THE KODAIRA VANISHING THEOREM VIA HODGE THEORY

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Abstract. Assuming a limited knowledge of complex manifolds and sheaf cohomology we provide an overview of Hodge Theory, leading to a proof of the Kodaira Vanishing Theorem. Our intention is to fill the gap in Hartshorne [3]. These lecture notes were produced to accompany a series of lectures which I gave at the Mathematics Institute of the University of Warwick in March, 1998.

Contents

1. Hodge Theory on a Compact Complex Manifold

1.1. **Differential Forms.** Let X denote a compact, connected complex manifold of complex dimension n. If TX is the tangent bundle of the underlying real manifold then

$$\mathcal{E}^r(X) := \Gamma(X, \textstyle\bigwedge^r TX^* \otimes_{\mathbb{R}} \mathbb{C})$$

denotes the complex-valued r-forms on X. The exterior derivative $d : \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+1}(X)$ induces a differential complex

$$0 \longrightarrow \{\text{constant functions}\} \longrightarrow \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \mathcal{E}^2(X) \xrightarrow{d} \dots \xrightarrow{d} \mathcal{E}^{2n}(X) \xrightarrow{d} 0 \tag{1.1}$$

The classical de Rham Theorem enables us to calculate cohomology groups using this complex:

$$H^r(X, \mathbb{C}) = \frac{\ker \{d : \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+1}(X)\}}{\operatorname{im} \{d : \mathcal{E}^{r-1}(X) \longrightarrow \mathcal{E}^r(X)\}}$$

Let X denote a compact, connected complex manifold of complex dimension n. If TX is the tangent bundle of the underlying real manifold then

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denotes the complex-valued r-forms on X. The exterior derivative d, defined apriori on $\mathbb{R}$-valued forms, is extended by complex-linearity to an operator on complex-valued forms

$$d : \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+1}(X)$$

The complex structure induces a splitting $TX \otimes_{\mathbb{R}} \mathbb{C} = T_X^{1,0} \oplus \overline{T_X^{1,0}}$ of the complexified tangent bundle into holomorphic and anti-holomorphic parts. The holomorphic cotangent bundle is $\Omega_X^1 := (T_X^{1,0})^*$, and

$$\mathcal{E}^{p,q}(X) := \Gamma(X, \textstyle\bigwedge^p \Omega_X^1 \otimes \bigwedge^q \overline{\Omega_X^1})$$

denotes the forms of type (p,q) on X. Notice that there's a decomposition

$$\mathcal{E}^r(X) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X)$$

This splitting enables us to decompose d as the sum of two complex-linear operators $d = \partial + \overline{\partial}$ which act on (p,q)-forms: the 'del' operator $\partial : \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p+1,q}(X)$ and, of particular interest to us, the 'delbar' operator

$$\overline{\partial} : \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p,q+1}(X)$$

The relations $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$ hold.

Our ultimate goal is to prove the Kodaira Vanishing Theorem and for this we require a generalisation of the (p,q)-forms and of the $\overline{\partial}$ operator. If E is a holomorphic vector bundle then

$$\mathcal{E}^{p,q}(X,E) := \Gamma(X, \mathcal{E}^{p,q} \otimes_{\mathbb{C}} E)$$

denotes the forms of type (p,q) on X with coefficients in E. Notice that

$$\mathcal{E}^{p,q}(X,E) = \mathcal{E}^{p,q}(X) \otimes_{\mathcal{E}^0(X)} \Gamma(X,E)$$

We now use $\overline{\partial}$ to create an operator

$$\overline{\partial}_E := \overline{\partial} \otimes \mathrm{id}_E : \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p,q+1}(X,E) : \sum \eta_i \otimes e_i \longrightarrow \sum \overline{\partial}(\eta_i) \otimes e_i$$

on E-valued (p,q)-forms.

1.2. **Sheaf Cohomology and Differential Complexes.** The bundles of r-forms define a resolution of the constant sheaf[1] $\mathbb{C}$ by acyclic sheaves

$$0 \longrightarrow \mathbb{C} \longrightarrow \mathcal{E}^0 \xrightarrow{d} \mathcal{E}^1 \xrightarrow{d} \mathcal{E}^2 \xrightarrow{d} \dots$$

To show exactness, note that the Poincaré Lemma shows that this sequence is exact beyond the second term, while everybody knows that the only smooth functions annihilated by d are the

[1]If we had defined $\mathcal{E}^r(X) := \Gamma(X, \bigwedge^r TX^*)$ to be the usual real-valued r-forms on X then this would be a resolution of the constant sheaf $\mathbb{R}$.

constants (!), so it's exact everywhere. If we apply the global sections functor to this resolution, giving a complex

$$0 \longrightarrow \Gamma(X, \mathbb{C}) \longrightarrow \mathcal{E}^0(X) \xrightarrow{d} \mathcal{E}^1(X) \xrightarrow{d} \mathcal{E}^2(X) \xrightarrow{d} \dots \tag{1.2}$$

then the theory of sheaf cohomology reveals that the groups $H^p(X, \mathbb{C})$ are simply 'kernel modulo image'. That is

Theorem 1.1 (de Rham)**.**

$$H^r(X, \mathbb{C}) = \frac{ker\,\{d\ :\ \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+1}(X)\}}{im\,\{d\ :\ \mathcal{E}^{r-1}(X) \longrightarrow \mathcal{E}^r(X)\}}$$

Similarly, the bundles of (p, q)-forms define a resolution of the sheaf $\Omega^p_X = \bigwedge^p \Omega^1_X$ of holomorphic p-forms by acyclic sheaves

$$0 \longrightarrow \Omega^p_X \longrightarrow \mathcal{E}^{p,0} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1} \xrightarrow{\overline{\partial}} \mathcal{E}^{p,2} \xrightarrow{\overline{\partial}} \dots$$

The Dolbeault-Grothendieck Lemma shows that this sequence is exact beyond the second term. Forms of type $(p, 0)$ annihilated by $\overline{\partial}$ are precisely the holomorphic p-forms[2] so the sequence is exact everywhere. Applying the global sections functor to this resolution gives a complex

$$0 \longrightarrow \Gamma(X, \Omega^p_X) \longrightarrow \mathcal{E}^{p,0}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,1}(X) \xrightarrow{\overline{\partial}} \mathcal{E}^{p,2}(X) \xrightarrow{\overline{\partial}} \dots \tag{1.3}$$

As above, sheaf cohomology theory shows that:

Theorem 1.2 (Dolbeault)**.**

$$H^{p,q}(X) := H^q(X, \Omega^p_X) = \frac{ker\,\{\overline{\partial}\ :\ \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p,q+1}(X)\}}{im\,\{\overline{\partial}\ :\ \mathcal{E}^{p,q-1}(X) \longrightarrow \mathcal{E}^{p,q}(X)\}}$$

Finally, one can show that there's a resolution of the sheaf $\Omega^p_X \otimes E$ of holomorphic p-forms with coefficients in E by acyclic sheaves

$$0 \longrightarrow \Omega^p_X \otimes E \longrightarrow \mathcal{E}^{p,0} \otimes E \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,1} \otimes E \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,2} \otimes E \xrightarrow{\overline{\partial}_E} \dots$$

which, when we apply the global sections functor, gives rise to a complex

$$0 \longrightarrow \Gamma(X, \Omega^p_X \otimes E) \longrightarrow \mathcal{E}^{p,0}(X, E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,1}(X, E) \xrightarrow{\overline{\partial}_E} \mathcal{E}^{p,2}(X, E) \xrightarrow{\overline{\partial}_E} \dots \tag{1.4}$$

from which we may calculate the cohomology of the sheaf $\Omega^p_X \otimes E$:

Theorem 1.3 (Generalised Dolbeault)**.**

$$H^q(X, \Omega^p_X \otimes E) = \frac{ker\,\{\overline{\partial}_E\ :\ \mathcal{E}^{p,q}(X, E) \longrightarrow \mathcal{E}^{p,q+1}(X, E)\}}{im\,\{\overline{\partial}_E\ :\ \mathcal{E}^{p,q-1}(X, E) \longrightarrow \mathcal{E}^{p,q}(X, E)\}}$$

We'll concentrate on complexes (1.1) and (1.4) since complex (1.3) is a special case of complex (1.4).

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[2]Recall that a smooth function $f\ :\ X \longrightarrow \mathbb{C}$ is holomorphic iff $\overline{\partial}(f) = 0$. More generally, a $(p, 0)$-form α is holomorphic iff $\overline{\partial}(\alpha) = 0$ iff in local co-ordinates $\alpha = \sum \alpha_I(z)\, dz_I$ with $\alpha_I(z)$ holomorphic functions.

1.3. **Hermitian Structure and the Hodge Inner Product on Forms.** In order to appeal to the results of elliptic operator theory we need to endow each of the spaces $\mathcal{E}^r(X)$ and $\mathcal{E}^{p,q}(X,E)$ with a global inner product. We begin, however, by defining a local inner product on the spaces $\mathcal{E}^{p,q}(X)$, and therefore on $\mathcal{E}^r(X) = \bigoplus \mathcal{E}^{p,q}(X)$.

Suppose hereafter that $T_X^{1,0}$ is endowed with a hermitian metric. This is always possible, since every smooth complex vector bundle admits a hermitian metric, but in general there's no canonical choice[3]. It follows that we have a complex inner product on the fibres $\bigwedge^p \Omega^1_{X,x} \times \bigwedge^q \overline{\Omega^1_{X,x}}$ of the bundles $\mathcal{E}^{p,q}$ varying smoothly with $x \in X$. In local holomorphic co-ordinates $\{z^\alpha\}_{\alpha=1}^n$ around a point $x \in X$ such a metric can be written as

$$h\langle\ ,\ \rangle_z = \sum h_{\alpha\overline{\beta}}(z)\ dz^\alpha \otimes \overline{dz^\beta}$$

where $(h_{\alpha\overline{\beta}}(z))$ is a positive definite hermitian symmetric matrix for each z near x. If we decompose h into its real and imaginery parts then

- the real part $\Re(h)$ can be viewed, via the natural isomorphism between $T_X^{1,0}$ and TX, defined by the map $\partial/\partial z^\alpha \to \frac{1}{2}(\partial/\partial x^\alpha - i\partial/\partial y^\alpha)$, as a Riemannian metric on the underlying real manifold with co-ordinates $\{x^\alpha, y^\alpha\}_{\alpha=1}^n$. When we refer to a Riemannian metric on a complex manifold with hermitian structure, it's this metric to which we are referring. In particular, X has a volume form denoted by vol.
- the imaginery part $\Im(h)$ is an alternating $(1,1)$-form. The form $\omega := -\frac{1}{2}\Im(h)$ is called the *fundamental form* of the hermitian metric. To fix the normalisation, in the local co-ordinates $\{z^\alpha\}_{\alpha=1}^n$ introduced above for which $h\langle\ ,\ \rangle_z = \sum h_{\alpha\overline{\beta}}(z)\ dz^\alpha \otimes \overline{dz^\beta}$ we have

$$\omega = \frac{i}{2}\sum h_{\alpha\overline{\beta}}\ dz^\alpha \otimes \overline{dz^\beta}$$

Now that we have a hermitian inner product on the vector spaces $\bigwedge^p \Omega^1_{X,x} \times \bigwedge^q \overline{\Omega^1_{X,x}}$ and a volume form on X, we define the *Hodge* $\overline{\star}$*-operator* on forms[4]. The map

$$\overline{\star}\ :\ \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{n-p,n-q}(X)\ :\ \beta \longrightarrow \overline{\star}(\beta)$$

is defined pointwise[5] via the relation

$$(\alpha \wedge \overline{\star}(\beta))(x) = \langle \alpha, \beta\rangle_x \mathrm{vol}(x) \qquad \text{for all } \alpha \in \mathcal{E}^{p,q}(X)$$

Finally we're in a position to define the *Hodge inner product* on the space of forms

$$\mathcal{E}^*(X) = \bigoplus_{r=0}^{2n} \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X)$$

[3]If, however, X admits a positive line bundle then there is a canonical choice. Stay tuned!

[4]I deliberately avoid use of the notation $\star$ which is traditionally used for the operator defined for real-valued forms on a Riemannian manifolds via the relation $(\alpha \wedge \star(\beta))(x) = \langle \alpha, \beta\rangle_x \mathrm{vol}(x)$. The operator which I refer to as $\overline{\star}$ is simply the extension of $\star$ to complex-valued r-forms, followed by the restriction of $\mathcal{E}^r(X)$ to $\mathcal{E}^{p,q}(X)$, so $\overline{\star}(\beta) = \star(\overline{\beta})$.

[5]The smooth variation of the hermitian structure with $x \in X$ ensures that a pointwise definition suffices here.

as follows. For $\alpha \in \mathcal{E}^r(X)$ and $\beta \in \mathcal{E}^s(X)$ we set

$$\langle \alpha, \beta \rangle = \begin{cases} \int_X \alpha \wedge \overline{\star}(\beta) & \text{if } r = s \\ 0 & \text{otherwise} \end{cases}$$

One can show that the decomposition $\mathcal{E}^r(X) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X)$ is orthogonal with respect to the Hodge inner product.

Remark It remains to extend the Hodge inner product to the spaces $\mathcal{E}^{p,q}(X, E)$. Since E is a holomorphic bundle we may choose a hermitian metric h_E on E. Now choose a conjugate-linear bundle isomorphism $\tau : E \to E^*$ and define

$$\overline{\star}_E : \mathcal{E}^{p,q}(X, E) \longrightarrow \mathcal{E}^{n-p,n-q}(X, E^*) : \alpha \otimes e \longrightarrow \overline{\star}(\alpha) \otimes \tau(e)$$

For $\alpha \in \mathcal{E}^r(X, E)$ and $\beta \in \mathcal{E}^s(X, E)$ we define the Hodge inner product to be

$$\langle \alpha, \beta \rangle = \begin{cases} \int_X \alpha \wedge \overline{\star}_E(\beta) & \text{if } r = s \\ 0 & \text{otherwise} \end{cases}$$

1.4. **Elliptic Complexes.** The resolutions which we encountered above are *elliptic complexes.* Rather than investigate the general theory of elliptic complexes we'll focus on the consequences of the general theory for our three resolutions. Notice that resolution (1.3) is a special case of resolution (1.4) so we'll waste no time on resolution (1.3). One should also remark that the following holds if $\mathcal{E}^*(X)$ is endowed with *any* hermitian inner product. The Hodge inner product will come into its own in section 1.5 and the later chapters.

1.4.1. *The de Rham Complex (1.1).* The exterior derivative $d : \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+1}(X)$ has an adjoint operator with respect to the inner product on $\mathcal{E}^*(X)$ denoted by $d^* : \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r-1}(X)$. Define the Laplacian

$$\Delta_d := d\, d^* + d^*\, d : \mathcal{E}^r(X) \longrightarrow \mathcal{E}^r(X)$$

which is a self-adjoint, elliptic operator. We let $\mathcal{H}^r(X, \mathbb{C}) := \{\alpha \in \mathcal{E}^r(X) : \Delta_d(\alpha) = 0\}$ denote the space of complex-valued r-forms annihilated by the Laplacian, called the space of *d-harmonic* r-forms. If you play with the adjoint property you'll soon convince yourself that for $\alpha \in \mathcal{E}^r(X)$

- $\Delta_d(\alpha) = 0 \Leftrightarrow d^*(\alpha) = d(\alpha) = 0$.
- $d(d^*(\alpha)) = 0 \Leftrightarrow d^*(\alpha) = 0$, so $\ker d \cap d^*\mathcal{E}^{r+1}(X) = \{0\}$.
- $d^*(d(\alpha)) = 0 \Leftrightarrow d(\alpha) = 0$, so $\ker d^* \cap d\mathcal{E}^{r-1}(X) = \{0\}$.

It follows almost immediately that

$$\mathcal{H}^r(X, \mathbb{C}) \oplus d^*\mathcal{E}^{r+1}(X) \oplus d\mathcal{E}^{r-1}(X) \subseteq \mathcal{E}^r(X)$$

Elliptic operator theory shows that these spaces are the same. That is:

Theorem 1.4 (Hodge)**.** *If X is a compact, complex manifold equipped with a hermitian metric then*

(1) $dim\ \mathcal{H}^r(X,\mathbb{C}) < \infty$.
(2) $\mathcal{E}^r(X) = \mathcal{H}^r(X,\mathbb{C}) \oplus d^*\mathcal{E}^{r+1}(X) \oplus d\mathcal{E}^{r-1}(X)$
(3) $\mathcal{H}^r(X,\mathbb{C}) \cong H^r(X,\mathbb{C})$

Notice that 3. follows from 2. with very little work: we've observed that $\ker d \cap d^*\mathcal{E}^{r+1}(X) = \{0\}$. Since $\Delta_d(\alpha) = 0 \Rightarrow d(\alpha) = 0$ and since $d^2\alpha = 0$ it follows that $\ker d = \mathcal{H}^r(X) \oplus d\mathcal{E}^{r-1}(X)$. We may therefore consider the harmonic forms as the quotient

$$\begin{aligned}\mathcal{H}^r(X,\mathbb{C}) &= \frac{\ker\, d}{d\mathcal{E}^{r-1}(X)} \\ &= \frac{\ker\, d\ :\ \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+1}(X)}{\mathrm{im}\ d\ :\ \mathcal{E}^{r-1}(X) \longrightarrow \mathcal{E}^r(X)} \\ &= H^r(X,\mathbb{C})\end{aligned}$$

by Theorem 1.1 above.

Key Observation *To prove any result involving the cohomology groups $H^r(X,\mathbb{C})$ it suffices to show that the result holds for the harmonic r-forms $\mathcal{H}^r(X,\mathbb{C})$.*

1.4.2. *The E-Dolbeault Complex (1.4).* The differential operator $\overline{\partial}_E\ :\ \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p,q+1}(X,E)$ has an adjoint with respect to the inner product on $\mathcal{E}^*(X)$ denoted by $\overline{\partial}_E^*\ :\ \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p,q-1}(X,E)$. Define the Laplacian

$$\Delta_{\overline{\partial}_E} := \overline{\partial}_E\, \overline{\partial}_E^* + \overline{\partial}_E^*\, \overline{\partial}_E\ :\ \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p,q}(X,E)$$

which is a self-adjoint, elliptic operator. Let $\mathcal{H}^{p,q}(X,E); = \{\alpha \in \mathcal{E}^{p,q}(X,E)\ :\ \Delta_{\overline{\partial}_E}(\alpha) = 0\}$ denote the $\overline{\partial}_E$*-harmonic* forms. As above we can prove that

$$\mathcal{H}^{p,q}(X,E) \oplus \overline{\partial}_E^*\mathcal{E}^{p,q+1}(X,E) \oplus \overline{\partial}_E\mathcal{E}^{p,q-1}(X,E) \subseteq \mathcal{E}^{p,q}(X,E)$$

using only the adjoint property. Applying elliptic operator theory gives the following theorem:

It follows from general elliptic operator theory that

Theorem 1.5. *If X is a compact, complex manifold equipped with a hermitian metric, and if E is a hermitian, holomorphic vector bundle over X then*

(1) $dim\ \mathcal{H}^{p,q}(X,E) < \infty$.
(2) $\mathcal{E}^{p,q}(X,E) = \mathcal{H}^{p,q}(X,E) \oplus \overline{\partial}_E^*\mathcal{E}^{p,q+1}(X,E) \oplus \overline{\partial}_E\mathcal{E}^{p,q-1}(X,E)$.
(3) $\mathcal{H}^{p,q}(X,E) \cong H^q(X,\Omega_X^p \otimes E)$.

Key Observation *To prove any result involving the cohomology groups $H^q(X,\Omega^p \otimes E)$ it suffices to show that the result holds for the harmonic forms $\mathcal{H}^{p,q}(X,E)$.*

1.5. **Applications of Hodge Theory.** In the previous section we remarked that d and $\overline{\partial}_E$ have adjoints with respect to the inner product on $\mathcal{E}^*(X)$. If we use the Hodge inner product on $\mathcal{E}^*(X)$ then one can calculate that

$$d^* = -\overline{\star}\, d\, \overline{\star}, \qquad \overline{\partial}_E^* = -\overline{\star}_{E^*}\, \overline{\partial}_E\, \overline{\star}_E$$

Using these explicit forms of the adjoint, one can show that these star operators commute with the relevant Laplacian operators:

$$\overline{\star}\, \Delta_d = \Delta_d\, \overline{\star}, \qquad \overline{\star}_E\, \Delta_{\overline{\partial}_E} = \Delta_{\overline{\partial}_E}\, \overline{\star}_E$$

Furthermore $\overline{\star}\,\overline{\star}(\beta) = (-1)^{p+q} \cdot \beta$ so that $\overline{\star}$ is a conjugate-linear isomorphism. A similar result holds for $\overline{\star}_E$ so we have conjugate-linear isomorphisms

$$\begin{aligned} \mathcal{E}^r(X) &\xrightarrow{\overline{\star}} \mathcal{E}^{2n-r}(X) \\ \mathcal{E}^{p,q}(X,E) &\xrightarrow{\overline{\star}_E} \mathcal{E}^{n-p,n-q}(X,E^*) \end{aligned}$$

As we've remarked, these star operators commute with the corresponding Laplacians so the maps $\overline{\star}$ and $\overline{\star}_E$ descend to conjugate-linear isomorphisms

$$\begin{aligned} \mathcal{H}^r(X,\mathbb{C}) &\xrightarrow{\overline{\star}} \mathcal{H}^{2n-r}(X,\mathbb{C}) \\ \mathcal{H}^{p,q}(X,E) &\xrightarrow{\overline{\star}_E} \mathcal{H}^{n-p,n-q}(X,E^*) \end{aligned}$$

So, by our work from section 1.4 we have conjugate-linear isomorphisms on cohomology. Finite dimensional complex vector spaces are conjugate-linear if and only if one is complex-linearly isomorphic to the dual of the other. So the following famous results drop into our laps!

Theorem 1.6 (Poincaré Duality). *If X is a compact, complex manifold equipped with a hermitian metric then*

$$H^r(X,\mathbb{C}) \cong H^{2n-r}(X,\mathbb{C})^*$$

Theorem 1.7 (Serre Duality). *If X is a compact, complex manifold equipped with a hermitian metric, and if E is a hermitian, holomorphic vector bundle over X then*

$$H^q(X,\Omega_X^p \otimes E) \cong H^{n-q}(X,\Omega_X^{n-p} \otimes E)^*$$

In particular, when $p = 0$ we have

$$H^q(X,E) \cong H^{n-q}(X,K_X \otimes E)^*$$

where $K_X := \bigwedge^n \Omega_X^1$ is the canonical line bundle of holomorphic n-forms on X.

2. The Hodge Decomposition on Kähler Manifolds

2.1. **Kähler Manifolds.** A compact, complex manifold X is said to be *Kähler* if it admits a metric h which satisfies one of the following equivalent conditions:

(1) The fundamental $(1,1)$-form ω associated to the metric h is d-closed.
(2) For each $x \in X$ there exist holomorphic co-ordinates $\{z^\alpha\}$ such that h is Euclidean to second order. That is, when we write the metric in terms of these local co-ordinates it takes the form

$$h\langle\ ,\ \rangle_z = \sum \left(\delta_{\alpha\overline{\beta}} + a_{\alpha\overline{\beta}rs} z^r z^s + \text{higher order terms}\right)\ dz^\alpha \otimes \overline{dz^\beta}$$

where $\delta_{\alpha\overline{\beta}}$ is the Kronecker delta.

This second statement is somewhat cumbersome but it lends itself well to a discussion of why Kähler manifolds are of special interest. If we consider the action of a *first order* differential operator on a Kähler manifold X then it sees only first order information. If we choose co-ordinates on X for which the metric is Euclidean to second order then the operator believes that the metric is the Euclidean metric

$$h\langle\ ,\ \rangle_z = \sum\ dz^\alpha \otimes \overline{dz^\alpha} \tag{2.1}$$

This simplifies calculations considerably. For instance, if we define local co-ordinates $\{u^j\}_{j=1}^{2n}$ on the underlying real manifold by the formula $z^\alpha = u^\alpha + i\, u^{n+\alpha}$ then

$$\begin{aligned} h\langle\ ,\ \rangle &= \sum (du^\alpha + i\, du^{n+\alpha}) \otimes (du^\alpha - i\, du^{n+\alpha}) \\ &= \sum du^\alpha \otimes du^\alpha + du^{n+\alpha} \otimes du^{n+\alpha} - 2i \sum du^\alpha \wedge du^{n+\alpha} \end{aligned}$$

Recall from lecture 1 that the real part of our hermitian metric is the Riemannian metric on the underlying real manifold, and that the imaginery part is -2 times the fundamental form ω. Therefore

- the Riemannian metric can be written in terms of local co-ordinates as

$$g = \sum_{\alpha=1}^{n} \left(du^\alpha \otimes du^\alpha + du^{n+\alpha} \otimes du^{n+\alpha}\right) = \sum_{j=1}^{2n}\ du^j \otimes du^j \tag{2.2}$$

  and the volume form is simply:

$$\text{vol} = du^1 \wedge du^{n+1} \wedge \ldots du^n \wedge du^{2n} = \prod_{\alpha\in\{1,\ldots n\}} du^\alpha \wedge du^{n+\alpha} \tag{2.3}$$

- by inspecting the imaginery part we see that the fundamental form is given by

$$\omega = \sum du^\alpha \wedge du^{n+\alpha} = \frac{i}{2} \sum dz^\alpha \wedge \overline{dz^\alpha} \tag{2.4}$$

  Notice also that $\omega^n = n!\, du^1 \wedge du^{n+1} \wedge \ldots du^n \wedge du^{2n} = n!\, \text{vol}$.

In summary, as long as the identities and formulae which we derive hereafter on a Kähler manifold X involve no differential operators of order two and higher then we may assume that (2.1), (2.2), (2.3) and (2.4) hold. This considerable simplification of the hermitian structure will enable us to prove some powerful results.

2.1.1. *Examples of Kähler Manifolds.*

- Compact Riemann surfaces (i.e. compact, complex manifolds of complex dimension 1) are Kähler because the underlying real manifold is two dimensional so every 2-form is closed.
- The $(1,1)$-form $\omega := \frac{i}{2\pi}\partial\overline{\partial}\log\left(\sum_{j=0}^{n} z^j\overline{z^j}\right)$ is the fundamental form associated to the Fubini-Study metric on $\mathbb{P}^n$. Notice that

$$d(\omega) = \frac{i}{2\pi}\partial^2\overline{\partial}\log\left(\sum_{j=0}^{n} z^j\overline{z^j}\right) + \frac{i}{2\pi}\overline{\partial}\partial\overline{\partial}\log\left(\sum_{j=0}^{n} z^j\overline{z^j}\right) = -\frac{i}{2\pi}\overline{\partial}^2\partial\log\left(\sum_{j=0}^{n} z^j\overline{z^j}\right) = 0$$

  where we've used the identities $d = \partial + \overline{\partial}$, $\partial^2 = \overline{\partial}^2 = 0$ and $\partial\overline{\partial} = -\overline{\partial}\partial$. Since ω is closed we see that $\mathbb{P}^n$ is Kähler.
- Submanifolds of Kähler manifolds are Kähler. To see this, suppose that X is Kähler and $Z \hookrightarrow X$ is a submanifold. The $(1,1)$-form η on Z associated to the induced hermitian metric is simply the pullback of the Kähler form on X. Therefore η is closed and the result follows. In particular, every manifold which admits an embedding into projective space is Kähler.

2.2. **The Hodge Decomposition.** Our goal in this section is to prove the famous decomposition theorem of Hodge. In order to prove this result we wish to show that, on a Kähler manifold X, the d-Laplacian $\Delta_d = dd^* + d^*d$, the $\overline{\partial}$-Laplacian $\Delta_{\overline{\partial}} = \overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\overline{\partial}$ and the ∂-Laplacian $\Delta_\partial = \partial\partial^* + \partial^*\partial$ are related via the identities

$$\Delta_d = 2\,\Delta_{\overline{\partial}} = 2\,\Delta_\partial \tag{2.5}$$

To this end, we introduce the *Lefschetz operator*

$$L \;:\; \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p+1,q+1}(X) \;:\; \eta \longrightarrow \omega\wedge\eta$$

and let L^* denote its adjoint with respect to the Hodge inner product. It is straightforward to show that, for $p + q = r \geq 2$, we can calculate L^* explicitly:

$$L^* \;:\; \mathcal{E}^{p,q}(X) \longrightarrow \mathcal{E}^{p-1,q-1}(X) \;:\; \eta \longrightarrow L^*(\eta) = (-1)^r\overline{\star}L\overline{\star}(\eta)$$

with L^* equal to the zero map when either p or q is zero. A key step in the proof of the Hodge decomposition is the derivation of the *Kähler identities*:

Proposition 2.1. *Let* $[\ ,\ ]$ *denote the commutator. Then*

$$[L^*, \overline{\partial}] = -i\,\partial^*; \qquad [L^*, \partial] = i\,\overline{\partial}^*$$

Sketch of the Proof Recall that the operators d, ∂ and $\overline{\partial}$ are first order differential operators and, with a knowledge of elliptic operator theory, one can show that the corresponding adjoints d^*, ∂^* and $\overline{\partial}^*$ are likewise first order differential operators. Clearly both L and L^* are zeroth order so that each of the operators

$$L^*\overline{\partial};\ \ \overline{\partial}L^*;\ \ -i\,\partial^*;\ \ L^*\partial;\ \ \partial L^*;\ \ i\,\overline{\partial}^*$$

which appear in the statement of the proposition is first order. It now follows from our observation in section 2.1 that we may employ the relations (2.1), (2.2), (2.3) and (2.4). Even in this considerably simplified situation the calculation is complicated so we avoid it here[6].

Armed with these identities it is relatively easy to prove that the relations (2.5) hold:

Proposition 2.2. *The relations* $\Delta_d = 2\,\Delta_{\overline{\partial}} = 2\,\Delta_\partial$ *hold on a Kähler manifold.*

Proof Firstly note that

$$i(\partial\overline{\partial}^* + \overline{\partial}^*\partial) = \partial[L^*, \partial] + [L^*, \partial]\partial = \partial L^*\partial - \partial L^*\partial = 0$$

using the second Kähler identity so $\partial\overline{\partial}^* + \overline{\partial}^*\partial = 0$. Now, the complex conjugate of the delbar operator $\overline{\partial}$ is ∂ by construction. Furthermore, using the explicit form $\overline{\partial}^* = -\overline{\star}\partial\overline{\star}$ of the adjoint of the delbar operator, one can show that the complex conjugate of $\overline{\partial}^*$ is ∂^*. Hence the complex conjugate of $\overline{\partial}\partial^* + \partial^*\overline{\partial}$ is $\partial\overline{\partial}^* + \overline{\partial}^*\partial$ which we've just shown to be zero. Therefore $\overline{\partial}\partial^* + \partial^*\overline{\partial}$ is zero. Using these two identities we expand

$$\begin{aligned}
\Delta_d &= (\partial + \overline{\partial})(\partial^* + \overline{\partial}^*) + (\partial^* + \overline{\partial}^*)(\partial + \overline{\partial}) \\
&= (\partial\partial^* + \partial^*\partial) + (\overline{\partial}\,\overline{\partial}^* + \overline{\partial}^*\,\overline{\partial}) + (\partial\overline{\partial}^* + \overline{\partial}^*\partial) + (\overline{\partial}\partial^* + \partial^*\overline{\partial}) \\
&= \Delta_\partial + \Delta_{\overline{\partial}}
\end{aligned}$$

It remains to prove that $\Delta_\partial = \Delta_{\overline{\partial}}$. Once again this is an exercise is keeping your cool(!):

$$\begin{aligned}
-i\Delta_\partial = -i(\partial\partial^* + \partial^*\partial) &= \partial[L^*, \overline{\partial}] + [L^*, \overline{\partial}]\partial \\
&= \partial L^*\overline{\partial} - \partial\overline{\partial}L^* + L^*\overline{\partial}\partial - \overline{\partial}L^*\partial \\
&= -\overline{\partial}L^*\partial + \overline{\partial}\partial L^* - L^*\partial\overline{\partial} + \partial L^*\overline{\partial} \\
&= -\overline{\partial}[L^*, \partial] - [L^*, \partial]\overline{\partial} = -i\overline{\partial}\overline{\partial}^* - i\overline{\partial}^*\overline{\partial} = -i\Delta_{\overline{\partial}}
\end{aligned}$$

Notice that we used the result $\partial\overline{\partial} = -\overline{\partial}\partial$. This concludes the proof of the proposition.

□

Corollary 2.3. *When* X *is a compact Kähler manifold,* $\Delta_{\overline{\partial}}$ *is a real operator*

[6] We do however calculate the operators L and L^* using the relations (2.1) to (2.4). These calculations will turn out to be vital in the proof of the Kodaira Vanishing Theorem, our ultimate goal.

Proof We've already done the hard work. It remains to notice that

$$\overline{\Delta_{\overline{\partial}}(\beta)} = \overline{\left((\overline{\partial}\overline{\partial}^* + \overline{\partial}^*\overline{\partial})(\beta)\right)} = (\partial\partial^* + \partial^*\partial)(\overline{\beta}) = \Delta_\partial(\overline{\beta}) = \Delta_{\overline{\partial}}(\overline{\beta})$$

□

Finally we are in a position to prove the famous decomposition theorem:

Theorem 2.4 (Hodge Decomposition)**.** *If X be a compact Kähler manifold then*

$$H^r(X;\mathbb{C}) = \bigoplus_{p+q=r} H^q(X,\Omega^p) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}(X)$$

Furthermore $H^{p,q}(X) \cong \overline{H^{q,p}(X)}$.

Proof We saw in lecture 1 that $H^r(X;\mathbb{C}) = \mathcal{H}^r(X;\mathbb{C})$ and, for the special case when E is equal to the trivial line bundle, $H^q(X,\Omega^p) = \mathcal{H}^{p,q}(X)$. It therefore suffices to prove that

$$\mathcal{H}^r(X,\mathbb{C}) = \bigoplus_{p+q=r} \mathcal{H}^{p,q}(X)$$

Write any given $\alpha \in \mathcal{H}^r(X,\mathbb{C}) \subset \mathcal{E}^r(X) = \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X)$ as a sum of bihomogeneous terms $\alpha = \sum \alpha^{p,q}$ with $\alpha^{p,q} \in \mathcal{E}^{p,q}(X)$. Now,

$$0 = \Delta_d(\alpha) = 2\Delta_{\overline{\partial}}(\alpha) = 2\sum_{p+q=r} \Delta_{\overline{\partial}}(\alpha^{p,q})$$

Since $\Delta_{\overline{\partial}}$ preserves type (i.e. maps (p,q)-forms to (p,q)-forms), and since the terms $\alpha^{p,q}$ have different type by construction, it follows that $\Delta_{\overline{\partial}}(\alpha^{p,q}) = 0$. That is, $\alpha^{p,q} \in \mathcal{H}^{p,q}(X)$ so the natural map

$$\mathcal{E}^r(X,\mathbb{C}) \longrightarrow \bigoplus_{p+q=r} \mathcal{E}^{p,q}(X) \ : \ \alpha \longrightarrow (\alpha^{r,0}, \alpha^{r-1,1}, \dots \alpha^{0,r})$$

restricts to a map

$$\mathcal{H}^r(X,\mathbb{C}) \longrightarrow \bigoplus_{p+q=r} \mathcal{H}^{p,q}(X) \ : \ \alpha \longrightarrow (\alpha^{r,0}, \alpha^{r-1,1}, \dots \alpha^{0,r})$$

which is clearly injective. Moreover, if $\alpha \in \mathcal{H}^{p,q}(X)$ then $0 = 2\Delta_{\overline{\partial}}(\alpha) = \Delta_d(\alpha)$ so in fact $\alpha \in \mathcal{H}^{p+q}(X) = \mathcal{H}^r(X)$ and we see that the map surjects. This proves the first part of the theorem. Complex conjugation gives an isomorphism $\mathcal{E}^{p,q}(X) \cong \mathcal{E}^{q,p}(X)$. Since the operator $\Delta_{\overline{\partial}}$ is a real operator it commutes with complex conjugation and the isomorphism descends to an isomorphism on harmonic forms $\mathcal{H}^{p,q}(X) \cong \overline{\mathcal{H}^{q,p}(X)}$ and hence on cohomology.

□

2.3. **Applications of the Hodge Decomposition.** Our first application is the result which makes proving the Riemann-Roch Theorem on complex curves a triviality:

Proposition 2.5. *If X is a Riemann surface then the genus of X is equal to $\dim_{\mathbb{C}} H^0(X, \Omega^1_X)$.*

Proof It's well known that the topological Euler characteristic $e(X) = 2 - 2g(X)$ where $g(X)$ is the genus of X. Furthermore, $H^0(X,\mathbb{C}) = H^2(X,\mathbb{C}) = \mathbb{C}$ so that

$$2g(X) = \dim_{\mathbb{C}} H^1(X,\mathbb{C}) = \dim_{\mathbb{C}} \left(H^{0,1}(X) \oplus H^{1,0}(X)\right) = 2\dim_{\mathbb{C}} H^{1,0}(X)$$

since $\dim_{\mathbb{C}} H^{0,1}(X) = \dim_{\mathbb{C}} \overline{H^{1,0}(X)} = \dim_{\mathbb{C}} H^{1,0}(X)$. The result follows by the Dolbeault Theorem which gives the isomorphism $H^0(X, \Omega^1_X) \cong H^{1,0}(X)$.

□

We now wish to construct a compact complex manifold which is not Kähler. We've already noted that all one-dimensional compact complex manifolds are Kähler, so there's some work to be done. Firstly we prove a result which gives a necessary topological condition for a compact complex manifold to be Kähler.

Proposition 2.6. *The odd Betti numbers of a Kähler manifold are even.*

Proof We adopt the standard notation $h^{p,q}(X) := \dim_{\mathbb{C}} H^{p,q}(X)$. Then the $(2r+1)^{st}$ Betti number is given by the sum

$$b_{2r+1}(X) = \sum_{p+q=2r+1} h^{p,q}(X)$$

There are $2r+2$ summands which we can pair off by the identity $h^{p,q}(X) = h^{q,p}(X)$, proving the result.

□

We now construct the example of a non-Kähler compact complex manifold, called the 'Hopf Surface'. Consider the 3-sphere $S^3 = \{(z_1, z_2) \in \mathbb{C}^2 \,:\, |z_1| = |z_2| = 1\}$. Then there's a diffeomorphism

$$S^3 \times \mathbb{R} \,:\longrightarrow \mathbb{C}^2 - \{0\} \,:\, (z_1, z_2, t) \longrightarrow (e^t z_1, e^t z_2)$$

Let $\mathbb{Z}$ act on the space $S^3 \times \mathbb{R}$ as $(z_1, z_2, t) \longrightarrow (z_1, z_2, t+n)$. The quotient is $S^3 \times S^1$. Via the diffeomorphism we see that this defines an action of $\mathbb{Z}$ on $\mathbb{C}^2 - \{0\}$ as

$$(e^t z_1, e^t, z_2) \longrightarrow (e^{t+n} z_1, e^{t+n} z_2)$$

One can show that the quotient map $\mathbb{C}^2 - \{0\} \longrightarrow (\mathbb{C}^2 - \{0\}/\mathbb{Z})$ is a regular cover which gives the four dimensional real manifold $(\mathbb{C}^2 - \{0\}/\mathbb{Z})$ a complex structure. Furthermore, there's a diffeomorphism

$$S^3 \times S^1 \longrightarrow \mathbb{C}^2 - \{0\}/\mathbb{Z}$$

which is compatible with the quotient maps. As a result, $(\mathbb{C}^2 - \{0\}/\mathbb{Z})$ is a compact, complex manifold whose cohomology is that of $S^3 \times S^1$. But $b_1(S^3 \times S^1) = 1$ so the space $(\mathbb{C}^2 - \{0\}/\mathbb{Z})$ cannot have a Kähler structure because one of its odd Betti numbers is an odd number.

3. The Lefschetz Decomposition on Kähler Manifolds

3.1. **The Operators Defining the Representation.** The Lefschetz operator L is more than simply a tool to help one prove that $\Delta_d = 2\Delta_{\overline{\partial}}$. In this section we show that the operators L, L^* and their commutator $[L, L^*]$ define a representation of $\mathfrak{sl}(2, \mathbb{C})$ on the the complex vector space $H^*(X, \mathbb{C})$.

The Lefschetz operator, defined to be an operator on $\mathcal{E}^{p,q}(X)$, may be considered as an operator on $\mathcal{E}^r(X) = \bigoplus \mathcal{E}^{p,q}(X)$

$$L \,:\, \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+2}(X) \,:\, \eta \longrightarrow \omega \wedge \eta$$

We've remarked that it's adjoint with respect to the Hodge inner product can be described explicitly by

$$L^* \,:\, \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r-2}(X) \,:\, \eta \longrightarrow L^*(\eta) = (-1)^r \overline{\star} L \overline{\star}(\eta)$$

for $r \geq 2$, where L^* equal to the zero map when $r \leq 1$. It's clear that the commutator will be a map

$$[L, L^*] \,:\, \mathcal{E}^r(X) \longrightarrow \mathcal{E}^r(X)$$

The precise form of this map will be the result upon which the proof of the Kodaira Vanishing Theorem hinges so we wish to calculate $[L, L^*]$ explicitly. To do this we need to make some calculations in local co-ordinates. For those readers who do not wish to get bogged down with notation I suggest that you skip section 3.2 entirely and move onto section 3.3!

3.2. **Local Calculations on a Kähler Manifold.** The operators L, L^* and $[L, L^*]$ involve no differential operators of order two and higher so, by the key observation from our work on Kähler manifolds we may assume that the hermitian metric takes the simplified form given by (2.1), (2.2), (2.3) and (2.4). As a result L is the map

$$L \,:\, \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+2}(X) \,:\, \eta \longrightarrow \left(\sum_\alpha du^\alpha \wedge du^{n+\alpha} \right) \wedge \eta$$

We wish to simplify our notation hereafter as follows. If an operator on the space $\mathcal{E}^r(X)$ is linear over functions it suffices to show how it acts on products of covectors $\{du^i\}$, and extend linearly over functions to the whole space. To further simplify our notation we define, for a subset $A \subseteq \{1, \ldots n\}$, the expression

$$\omega^A := \prod_{\alpha \in A} \omega^\alpha$$

Lemma 3.1. *Every product of covectors, up to sign, is of the form* $\omega^A du^B du^{n+C}$ *where* $A, B, C \subset \{1, \ldots n\}$ *are disjoint.*

Proof of Lemma Fix a covector du^I with $I \subseteq \{1, \ldots 2n\}$ and consider some $i \in \{1, \ldots n\}$. If $i \in I$ then either $n+i \in I$ in which case $i \in A$ or not, in which case we can put $i \in B$. Similarly, for $i \in \{n+1, \ldots 2n\}$ then either $i \in I$ in which case $n+1 \in A$ or not, in which case $i \in C$. This proves the lemma.

As a result it suffices to define our operators on the product $\omega^A du^B du^{n+C}$. The Lefschetz operator may now be considered as the map

$$L : \mathcal{E}^r(X) \longrightarrow \mathcal{E}^{r+2}(X) : \omega^A du^B du^{n+C} \longrightarrow \sum_{\gamma \notin A \cup B \cup C} \omega^\gamma \omega^A du^B du^{n+C}$$

which is extended to all of $\mathcal{E}^r(X)$ by function-linearity.
We now turn our attention to the operator $L^* = (-1)^r \overline{\star} L \overline{\star}$. The $\overline{\star}$-operator is conjugate-linear with respect to functions but, since we wish to apply the operator twice, we see that the composite $L^* = (-1)^r \overline{\star} L \overline{\star}$ is indeed linear, so to describe L^* locally it suffices to define its action on the product $\omega^A du^B du^{n+C}$. It is convenient at this point to introduce the notation $N := \{1, 2, \ldots n\}$.

Proposition 3.2. $L^*(\omega^A du^B du^{n+C}) = \sum_{\alpha \in A} \omega^{A - \{\alpha\}} du^B du^{n+C}$

Proof The $\overline{\star}$-operator acts on covectors in the same way as the $\star$-operator of Riemannian geometry. A little thought reveals that

$$\overline{\star}(\omega^A du^B du^{n+C}) = \pm \omega^{N-(A \cup B \cup C)} du^{n+B} du^C$$

We need not be accurate with the sign since we wish to apply $\overline{\star}$ twice. Therefore if $\omega^A du^B du^{n+C} \in \mathcal{E}^r(X)$

$$\begin{aligned}
L^*(\omega^A du^B du^{n+C}) &= (-1)^r \overline{\star} L \overline{\star} \left(\omega^A du^B du^{n+C}\right) \\
&= (-1)^r \overline{\star} L \left(\pm \omega^{N-(A \cup B \cup C)} du^{n+B} du^C\right) \\
&= (-1)^r \overline{\star} \left(\pm \sum_{\gamma \notin N-(A \cup B \cup C) \cup B \cup C} \omega^\gamma \omega^{N-(A \cup B \cup C)} du^{n+B} du^C\right) \\
&= (-1)^r \overline{\star} \left(\pm \sum_{\gamma \notin N-A} \omega^\gamma \omega^{N-(A \cup B \cup C)} du^{n+B} du^C\right) \\
&= (-1)^r \overline{\star} \left(\pm \sum_{\gamma \in A} \omega^\gamma \omega^{N-(A \cup B \cup C)} du^{n+B} du^C\right) \\
&= (-1)^r \overline{\star} \left(\pm \sum_{\gamma \in A} \omega^{N-(A-\{\gamma\}) \cup B \cup C)} du^{n+B} du^C\right)
\end{aligned}$$

Up to this point we're unsure about the sign inside the brackets, but we know that $\overline{\star}$ acts on r-forms as $-1)^r$ times the identity. Since L is linear on functions, and hence on constants, we conclude that our sign is $(-1)^r$ after applying $\overline{\star}$ a second time. That is

$$\begin{aligned} L^*(\omega^A du^B du^{n+C}) &= (-1)^r \cdot (-1)^r \sum_{\gamma \in A} \omega^{N-(N-(A-\{\gamma\})\cup B\cup C)\cup B\cup C} du^B du^{n+C} \\ &= \sum_{\gamma \in A} \omega^{A-\{\gamma\}} du^B du^{n+C} \end{aligned}$$

proving the result.

Proposition 3.3. *If n is the complex dimension of the Kähler manifold then*

$$[L, L^*] \,:\, \mathcal{E}^r(X) \longrightarrow \mathcal{E}^r(X) \,:\, \eta \longrightarrow (r-n)\cdot\eta$$

That is, the commutator $[L, L^*]$ acts on r-forms by simply multiplying by the constant $(r-n)$.

Proof It suffices to prove the result on elements of the form $\omega^A du^B du^{n+C}$ because we may then extend the result to all r-forms by the function linearity of L and L^*. Now

$$\begin{aligned} LL^*(\omega^A du^B du^{n+C}) &= L\left(\sum_{\alpha\in A} \omega^{A-\{\alpha\}} du^B du^{n+C}\right) \\ &= \sum_{\gamma\notin(A-\{\alpha\})\cup B\cup C} \sum_{\alpha\in A} \omega^\gamma \omega^{A-\{\alpha\}} du^B du^{n+C} \end{aligned}$$

We break this sum up into the case when $\alpha = \gamma$ and otherwise. That is

$$\begin{aligned} LL^*(\omega^A du^B du^{n+C}) &= \sum_{\alpha\in A} \omega^\alpha \omega^{A-\{\alpha\}} du^B du^{n+C} + \sum_{\gamma\notin A\cup B\cup C} \sum_{\alpha\in A} \omega^\gamma \omega^{A-\{\alpha\}} du^B du^{n+C} \\ &= |A|\cdot \omega^A du^B du^{n+C} + \sum_{\gamma\notin A\cup B\cup C} \sum_{\alpha\in A} \omega^{A\cup\{\gamma\}-\{\alpha\}} du^B du^{n+C} \end{aligned}$$

On the other hand

$$\begin{aligned} L^*L(\omega^A du^B du^{n+C}) &= L^*\left(\sum_{\gamma\notin A\cup B\cup C} \omega^\gamma \omega^A du^B du^{n+C}\right) \\ &= \sum_{\alpha\in A} \sum_{\gamma\notin A\cup B\cup C} \omega^{A\cup\{\gamma\}-\{\alpha\}} du^B du^{n+C} \\ &= \sum_{\gamma\notin A\cup B\cup C} \omega^A du^B du^{n+C} + \sum_{\alpha\in A} \sum_{\gamma\notin A\cup B\cup C} \omega^{A\cup\{\gamma\}-\{\alpha\}} du^B du^{n+C} \\ &= K\cdot \omega^A du^B du^{n+C} + \sum_{\alpha\in A} \sum_{\gamma\notin A\cup B\cup C} \omega^{A\cup\{\gamma\}-\{\alpha\}} du^B du^{n+C} \end{aligned}$$

where $K := |N - A\cup B\cup C| = n - |A| - |B| - |C|$ because A, B and C are disjoint. Therefore

$$\begin{aligned} (LL^* - L^*L)(\omega^A du^B du^{n+C}) &= (n - 2|A| - |B| - |C|)\cdot \omega^A du^B du^{n+C} \\ &= (n-r)\omega^A du^B du^{n+C} \end{aligned}$$

because $\omega^A du^B du^{n+C} \in \mathcal{E}^r(X)$ ensures[7] that $2|A| + |B| + |C| = r$. The Proposition is now proven.

3.3. **A Representation of $\mathfrak{sl}(2,\mathbb{C})$ on $H^*(X,\mathbb{C})$.** Thus far, we've seen that the operators L, L^* and $[L, L^*]$ act on $\mathcal{E}^*(X) = \bigoplus \mathcal{E}^r(X)$.

Proposition 3.4. *L, L^* and $[L, L^*]$ act on $\mathcal{H}^*(X,\mathbb{C})$ and so, by the Hodge Theorem, on $H^*(X,\mathbb{C})$.*

Proof We saw in Proposition 3.3 that $[L, L^*]$ is a multiple of the identity so it certainly acts on $\mathcal{H}^*(X,\mathbb{C})$. To prove that L, L^* also act on $\mathcal{H}^*(X,\mathbb{C})$ it suffices to show that they commute with the d-Laplacian Δ_d. If we can show that $[L, \Delta_d] = 0$ then, by taking adjoints, it will follow that $[L^*, \Delta_d] = 0$, since Δ_d is self-adjoint. So it suffices to prove that $[L, \Delta_d] = 0$.
Since ω is closed

$$d\, L(\eta) = d(\omega \wedge \eta) = \omega \wedge d\eta = L\, d(\eta)$$

so $[L, d] = 0$. Furthermore, using the Kähler identities we see that

$$[L^*, d] = [L^*, \partial + \overline{\partial}] = [L^*, \partial] + [L^*, \overline{\partial}] = i(\overline{\partial}^* - \partial^*)$$

Taking adjoints once again gives $[L, d^*] = i(\overline{\partial} - \partial)$. Therefore

$$\begin{aligned}
[L, \Delta_d] &= Ldd^* + Ld^*d - dd^*L - d^*dL \\
&= dLd^* - dd^*L + Ld^*d - d^*Ld \\
&= d[L, d^*] + [L, d^*]d \\
&= i\, d(\overline{\partial} - \partial) + i(\overline{\partial} - \partial)d \\
&= i\, (\partial + \overline{\partial})(\overline{\partial} - \partial) + i(\overline{\partial} - \partial)(\partial + \overline{\partial}) \\
&= 2i\partial\overline{\partial} - 2i\partial\overline{\partial} = 0
\end{aligned}$$

which proves the proposition. □

In fact, L, L^* and $[L, L^*]$ act on $\mathcal{H}^*(X,\mathbb{C})$ in a very precise way. We wish to recall the following notion from Representation Theory: a *representation of the Lie Algebra* $\mathfrak{sl}(2,\mathbb{C})$ on a complex vector space V is a map of Lie Algebras

$$\rho \;:\; \mathfrak{sl}(2,\mathbb{C}) \longrightarrow \mathrm{End}_{\mathbb{C}}(V)$$

That is, a linear map ρ such that

$$\rho\,([X, Y]) = [\rho(X), \rho(Y)] \tag{3.1}$$

where, in this particular case, the bracket [,] is the commutator in both $\mathfrak{sl}(2,\mathbb{C})$ and $\mathrm{End}_{\mathbb{C}}(V)$. The matrices

$$E_+ := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \;,\; E_- := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \;,\; A := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

[7]Each index in $\alpha \in A$ contributes a double covector $du^\alpha du^{n+\alpha}$ to $\omega^A du^B du^{n+C}$, whereas each index in B and C contributes a single covector.

form a basis on $\mathfrak{sl}(2:\mathbb{C})$ and satisfy the relations

$$[A,E_+] = 2E_+ \quad [A,E_-] = -2E_- \quad [E_+,E_-] = A \tag{3.2}$$

Theorem 3.5. *The map*

$$\rho \ : \ \mathfrak{sl}(2,\mathbb{C}) \longrightarrow End_{\mathbb{C}}(H^*(X,\mathbb{C})$$

defined by $\rho(E_+) = L$, $\rho(E_-) = L^*$ and $\rho(A) = [L,L^*]$ is a representation of $\mathfrak{sl}(2,\mathbb{C})$ on $H^*(X,\mathbb{C})$.

Proof We have to show that the relation (3.1) holds. The commutator relations of the basis $\{E_+,E_-,A\}$ of $\mathfrak{sl}(2,\mathbb{C})$ are listed above on line (3.2). To prove the theorem we must therefore show that the relations

$$[[L,L^*],L] = 2L \quad [[L,L^*],L^*] = -2L^* \quad [L,L^*] = [L,L^*](!)$$

hold. Clearly the last one is a tautology! For $\eta \in \mathcal{E}^r(X)$ we have

$$\begin{aligned}
[[L,L^*],L]\,(\eta) &= [L,L^*]L(\eta) - L[L,L^*](\eta) \\
&= (r+2-n)L(\eta) - L((r-n)\eta) && \text{as } L(\eta) \in \mathcal{E}^{r+2}(X) \\
&= (r+2-n)L(\eta) - (r-n)L(\eta) \\
&= 2\,L(\eta)
\end{aligned}$$

proving the first relation. Similarly

$$\begin{aligned}
[[L,L^*],L^*]\,(\eta) &= [[L,L^*],L^*]\,(\eta) \\
&= [L,L^*]L^*(\eta) - L^*[L,L^*](\eta) \\
&= (r-2-n)L^*(\eta) - L^*((r-n)\eta) && \text{as } L^*(\eta) \in \mathcal{E}^{r-2}(X) \\
&= (r-2-n)L^*(\eta) - (r-n)L^*(\eta) \\
&= -2\,L^*(\eta)
\end{aligned}$$

proving the theorem. □

The general theory of representations of $\mathfrak{sl}(2,\mathbb{C})$ enables us to conclude the following result:

Theorem 3.6 ('Strong' Lefschetz)**.** *If X is a Kähler manifold of complex dimension n then the map*

$$L^{n-p} \ : \ H^r(X,\mathbb{C}) \longrightarrow H^{2n-r}(X,:\mathbb{C}) \ : \ \eta \longrightarrow \omega^{n-r}\wedge\eta$$

is an isomorphism for all p. Furthermore, if we define the primitive cohomology by

$$H^r_p(X,\mathbb{C}) = ker(L^*) \cap H^r(X,\mathbb{C})$$

then we have the so-called 'Lefschetz Decomposition'

$$H^r(X,\mathbb{C}) = \sum_{k\geq 0} L^k\left(H^{r-2k}_p(X,\mathbb{C})\right)$$

4. The Kodaira Vanishing Theorem

Our goal is to prove the famous Kodaira Vanishing Theorem. We begin by reviewing some standard ideas so that we may establish our notation.

4.1. **The First Chern Class of Holomorphic Vector Bundles.** Let E be a smooth, complex vector bundle of rank r over X. Recall that a *connection* ∇ on E is an operator

$$\nabla \,:\, \mathcal{E}^0(X,E) \longrightarrow \mathcal{E}^1(X,E)$$

for which the Leibnitz rule holds $\nabla(f\cdot\eta) = df\otimes\eta + f\cdot\nabla\eta$ for $f\in\mathcal{E}^0(X)$ and $e\in\mathcal{E}^0(X,E)$. If $\{e_1,\dots e_r\}$ is a local frame for E then clearly $\nabla e_i = \sum \theta_i{}^j\otimes e_j$ for some complex-valued 1-forms $\{\theta_i{}^j\}$. It follows from the Leibnitz rule that $\nabla\tau = (d\,\tau^i + \sum\tau^j\theta_j{}^i)\otimes e_i$ for $\tau = \sum\tau^i e_i$. The curvature of the connection ∇ on E is the $\mathcal{E}^0(X)$-linear map

$$\Theta := d^\nabla\circ\nabla \,:\, \mathcal{E}^0(X,E)\longrightarrow\mathcal{E}^2(X,E)$$

where $d^\nabla \,:\, \mathcal{E}^1(X,E)\longrightarrow\mathcal{E}^2(X,E)$ is the covariant derivative[8] corresponding to ∇. In terms of the local frame $\{e_1,\dots e_r\}$ for E

$$\Theta(e_i) = d^\nabla\circ\nabla(e_i) = d^\nabla\left(\textstyle\sum_j\theta_i{}^j e_j\right) = (\textstyle\sum_j d\,\theta_i{}^j - \sum_k\theta_i{}^k\wedge\theta_k{}^j)\otimes e_j$$

for some complex-valued two-forms $\{\Theta_i{}^j\}$. We extend Θ to all of $\mathcal{E}^0(X,E)$ via function-linearity. The entries in the $r\times r$ *curvature matrix* $\Theta := (\Theta_i{}^j)$ are $\Theta_i{}^j = d\,\theta_i{}^j - \theta_i{}^k\wedge\theta_k{}^j$. Notice that, since Θ is linear over $\mathcal{E}^0(X)$ the entries in the curvature matrix are globally defined two-forms.

Since $\mathcal{E}^1(X,E) = \Gamma(X, TX^*\otimes_{\mathbb{C}} E)$, the splitting $TX^*\otimes_{\mathbb{R}}\mathbb{C} = \Omega^1_X\oplus\overline{\Omega^1_X}$ means that we can decompose ∇ as a sum $\nabla = \nabla' + \nabla''$ where

$$\nabla' \,:\, \Gamma(X,E)\longrightarrow\mathcal{E}^{1,0}(X,E) \quad ; \quad \nabla'' \,:\, \Gamma(X,E)\longrightarrow\mathcal{E}^{0,1}(X,E)$$

The connection is said to be *compatible with the complex structure* if $\nabla'' = \overline{\partial}_E$, the operator introduced in lecture 1. If E has been endowed with a hermitian metric, which is always possible, then we say that ∇ is *compatible with the metric* if $d\langle\xi,\eta\rangle = \langle\nabla\xi,\eta\rangle + \langle\xi,\nabla\eta\rangle$. The following result is standard:

Theorem 4.1. *Every* holomorphic *vector bundle of rank* r *endowed with a hermitian metric admits a unique connection* ∇*, called the canonical connection, which is compatible with both the complex structure and the metric. Furthermore, the curvature matrix* $(\Theta_i{}^j)$ *with respect to any local frame is an* $r\times r$ *hermitian matrix of* $(1,1)$*-forms.*

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[8]For each r we define an operator $d^\nabla : \mathcal{E}^r(X,E)\longrightarrow\mathcal{E}^{r+1}(X,E)$ by requiring that $d^\nabla(\eta\otimes e) = d\eta\otimes e + (-1)^r\eta\wedge\nabla(e)$ for $\eta\in\mathcal{E}^r(X)$ and $e\in\mathcal{E}^0(X,E)$.

We now restrict ourselves to the case when E is a holomorphic line bundle with a chosen hermitian metric, so that the curvature matrix of $(1,1)$-forms of the canonical connection is a single $(1,1)$-form Θ. We define the first Chern class of a line bundle E by the formula

$$c_1(E) = \left[\frac{i}{2\pi}\Theta\right] \in H^{1,1}(X) \tag{4.1}$$

Alternatively, one can show that the first Chern class coincides with the image of $E \in \mathrm{Pic}(X) \xrightarrow{c_1} c_1(E) \in H^2(X,\mathbb{Z})$ under the coboundary map in the long exact sequence on cohomology induced from the exponential sequence. As a result, any representative of the class $c_1(E)$ is an integer-valued form, and the definition is independent of the choice of metric.

4.2. **Positive Line Bundles.** Recall that

- If X is a Kähler manifold then there exists a Kähler metric on X, which we'll denote by h_X, for which the corresponding real-valued skew $(1,1)$-form ω is closed.
- If E is a holomorphic line bundle over a compact, complex manifold X, which has been endowed with a hermitian metric h_E then, by Theorem 4.1, there's a canonical connection on E for which the form $\frac{i}{/2\pi}\Theta$ is a closed, integer-valued $(1,1)$-form on X.

This raises the following question: *given a holomorphic line bundle E over X, when does there exist a hermitian metric h_X on X and a hermitian metric h_E on E for which*

$$\omega = \frac{i}{2\pi}\Theta \tag{4.2}$$

where ω is the fundamental form associated to h_X, and where Θ is the curvature form of the canonical connection on E associated to h_E ?

Defintiion Let E be a holomorphic line bundle over a compact, complex manifold X. Then E is said to be a *positive line bundle* if and only if there exists a hermitian metric h_X on X and a hermitian metric h_E on E for which equation (4.2) holds. Notice then that our motivating question has an affirmative answer if and only if the line bundle is positive.

In the literature this definition is stated as follows: 'A holomorphic line bundle E over a compact, complex manifold X is positive if and only if there exists a hermitian metric on E such that $i\Theta$ is a positive differerential form, where Θ is the curvature of E with respect to the canonical connection induced by h'. This explains the terminology 'positive'.

4.3. **The Kodaira Vanishing Theorem.** In this section we will be working with E-valued (p,q)-forms. We wish to make use of the operator L and the Kähler identity $[L^*, \overline{\partial}] = -i\partial^*$, but these operatots are defined on $\mathcal{E}^{p,q}(X)$. We must therefore define analogues of the operators $L, L^*, \overline{\partial}$ and

∂^* on $\mathcal{E}^{p,q}(X, E)$. We have already done so for $\overline{\partial}$:

$$\overline{\partial}_E \ : \ \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p,q+1}(X,E) \ : \ \sum_i \eta_i \otimes e_i \longrightarrow \sum_i \overline{\partial}(\eta_i) \otimes e_i$$

Similarly, we define

$$L_E \ : \ \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p+1,q+1}(X,E) \ : \ \sum_i \eta_i \otimes e_i \longrightarrow \sum_i \omega \wedge \eta_i \otimes e_i$$

$$L_E^* \ : \ \mathcal{E}^{p,q}(X,E) \longrightarrow \mathcal{E}^{p-1,q-1}(X,E) \ : \ \sum_i \eta_i \otimes e_i \longrightarrow \sum_i L^*(\eta_i) \otimes e_i$$

Notice that as before if $\xi \in \mathcal{E}^{p,q}(X, E)$ then we have the relation

$$[L_E, L_E^*](\xi) = (p+q-n) \cdot \xi \tag{4.3}$$

By extending ∂ in the same way we do not arrive at a well-defined operator. Recall however that a connection is the analogue on a smooth bundle of the exterior derivative. ∂ was defined via the splitting $d = \partial + \overline{\partial}$, and we have already remarked that a canonical connection can be decomposed as $\nabla = \nabla' + \overline{\partial}_E$. The analogue of the operator ∂ is therefore ∇' and, using the original Kähler identity $[L^*, \overline{\partial}] = -i\partial^*$, one can show that the following identity holds:

$$[L_E^*, \overline{\partial}_E] = -i(\nabla')^* \tag{4.4}$$

We are now in a position to state and prove our main theorem:

Theorem 4.2 (Kodaira-Nakano Vanishing). *If E is a positive line bundle over a compact, complex manifold X, then*

$$H^q(X, \Omega_X^p \otimes E) = 0 \quad \textit{for } p+q > n$$

Proof By the Hodge Theorem discussed in lecture 1 it suffices to prove that $\mathcal{H}^{p,q}(X, E) = 0$ for $p + q > n$. That is, we must show that when $p + q > n$, every form $\xi \in \mathcal{E}^{p,q}(X, E)$ such that $\Delta_{\overline{\partial}_E}(\xi) = 0$ is necessarily zero.
By hypothesis there is a hermitian metric h_E on E such that $\Theta = -2\pi i\omega$, where Θ is the curvature of E with respect to the canonical connection ∇ induced by h_E, and where ω is the fundamental form of some hermitian metric h on X. The canonical connection can be written as a sum $\nabla = \nabla' + \overline{\partial}_E$ and the identity (4.4) holds. We require the following lemma:

Lemma 4.3 (Nakano Inequalities). *For $\xi \in \mathcal{H}^{p,q}(X, E)$ we have*

(1) $\dfrac{i}{2\pi} \langle \Theta \wedge L_E^*(\xi), \xi \rangle \leq 0$

(2) $\dfrac{i}{2\pi} \langle L_E^*(\Theta \wedge \xi), \xi \rangle \geq 0$

Proof of Lemma $\Theta = \nabla'\overline{\partial}_E + \overline{\partial}_E\nabla'$. Now, by assumption $\Delta_{\overline{\partial}_E}(\xi) = 0$ which, by a remark from lecture 1, is equivalent to $\overline{\partial}_E(\xi) = \overline{\partial}_E^*(\xi) = 0$. Therefore $\Theta(\xi) = \overline{\partial}_E\nabla'(\xi)$. We now wish to consider

$$\begin{aligned}
i \parallel (\nabla')^*(\xi) \parallel^2 &= i\left\langle(\nabla')^*(\xi), (\nabla')^*(\xi)\right\rangle \\
&= \left\langle -[L_E^*, \overline{\partial}_E](\xi), (\nabla')^*(\xi)\right\rangle && \text{by (4.4)} \\
&= \left\langle \overline{\partial}_E L_E^*(\xi) - L_E^*\overline{\partial}_E(\xi), (\nabla')^*(\xi)\right\rangle \\
&= \left\langle \overline{\partial}_E L_E^*(\xi), (\nabla')^*(\xi)\right\rangle && \text{as } \overline{\partial}_E(\xi) = 0 \\
&= \left\langle L_E^*(\xi), \overline{\partial}_E^*(\nabla')^*(\xi)\right\rangle && \text{by taking adjoints} \\
&= \left\langle L_E^*(\xi), (\overline{\partial}_E^*(\nabla')^* + (\nabla')^*\overline{\partial}_E^*)(\xi)\right\rangle && \text{as } \overline{\partial}_E^*(\xi) = 0 \\
&= \left\langle (\nabla'\overline{\partial}_E + \overline{\partial}_E\nabla')L_E^*(\xi), \xi)\right\rangle && \text{by taking adjoints} \\
&= \langle \Theta \wedge L_E^*(\xi), \xi)\rangle
\end{aligned}$$

Now multiply through by i and the first inequality holds. The second is similar. It is convenient to introduce the factor $1/(2\pi)$ in each inequality.

We can now complete the proof of the theorem. Recall that we wish to show that when $p+q > n$, every form $\xi \in \mathcal{E}^{p,q}(X, E)$ such that $\Delta_{\overline{\partial}_E}(\xi) = 0$ is necessarily zero. The Nakano inequalities hold for ξ and subtracting *1* from *2* shows that

$$\frac{i}{2\pi}\,\langle (L_E^*(\Theta \wedge \xi) - \Theta \wedge L_E^*(\xi), \xi\rangle \geq 0$$

But $L_E(\xi) = \omega \wedge \xi = \frac{i}{2\pi}\Theta \wedge \xi$, when we endow X with the hermitian metric h induced by the positive line bundle E, so we may rewrite the inequality as

$$\langle [L_E^*, L_E](\xi), \xi\rangle = \langle L_E^* L_E(\xi) - L_E L_E^*(\xi), \xi\rangle \geq 0$$

By multiplying equation (4.3) by -1 we see that

$$(n - p - q)\,\langle \xi, \xi\rangle = \langle [L_E^*, L_E](\xi), \xi\rangle \geq 0$$

As a result, if $p + q > n$ the form $\xi \in \mathcal{H}^{p,q}(X, E)$ is necessarily zero.

□

The special case $p = n$ is the Kodaira Vanishing Theorem:

Theorem 4.4 (Kodaira Vanishing)**.** *If E is a positive line bundle over a compact, complex manifold X, then*

$$H^q(X, K_X \otimes E) = 0 \quad \textit{for } p > 0$$

where $K_X := \bigwedge^n \Omega_X^1$.

Using the Kodaira embedding theorem one can show that this theorem is equivalent to the following statement in the algebraic category:

Theorem 4.5 (Kodaira Vanishing)**.** *If E is an ample line bundle over a smooth projective variety X, then*

$$H^q(X, K_X \otimes E) = 0 \quad \textit{for } p > 0$$

where $K_X := \bigwedge^n \Omega^1_X$.

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