho_i = \bigoplus_{j=0}^{k} \rho_j^{\oplus a_{ij}}.$$

The *McKay quiver* for G has a vertex for each ρ_i and a_{ij} arrows from vertex i to vertex j.

Remark. Note that the construction of the integers a_{ij} is the same as in the construction of the McKay graph above Definition 5.4.

Definition 6.3. A (non-trivial) *oriented cycle* in a quiver is a concatenation of arrows $\gamma = a_k a_{k-1} ... a_1$ where $a_1, ..., a_k \in Q_1$ such that $h(a_i) = t(a_{i+1}) \forall i = 1, ..., k-1$ and $h(a_k) = t(a_1)$.
For a given representation $S \in \text{Rep}(Q, \delta)$ we can define the *trace* of an oriented cycle γ based at q_1 in Q by the following commutative diagram;



The representation S associates linear maps to each of the arrows which constitute γ and so $\gamma(S)$ is the endomorphism obtained by composing the linear maps associated to the arrows in the order γ follows them.

Remark. An important property of the trace is that it is cyclic. That is, $\text{Tr}(ab) = \text{Tr}(ba)$ and so on with any cyclic permutation yielding the same trace. The trace of a cycle doesn't depend on which base point we choose. The trace of similar matrices is the same and so the trace is change of basis invariant.

6.2 Le Bruyn-Procesi process for the cyclic group

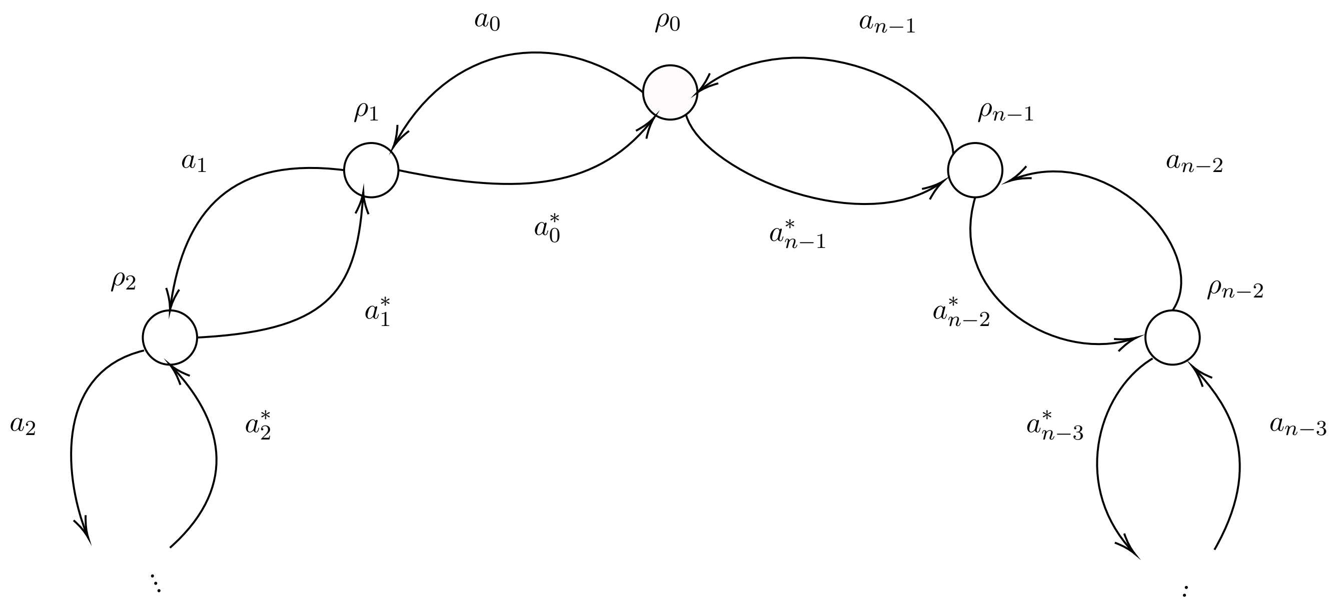


Figure 3: McKay quiver for cyclic group of order n with vertices labelled by their corresponding irreducible representations.

We turn our attention to the cyclic groups, or quivers of type $\tilde{A}_{n-1}$. Let δ be the dimension vector, $\delta_i = \dim(\rho_i)$. So in this case $\delta = (1, 1, ..., 1) \in \mathbb{N}^n$. We introduce the relations between arrows as labelled in Figure 3:

$$a_i^* a_i = a_{i-1} a_{i-1}^* \quad i \in \mathbb{Z}/n\mathbb{Z} \tag{*}$$

Proposition 6.4. The cycles

$$\begin{aligned} \gamma_x &= a_{n-1} a_{n-2} ... a_0 \\ \gamma_y &= a_0^* a_1^* ... a_{n-1}^* \\ \gamma_z &= a_0^* a_0 \end{aligned}$$

satisfy the relationship $\gamma_x \gamma_y - \gamma_z^n = 0$ modulo the relations (*).

Proof.

$$\begin{aligned} \gamma_x \gamma_y &= (a_{n-1} ... a_1 a_0)(a_0^* a_1^* ... a_{n-1}^*) \\ &= (a_{n-1} ... a_1)(a_0 a_0^*)(a_1^* ... a_{n-1}^*) \\ &= (a_{n-1} ... a_2)(a_1 a_1^*)(a_1 a_1^*)(a_2^* ... a_{n-1}^*) \\ &= (a_{n-1} ... a_2)(a_1 a_1^*)^2 (a_2^* ... a_{n_1}^*) \\ &... \\ &= (a_{n-1} a_{n-1}^*)^n = (a_0^* a_0)^n = \gamma_z^n \end{aligned}$$

Rearranging yields the desired equality. □

Proposition 6.5. $\mathbb{C}[\text{Tr}(\gamma_x), \text{Tr}(\gamma_y), \text{Tr}(\gamma_z)] = \mathbb{C}[\text{traces of cycles in } Q \text{ mod relations } (*)]$

Proof. We prove this by induction.
Suppose γ is a cycle in Q of length 2 which starts at vertex ρ_i. By the relations (*) we may assume that $\gamma = a_i^* a_i$. Then

$$\begin{aligned}\text{Tr}(\gamma) &= \text{Tr}(a_i^* a_i) \\ &= \text{Tr}(a_{i-1} a_{i-1}^*) \quad \text{by relations } (*) \\ &= \text{Tr}(a_{i-1}^* a_{i-1}) \quad \text{by the cyclic property of trace} \\ &= ... \\ &= \text{Tr}(a_0^* a_0).\end{aligned}$$

Now suppose that the claim holds for all cycles of length less than $2k$ and let γ be a cycle of length $2k$. Since $\delta = (1, 1, ..., 1)$ we have the property that $\text{Tr}(\gamma_1\gamma_2) = \text{Tr}(\gamma_1)\text{Tr}(\gamma_2)$ for any oriented cycles γ_1, γ_2. This is because for any choice of representation, the linear maps will be scalar multiples of the identity and the trace of a concatenation of arrows applied to such a representation simply gives the product of the scalars. Then we can either write $\gamma = \gamma_1 a_m^* a_m \gamma_2$ or $\gamma = \gamma_1 a_m a_m^* \gamma_2$ for some oriented cycles γ_1, γ_2 with lengths strictly less than $2k$ and some $m \in \{0, ..., n-1\}$. In either case we have

$$\text{Tr}(\gamma) = \text{Tr}(\gamma_1)\text{Tr}(a_m a_m^*)\text{Tr}(\gamma_2)$$

so the claim follows from the inductive hypothesis. □

Remark. For some intuition, one can think of this result in terms of the fundamental group of S^1 which is isomorphic to $\mathbb{Z}$. As in the proof above, we can use the cycle γ_z and the relations (*) to 'contract' any cycles which don't make a full loop around the quiver. Either of γ_x or γ_y could be used to generate $\pi_1(S^1)$ but since we can't take the inverse of a cycle in a quiver we need both.
From this we obtain a surjection

$$\mathbb{C}[x,y]^G \twoheadrightarrow \mathbb{C}[\text{Tr}(\gamma_x), \text{Tr}(\gamma_y), \text{Tr}(\gamma_z)] = \mathbb{C}[\text{traces of cycles in } Q \text{ mod relations } (*)] =: R$$

and thus an injection

$$\text{Spec}(R) \hookrightarrow \mathbb{C}^2/G.$$

where $\text{Spec}(R)$ is the affine algebraic set in $\mathbb{C}^3$ which has R as coordinate ring. If we show that the dimension of $\text{Spec}(R)$ is two then this injection is an isomorphism by Proposition A.3 and the fact that we know $\mathbb{C}^2/G$ is already an hypersurface. For any $\lambda, \mu \in \mathbb{C}^\times$ we have an element of $\text{Rep}(Q, \delta)$ which satisfies the relations (*). Namely, associating to each a_i the linear map λId and to each a_i^* the linear map μId. It can be shown that the map sending (λ, μ) to the element of $\text{Spec}(R)$ associated to this representation determines an embedding of the algebraic torus $(\mathbb{C}^\times)^2$ into $\text{Spec}(R)$, so $\text{Spec}(R)$ must have dimension two. Hence we obtain an isomorphism $\text{Spec}(R) \cong \mathbb{C}^2/G$.

A similar process may be carried out in the cases of the other finite subgroups, in which we introduce certain relations of the McKay quiver and then find three special oriented cycles whose traces generate the $\mathbb{C}$-algebra, $\mathbb{C}[\text{traces of cycles in } Q \text{ mod relations } (*)] = R$. One can then show that $\dim(\text{Spec}(R)) = 2$ and obtain isomorphisms

$$R \cong \mathbb{C}[x,y]^G, \text{Spec}(R) \cong \mathbb{C}^2/G.$$

Hence we have constructed the Kleinian singularity $\mathbb{C}^2/G$ directly from the McKay quiver of G as we had set out to do.

Appendix

Definition A.1. A $\mathbb{C}$-algebra is a ring, R, with a ring homomorphism $\mu : \mathbb{C} \to R$, so that for $f \in R, c \in \mathbb{C}$ we can define a scalar multiplication $c \cdot f = \mu(c) \cdot f$. For example, $\mathbb{C}[x_1, x_2, ..., x_k]$ is a $\mathbb{C}$-algebra for any $k \in \mathbb{N}$.

A $\mathbb{C}$-algebra homomorphism is a ring homomorphism between two $\mathbb{C}$-algebras which preserves the scalar multiplication structure.

Remark. Since $\mathbb{C}$-algebras have a well-defined scalar multiplication, they also have the structure of $\mathbb{C}$-vector spaces.

Theorem A.2. Let $X \subset \mathbb{A}^n, Y \subset \mathbb{A}^m$ be algebraic sets. For every $\mathbb{C}$-algebra homomorphism $\Phi : \mathbb{C}[Y] \to \mathbb{C}[X]$, there exists a unique polynomial map $\varphi : X \to Y$ such that Φ is the pullback of φ.

Proposition A.3. A variety $Y \subset \mathbb{C}^n$ has dimension $n-1$ if and only if it is the zero locus of a single nonconstant irreducible polynomial in $\mathbb{C}[x_1, ..., x_n]$.

Proposition A.4. There are precisely n irreducible representations of C_n all of dimension 1.

Proof. Write $C_n = \langle \alpha \mid \alpha^n = 1 \rangle$. Let V be an irreducible C_n module, given by the representation:

$$\rho : C_n \to \mathrm{GL}(V).$$

Then $\rho(\alpha) : V \to V$ has an eigenvector v with eigenvalue λ. Since ρ is a group homomorphism, we have that $\rho(\alpha)^n = \rho(\alpha^n) = \mathrm{Id}_V$, meaning that $\lambda^n = 1$. Thus, $\lambda = e^{\frac{2\pi i k}{n}}$ for some $k \in \{0, ..., n-1\}$. The C_n-action on V defined by ρ is linear, and so span$\{v\}$ is closed under the action of α, which generates C_n and so span$\{v\}$ is closed under the action of C_n. This means precisely that span$\{v\}$ is a submodule of V, which is irreducible, so $V = \text{span}\{v\}$, i.e. V is 1-dimensional. Each value of $k \in \{0, ..., n-1\}$ yields a pairwise non-equivalent irreducible representation of C_n. □

Theorem A.5. The binary dihedral group of order $4n$ has precisely four irreducible representations of dimension 1 and $n-1$ representations of dimension 2.

Proof. Write $\mathcal{B}D_{4n} = \langle \alpha, \beta \mid \alpha^{2n} = 1, \beta^2 = \alpha^n, \beta^{-1}\alpha\beta = \alpha^{-1} \rangle$. Let V be an irreducible $\mathcal{B}D_{4n}$ module, given by the representation:

$$\rho : \mathcal{B}D_{4n} \to \mathrm{GL}(V).$$

Then $\rho(\alpha) : V \to V$ has eigenvector v with eigenvalue λ. Since ρ is a group homomorphism, we have that $\rho(\alpha)^{2n} = \mathrm{Id}_V$ so that $\lambda^{2n} = 1$. Thus, $\lambda = e^{\frac{\pi i k}{n}}$ for some $k \in \{0, ..., 2n-1\}$. We calculate:

$$\alpha^{-1} \cdot v = \rho(\alpha^{-1})(v) = \rho(\alpha)^{-1}(v) = \lambda^{-1} v,$$

which means that:

$$\alpha \cdot (\beta \cdot v) = (\alpha\beta) \cdot v = (\beta\alpha^{-1}) \cdot v = \beta \cdot (\alpha^{-1} \cdot v) = \beta \cdot (\lambda^{-1} v) = \lambda^{-1}(\beta \cdot v),$$

so $\beta \cdot v$ is also an eigenvector of $\rho(\alpha)$, this time with eigenvalue λ^{-1}. Also:

$$\beta \cdot (\beta \cdot v) = \beta^2 \cdot v = \alpha^n \cdot v = \lambda^n v,$$

so span$\{v, \beta \cdot v\}$ is closed under the action of α and β, which generate $\mathcal{B}D_{4n}$ so this span is closed under the action of the whole group. This means that span$\{v, \beta \cdot v\}$ is a proper submodule of V, so irreducibility forces $V = \text{span}\{v, \beta \cdot v\}$. In particular, V is at most 2-dimensional.

Case 1: $k = 0$**.** Here, $\lambda = \lambda^{-1} = 1$ so $\beta \cdot v \in \text{span}\{v\}$, write $\beta \cdot v = xv$ for some $x \in \mathbb{C}$ and V is 1-dimensional. Then:

$$v = 1 \cdot v = \alpha^n \cdot v = \beta^2 \cdot v = x^2 v,$$

so $x^2 = 1$, i.e. $x \in \{\pm 1\}$. Since 1-dimensional representations are equivalent iff they are equal everywhere, we get two non-equivalent irreducible representations of $\mathcal{B}D_{4n}$ of degree 1.

Case 2: $k = n, 2 \mid n$ **.** In this case note that $\lambda^n = 1$, and also $\lambda = \lambda^{-1} = -1$. Similarly to Case 1, if we write $\beta \cdot v = xv$, then:

$$v = 1 \cdot v = \alpha^n \cdot v = \beta^2 \cdot v = x^2 v$$

so $x \in \{\pm 1\}$. Hence when n is even we have 4 irreducible representations of $\mathcal{B}D_{4n}$ of degree 1.

Case 3: $k = n, 2 \nmid n$ **.** Now $\lambda^n = -1$ and $\lambda = \lambda^{-1} = -1$. Write $\beta \cdot v = xv$. Then:

$$x^2 v = \beta^2 \cdot v = \alpha^n v = -v$$

Then $x \in \{\pm i\}$ and we have 2 additional irreducible representations of degree 1. So then when n is odd there are 4 irreducible degree 1 representations of $\mathcal{B}D_{4n}$.

Case 4: $k \notin \{0, n\}$**.** In this case, $\lambda \neq \lambda^{-1}$. This implies that v and $\beta \cdot v$ are linearly independent, since they are eigenvectors of $\rho(\alpha)$ with different eigenvalues and thus V is 2-dimensional. Let:

$$\rho_1 : G \to \text{GL}(V_1)$$
$$\rho_2 : G \to \text{GL}(V_2)$$

be two distinct representations of this form, where $\rho_1(\alpha)$ has eigenvectors v_1 and $\beta \cdot v_1$ with eigenvalues λ_1 and λ_1^{-1} respectively and $\rho_2(\alpha)$ has eigenvectors v_2 and $\beta \cdot v_2$ with eigenvalues λ_2 and λ_2^{-1} respectively. If ρ_1 and ρ_2 are equivalent, then there exists a G-linear map:

$$\theta : V_1 \to V_2, \theta\rho_1(g) = \rho_2(g)\theta \quad \forall g \in G.$$

In particular, consider:

$$\theta(\rho_1(\alpha)(v_1)) = \lambda_1 \theta(v_1) = \rho_2(\alpha)(\theta(v_1)) \tag{13}$$
$$\theta(\rho_1(\alpha)(\beta \cdot v_1)) = \lambda_1^{-1} \theta(\beta \cdot v_1) = \rho_2(\alpha)(\theta(\beta \cdot v_1)). \tag{14}$$

Then equation (13) implies that $\theta(v_1)$ is an eigenvector of $\rho_2(\alpha)$ with eigenvalue λ_1 and equation (14) implies that $\theta(\beta \cdot v_1)$ is an eigenvector of $\rho_2(\alpha)$ with eigenvalue λ_1^{-1}. There are two possibilities:

1. $\theta(v_1) = v_2, \theta(\beta \cdot v_1) = \beta \cdot v_2 \implies \lambda_1 = \lambda_2$. In this case we have that $V_1 = V_2$ as G-modules, which we assumed was not true.

2. $\theta(v_1) = \beta \cdot v_2, \theta(\beta \cdot v_1) = v_2 \implies \lambda_1 = \lambda_2^{-1}$.

Hence for any $k \notin \{0, n\}$, there is an equivalent representation given by $2n - k$. This means that there are precisely $n - 1$ non-equivalent irreducible representations of $\mathcal{B}D_{4n}$ of degree 2. □

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