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Toric Geometry

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Abstract. Toric Geometry plays a major role where a wide variety of mathematical fields intersect, such as algebraic and symplectic geometry, algebraic groups, and combinatorics. The main feature of this workshop was to bring people from these area together to learn about mutual, possibly up till now unnoticed similarities in their respective research.

Mathematics Subject Classification (2000): 14M25, 53D37, 52B20.

Introduction by the Organisers

The workshop "Toric Geometry" was attended by 53 people including many young participants. The idea was to shed light on the subject from many different points of view – toric geometry involves methods from algebraic and symplectic geometry, algebraic groups, and discrete mathematics.

A major driving force combining all these directions is still provided by the different flavours of mirror symmetry. So it is quite natural that related subjects like Lagrangians in symplectic manifolds showed up in many talks (Abreu, Woodward, Ono, Lau, Sjamaar).

A very common feature that appeared in many talks was the attempt to weaken assumptions in the setting of algebraic or symplectic toric varieties. This was done by either considering higher complexities of torus actions, or by relaxing the demands on the symplectic forms, or by studying non-algebraic situations or more general algebraic groups than just tori (Timashev, Hausen, Süß, Knop, Tolman, Holm, Masuda).

Polyhedral methods and their interplay with resolutions and deformations or degenerations is a classical feature of toric geometry. Recently this was extended

to non toric varieties by the notion of Okounkov bodies. Talks widely fitting into this area were given by Kiritchenko, Teissier, Tevelev, Kaveh, Ilten, Nill.

Finally, there were talks dealing with homological, K-theoretical or derived methods (Craw, Anderson, Ploog) or talks with the classical topics of syzygies or projective duality (Schenck, di Rocco).

The informal discussions in addition to the talks brought algebraic and symplectic geometers together – for instance the different languages for studying complexity one T-varieties were mutually recognized. Moreover, on Wednesday night a special session of short talks took place. Everybody was allowed to speak, but each contribution was strictly limited to ten minutes plus discussion. This was adroitly moderated by Christian Haase and became a very successful and energetic evening.

Workshop: Toric Geometry

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Abstracts

Divided difference operators on convex polytopes

Valentina Kiritchenko

I describe a convex geometric procedure for building generalized Newton polytopes of Schubert varieties. One of the goals is to extend to arbitrary reductive groups our joint work with Evgeny Smirnov and Vladlen Timorin on Schubert calculus (in type A) in terms of Gelfand–Zetlin polytopes [6].

Newton (or moment) polytopes of projective toric varieties play a prominent role in toric geometry. Analogs of Newton polytopes (*Newton–Okounkov convex bodies*) were defined for an arbitrary variety together with a line bundle [3]. If the variety enjoys an action of a reductive group G and the line bundle is G-linearized, then a more precise description of the Newton–Okounkov body can be given (in particular, for spherical varieties this convex body is a polytope). In this case, major building blocks in the construction of Newton-Okounkov bodies are *string polytopes*. String polytopes first appeared in representation theory [7] and are associated with the irreducible representations of G, e.g. a classical example of a string polytope for $G = GL_n$ is a Gelfand-Zetlin polytope. The integer points inside and at the boundary of a string polytope parameterize a canonical basis in the associated representation, in particular, string polytopes can be regarded as Newton polytopes for projective embeddings of the complete flag variety for the group G.

The Gelfand–Zetlin polytopes can be constructed from a single point by iterating a simple convex-geometric operator on polytopes that mimics the well-known *divided difference* or *Demazure operators* from representation theory [5]. Each operator acts on convex polytops (more generally, on *convex chains*) and takes a polytope to a polytope of dimension one greater. In the case of GL_n, these operators were used to calculate Demazure characters of Schubert varieties in terms of the exponential sums over unions of faces of Gelfand-Zetlin polytopes [6]. In particular, they yield generalized Newton polytopes (*convex chains*) for Schubert subvarieties in the variety of complete flags in $\mathbb{C}^n$.

The convex-geometric Demazure operators are defined not only for arbitrary reductive group but in a more general setting. They are well suited for inductive descriptions of Newton–Okounkov polytopes for line bundles on Bott towers and on Bott–Samelson varieties [5]. The former polytopes were described by Grossberg and Karshon [2] and the latter are currently being computed by Anderson [1].

Below I give a definition of Demazure operators on convex polytopes and formulate some results. A *root space* of rank n is a coordinate space $\mathbb{R}^d$ together with a direct sum decomposition

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_n}$$

and a collection of linear functions $l_1, \ldots, l_n \in (\mathbb{R}^d)^*$ such that l_i vanishes on $\mathbb{R}^{d_i}$. We always assume that the summands are coordinate subspaces (so that $\mathbb{R}^{d_1}$ is

spanned by the first d_1 basis vectors etc.). The coordinates in $\mathbb{R}^d$ will be denoted by $(x_1^1, \dots, x_{d_1}^1; \dots; x_1^n, \dots, x_{d_n}^n)$.

Let $P \subset \mathbb{R}^d$ be a convex polytope in the root space. It is called a *parapolytope* if for all $i = 1,\dots, n$, the intersection of P with any parallel translate of $\mathbb{R}^{d_i}$ is a *coordinate parallelepiped*, that is, the parallelepiped

$$\Pi(\mu, \nu) = \{\mu_k \leqslant x_k^i \leqslant \nu_k,\ k = 1, \dots, d_i\},$$

where $\mu_1, \dots, \mu_{d_i}, \nu_1, \dots, \nu_{d_i}$ are real numbers. For instance, if $d = n$ (i.e. $d_1 = \dots = d_n = 1$) then every polytope is a parapolytope. A less trivial example of a parapolytope is the classical Gelfand–Zetlin polytope Q_λ (where $\lambda = (\lambda_1, \dots, \lambda_n)$ is an increasing collection of integers) in the root space

$$\mathbb{R}^d = \mathbb{R}^{n-1} \oplus \mathbb{R}^{n-2} \oplus \dots \oplus \mathbb{R}^1 \tag{1}$$

of rank $(n-1)$. The polytope Q_λ is given by inequalities $x_j^{i-1} \leqslant x_j^i \leqslant x_{j+1}^{i-1}$ for all $i = 1,\dots, n-1$ and $1 \leqslant j \leqslant (n-i)$ (we put $x_j^0 = \lambda_j$ for $j = 1,\dots, n$).

For each $i = 1,\dots, n$, we now define a *divided difference operator* D_i on parapolytopes. In general, the operator D_i takes values in *convex chains* (or *virtual polytopes*) in $\mathbb{R}^d$ (see [4] for a definition).

First, consider the case where $P \subset (c + \mathbb{R}^{d_i})$ for some $c \in \mathbb{R}^d$, i.e. $P = P(\mu, \nu)$ is a coordinate parallelepiped. Choose the smallest $j = 1,\dots, d_i$ such that $\mu_j = \nu_j$. Define $D_i(P)$ as the coordinate parallelepiped $\Pi(\mu, \nu')$, where $\nu'_k = \nu_k$ for all $k \neq j$ and ν'_j is defined by the equality

$$\sum_{k=1}^{d_i} (\mu_k + \nu'_k) = l_i(c).$$

For an arbitrary parapolytope $P \subset \mathbb{R}^d$ define $D_i(P)$ as the union of $D_i(P \cap (c + \mathbb{R}^{d_i}))$ over all $c \in \mathbb{R}^d$:

$$D_i(P) = \bigcup_{c \in \mathbb{R}^d} \{D_i(P \cap (c + \mathbb{R}^{d_i}))\}$$

(assuming that $\dim(P \cap (c + \mathbb{R}^{d_i})) < d_i$ for all $c \in \mathbb{R}^d$). That is, we first slice P by subspaces parallel to $\mathbb{R}^{d_i}$ and then replace each slice $\Pi(\mu, \nu)$ with $\Pi(\mu, \nu')$. Note that P is a facet of $D_i(P)$ unless $D_i(P) = P$.

Example 1 (case of GL_n): Consider the root space (1) with the functions l_i given by the formula:

$$l_i(x) = \sigma_{i-1}(x) + \sigma_{i+1}(x),$$

where $\sigma_i(x) = \sum_{k=1}^{d_i} x_k^i$ for $i = 1,\dots, n-1$ and $\sigma_0 = \sigma_n = 0$. It is not hard to show that the Gelfand–Zetlin polytope Q_λ defined above coincides with the polytope

$$[(D_1 \dots D_{n-1})(D_1 \dots D_{n-2}) \dots (D_1)]\,(p),$$

where $p \in \mathbb{R}^d$ is the point $(\lambda_2, \dots, \lambda_n; \lambda_3, \dots, \lambda_n; \dots; \lambda_n)$.

Example 2 (arbitrary reductive groups): Let G be a connected reductive group of semisimple rank n, $\alpha_1,\dots, \alpha_n$ simple roots of G, and $s_1,\dots, s_n$ the corresponding simple reflections. Fix a reduced decomposition $w_0 = s_{i_1} \dots s_{i_d}$ of

the longest element in the Weyl group of G. Let d_i be the number of s_{i_j} in this decomposition such that $i_j = i$. Consider the root space

$$\mathbb{R}^d = \mathbb{R}^{d_1} \oplus \ldots \oplus \mathbb{R}^{d_n} \tag{2}$$

with the functions l_i given by the formula:

$$l_i(x) = \sum_{k \neq i} (\alpha_k, \alpha_i)\sigma_k(x),$$

where (α_k, α_i) is the element of the Cartan matrix of G, that is, $s_i(\alpha_k) = \alpha_k - (\alpha_k, \alpha_i)\alpha_i$. In particular, for $G = GL_n$ and $w_0 = (s_1 \ldots s_{n-1})(s_1 \ldots s_{n-2}) \ldots (s_1)$ we get the root space of Example 1. Define the projection of the root space to the real span of the weight lattice of G by the formula $p(x) = \sigma_1(x)\alpha_1 + \ldots + \sigma_n(x)\alpha_n$.

Theorem: *For each dominant weight λ of G there exists a point $p_\lambda \in \mathbb{R}^d$ such that the polytope*

$$P := D_{i_1} \ldots D_{i_d}(p_\lambda)$$

yields the Weyl character $\chi(V_\lambda)$ of the irreducible G-module V_λ, namely,

$$\chi(V_\lambda) = \sum_{x \in P \cap \mathbb{Z}^d} e^{p(x)}.$$

Similarly, the face of P defined as $D_{i_l} \ldots D_{i_d}(p_\lambda)$ yields the Demazure character corresponding to $w = s_{i_l} \ldots s_{i_d}$ and λ for any $l \leqslant d$. Demazure characters for the other elements of the Weyl group can be represented by unions of faces of P.

If we replace each $\mathbb{R}^{d_i}$ in root decomposition (2) with the sum of d_i one-dimensional coordinate subspaces with the same function l_i (the result is the root decomposition of rank d), then the (usually virtual) polytope $D_1 \ldots D_d(p_\lambda)$ is exactly the *skew* polytope constructed in [2]. The theorem above holds for this polytope by [2] as well as for the other polytopes constructed via arbitrary subdivisions of root decomposition (2) (that is, when $\mathbb{R}^{d_i}$ is split into several coordinate subspaces, not necessarily one-dimensional). Conjecturally, string polytopes can also be obtained using either (2) or one of its subdivisions.

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Symplectic varieties with invariant Lagrangian subvarieties

Dmitry A. Timashev
(joint work with Vladimir S. Zhgoon)

We consider symplectic algebraic varieties equipped with a Hamiltonian reductive group action which contain an invariant Lagrangian subvariety. A basic example is the cotangent bundle of an algebraic variety acted on by a reductive group; here the zero section is an invariant Lagrangian subvariety. The main thesis of this talk is that, though this basic example does not exhaust all possible cases (even locally), Hamiltonian symplectic varieties with invariant Lagrangian subvarieties behave very similar to cotangent bundles. See [4], [5] for more details.

Let (M,ω) be a symplectic manifold (real or complex) or smooth complex algebraic variety. (In the latter case the symplectic 2-form ω is assumed to be algebraic, of course.) We denote by ∇f the skew gradient of a function f, given by the formula: $df(\nu)=\omega(\nabla f,\nu)$, $\forall \nu\in TM$. Let $\{f,g\}=\omega(\nabla f,\nabla g)$ denote the Poisson bracket of functions on M. Recall that a submanifold or smooth subvariety $S\subseteq M$ is said to be:

- *isotropic* if $\omega|_{T_pS}=0$, $\forall p\in S$;
- *coisotropic* if $\omega|_{(T_pS)^\angle}=0$, $\forall p\in S$;
- *Lagrangian* if it is both isotropic and coisotropic.

Here and below, the superscript $^\angle$ denotes the skew-orthogonal complement.

Let G be a Lie or algebraic group acting on M. Recall that the action $G\curvearrowright M$ is said to be *Hamiltonian* if the following conditions are satisfied:

- G-action preserves ω;
- there exists a G-equivariant map $\Phi: M\to\mathfrak{g}^*$, called *moment map*, such that $\nabla(\Phi^*\xi)=\xi_*$, $\forall\xi\in\mathfrak{g}$, i.e., $\langle d_p\Phi(\nu),\xi\rangle=\omega(\xi p,\nu)$, $\forall p\in M$, $\nu\in T_pM$;
- $\{\Phi^*\xi,\Phi^*\eta\}=\Phi^*([\xi,\eta])$, $\forall\xi,\eta\in\mathfrak{g}$.

Here $\mathfrak{g}=\operatorname{Lie}G$, Φ^* denotes the pullback of functions along the map Φ, and ξ_* is the velocity vector field of ξ on M, i.e., $\xi_*(p)=\xi p=\frac{d}{dt}|_{t=0}\exp(t\xi)p$. (The above conditions are not independent, but it is convenient to collect them all together.)

Here is the basic example of a Hamiltonian action. Given a smooth manifold or algebraic variety X, the cotangent bundle $M=T^*X$ comes equipped with the canonical symplectic structure: $\omega=\sum_i dx_i\wedge dy_i$, where x_i are local coordinates on X and y_i are the dual coordinates in cotangent spaces. If G acts on X, then the induced action $G\curvearrowright T^*X$ is Hamiltonian, with the moment map given by $\langle\Phi(p),\xi\rangle=\langle p,\xi x\rangle$, $\forall x\in X$, $p\in T^*_xX$, $\xi\in\mathfrak{g}$.

The zero section $S\subset T^*X$ is a Lagrangian submanifold/subvariety. It is G-stable whenever G acts on X. A more general example of a (G-stable) Lagrangian submanifold/subvariety in T^*X is the conormal bundle $N^*(X/Y)$ of a (G-stable) submanifold/subvariety $Y\subseteq X$.

It is well known, by the Darboux–Weinstein theorem, that any symplectic manifold M is isomorphic to T^*S in a neighborhood of a Lagrangian submanifold $S\subset M$ (regarded as the zero section in T^*S). If $G\curvearrowright M$ is a Hamiltonian action

of a compact Lie group and S is G-stable, then the above local symplectomorphism can be chosen even G-equivariant. These observations are due to Kostant [1, Chap. IV, §1], [5, §22].

However, if we pass to the algebraic category and replace G with a reductive algebraic group assuming M be a Hamiltonian G-variety with a G-stable Lagrangian subvariety S, then the above equivariant local symplectomorphism may not exist, even analytically. The reason is in the structure of isotropy representations.

Namely, for any $p \in S$, the isotropy representation $G_p \curvearrowright T_p(T^*S) = T_pS \oplus T_p^*S$ splits, i.e., T_pS has a G_p-stable complement. (Here S is regarded as the zero section in T^*S.) However, it may happen that T_pS has no G_p-stable complement in T_pM.

Example. *Let X be the variety of complete conics, which is the blowup of the space $\mathbb{P}^5$ of plane conics in $\mathbb{P}^2$ along the set of double lines. The group $G = SL_3(\mathbb{C})$ acts on X in a natural way, with the unique closed orbit Y. Now put $M = T^*X$, $S = N^*(X/Y)$. It can be shown that, for a general point $p \in S$, T_pS has no G_p°-stable complement in T_pM, where $G_p^\circ \subseteq G_p$ is the identity component.*

Despite this example, our main result states that certain important invariants of a symplectic variety with a Hamiltonian reductive group action coincide with those for the cotangent bundle of any invariant Lagrangian subvariety. Let us introduce these invariants.

Definition. *The* corank *of $G \curvearrowright M$ is* $\operatorname{cork} M = \operatorname{rk} \omega|_{(\mathfrak{g}p)^\angle}$ *and the* defect *of $G \curvearrowright M$ is* $\operatorname{def} M = \dim(\mathfrak{g}p) \cap (\mathfrak{g}p)^\angle$, *where $p \in M$ is a general point and $\mathfrak{g}p = T_p(Gp)$.*

These two invariants are closely related to the moment map. Here are the basic properties:

(1) $\operatorname{Ker} d_p\Phi = (\mathfrak{g}p)^\angle$ for any $p \in M$;
(2) $\operatorname{Im} d_p\Phi = (\mathfrak{g}_p)^\perp$, where $\mathfrak{g}_p = \operatorname{Lie} G_p$ and $^\perp$ denotes the annihilator;
(3) $\dim \overline{\Phi(M)} = \underline{\dim} Gp$ for general $p \in M$;
(4) $\operatorname{def} M = \dim \overline{\Phi(M)}/G$, i.e., the codimension of a general G-orbit in $\overline{\Phi(M)}$;
(5) $\operatorname{cork} M = \dim M - \dim \overline{\Phi(M)} - \dim \overline{\Phi(M)}/G$;
(6) $\operatorname{cork} M + \operatorname{def} M = \dim M/G$.

Now let G be a connected reductive group, M an irreducible Hamiltonian G-variety, and $S \subset M$ an irreducible G-stable Lagrangian subvariety. It follows from (1) that $\Phi(S)$ is a G-fixed point in $\mathfrak{g}^*$. Since the moment map is defined up to a shift by a G-invariant vector, we may and will assume that $\Phi(S) = \{0\}$.

Theorem 1. $\overline{\Phi(M)} = \overline{\Phi(T^*S)}$.

Corollary. $\operatorname{cork} M = \operatorname{cork} T^*S$, $\operatorname{def} M = \operatorname{def} T^*S$, *and* $\dim M/G = \dim(T^*S)/G$.

Here are the main ideas of the proof. The first one is to deform (or, rather, contract) M to the normal bundle $N = N(M/S)$, which appears to be isomorphic to T^*S. This deformation is quite standard in algebraic geometry. What is less standard is that the Hamiltonian structure on M deforms to the canonical Hamiltonian structure on T^*S as well.

The second idea is to use a foliation of horospheres on S in the description of the images of moment maps. By definition, a *horosphere* is an orbit of a maximal unipotent subgroup $U \subset G$. Suppose for simplicity that S is quasiaffine and let $\mathcal{U} \subset T^*S$ be the conormal bundle to the foliation of generic horospheres. Knop proved in [3] that $\overline{\Phi(\mathcal{U})} = \mathfrak{p}_0^\perp$, $\overline{G\mathcal{U}} = T^*S$, and $\overline{\Phi(T^*S)} = G\mathfrak{p}_0^\perp$, where $P_0 \subset G$ is the normalizer of a generic horosphere and $\mathfrak{p}_0 = \operatorname{Lie} P_0$. In order to extend these results to M (instead of T^*S), one needs a substitute for $\mathcal{U}$. It is constructed by choosing P_0-invariant functions $F_1, \dots, F_m$ on an open subset $\mathring{M} \subset M$ intersecting S whose skew gradients $\nabla F_1, \dots, \nabla F_m$ are linearly independent in $N|_S$ and spreading S along the trajectories of these Hamiltonian vector fields. It is possible to perform this construction in a pure algebraic way and thus produce a smooth subvariety $\mathcal{W} \subset M$ containing an open subset of S. By comparing $\mathcal{W}$ with $\mathcal{U}$ via deformation, we deduce the following result, which clearly implies Theorem 1.

Theorem 2. *$\overline{\Phi(\mathcal{W})} = \mathfrak{p}_0^\perp$, $\overline{G\mathcal{W}} = M$, and $\overline{\Phi(M)} = G\mathfrak{p}_0^\perp$.*

We conclude this report by discussing a generalization of our results to coisotropic S. Assume additionally that $\mathfrak{g}x \subset (T_xS)^\angle$, $\forall x \in S$. Then deformation of M to N still exists, but N is no longer symplectic, just Poisson. However, using similar arguments, we can prove

Theorem 3. *$\overline{\Phi(M)} = \overline{\Phi(T^*S)}$, $\dim M/G = \dim N/G$.*

A generalization of this theorem to a wider subclass of coisotropic subvarieties would give a simple proof of the following conjecture of Elashvili, which was verified using case by case arguments by Yakimova, de Graaf, and Charbonnel–Moreau.

Claim. *$\operatorname{ind} \mathfrak{g}_p = \operatorname{ind} \mathfrak{g}$, $\forall p \in \mathfrak{g}^*$.*

Recall that the *index* of a Lie algebra $\mathfrak{h} = \operatorname{Lie} H$ is defined as $\operatorname{ind} \mathfrak{h} = \dim \mathfrak{h}^*/H$. To deduce the claim from a generalization of Theorem 3, it suffices to note that $S = (G \times G)p$ is a coisotropic subvariety in $M = T^*G$ acted on by $G \times G$ (via left and right multiplication on G) and $M/(G \times G) \simeq \mathfrak{g}^*/G$, $N/(G \times G) \simeq \mathfrak{g}_p^*/G_p$.

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Syzygies and toric varieties

Hal Schenck

We describe three questions on syzygies and toric varieties. The first two questions concern the interplay between the geometry of an embedding and vanishing of cohomology. The third question concerns a recent application: computing the syzygies of a (possibly incomplete) linear system on a toric variety is a crucial step in applying the method of approximation complexes to find the implicit equation(s) of the image. A Cartier divisor D on a toric variety X is given by a collection of rays of the fan Σ satisfying certain conditions; the global sections of the line bundle associated to D correspond to lattice points in a polytope P_D. These lattice points correspond to Laurant monomials, yielding a map $X_\Sigma \longrightarrow \mathbb{P}(H^0(D))$. The divisor D is very ample if this map is an embedding, and projectively normal if $S = Sym(H^0(\mathcal{O}_X(D)))$ surjects onto $R = \bigoplus_{m\in\mathbb{Z}} H^0(mD)$.

Problem: The Eisenbud-Goto conjecture. Let $S = k[x_0, \ldots, x_n]$ and assume $char(k) = 0$. A main invariant of a projective variety $X \subseteq Proj(S)$ is the Castelnuovo-Mumford regularity, defined as the smallest j such that

$$H^i(\mathcal{O}_X(j-i)) = 0, \text{ for all } i \geqslant 1.$$

This condition may be rephrased as $Tor_i^S(S/I_X, k)_t = 0$ for all $t \geqslant i + j$ and $i \geqslant 1$. In [8], Eisenbud and Goto conjecture that if I_X is a prime ideal containing no linear form, then the regularity of S/I_X is bounded by $\deg(X) - \operatorname{codim}(X)$. This is true for curves [11] and smooth surfaces [14], but open in general. In the setting of toric varieties, results of Peeva-Sturmfels [20] show the conjecture holds in codimension two.

In [15], L'vovsky gives a combinatorial interpretation of the bound for monomial curves: regularity is bounded by the sum of the two largest gap sequences. This suggests that for general toric varieties, there might be a combinatorial proof of the conjecture. Two ingredients will probably be useful here. First, in [16], Maclagan-Smith define a multigraded version of regularity, which captures the finer structure in the toric case. Second, in [13], Hochster gives a formula for computing $Tor_i(S/I, k)_b$, where $b \in \mathbb{Z}^n$. This involves associating a simplicial complex Δ_b to I, such that:

$$Tor_i(I, k)_b \simeq \widetilde{H}^i(\Delta_b).$$

An explicit description of Δ_b may be found in [24].

Problem: Property N_p. If X_Σ is smooth and D is very ample, is the resulting embedding by the complete linear system projectively normal? The answer is yes if P_D has a unimodular cover [4] or is Frobenius split [19], but unknown in general. Examples [3] show that the smoothness hypothesis is necessary, even for threefolds (the case of surfaces is easy). Taking this question one step further, if the embedding is projectively normal, is the resulting ideal generated by quadrics (again, assuming smoothness)? These are the first steps in a more general construction. Suppose $S = Sym(H^0(\mathcal{O}_X(D)))$ surjects onto $\bigoplus_{m\in\mathbb{Z}} H^0(mD) = R$, and let $F_\bullet$ be a minimal free resolution of R over S.

Definition 1. *A very ample divisor is said to satisfy property N_p if $F_0 = S$, and $F_q \simeq \oplus S(-q-1)$ for all $q \in \{1, \ldots, p\}$.*

Thus, N_0 means projectively normal, N_1 means that the homogeneous ideal is generated by quadrics, N_2 means the minimal syzygies on the quadrics are linear, and so on. In [10], Green used Koszul cohomology to show that on a curve of genus g, if $\deg(D) \geqslant 2g + p + 1$ then D satisfies N_p.

In the toric setting, in [9], Ewald-Wessels show that if D is an ample divisor, then $(\dim(X) - 1)D$ satisfies property N_0; this is also established by Liu-Trotter-Ziegler, Ogata-Nakagawa, and Bruns-Gubeladze-Trung. Building on this, Hering-Schenck-Smith show in [12] that $(\dim(X)-1+p)D$ satisfies property N_p, and prove this bound is in general tight. With the additional assumption that $\dim(X) \geqslant 3$, Ogata [17] improves the bound by one. However, in general these bounds are far from optimal, and perhaps finer combinatorial data can be used to improve the picture. For example, [22] shows that a lattice polygon $P = H^0(\mathcal{O}_X(D))$ satisfies property N_p if $p \leqslant |\partial(P)| - 3$, where $|\partial(P)|$ is the number of lattice points in the boundary of P; while [21] uses the complex Δ_b discussed above to obtain results on N_p for Segre embeddings. To give an idea of just how far current knowledge is from being optimal, we close with a conjecture of Ottaviani-Paoletti [18], who show that on $\mathbb{P}^n$, $\mathcal{O}(d)$ does not satisfy N_{3d-2}, and they make

Conjecture 1. *The d-uple Veronese embedding of $\mathbb{P}^n$ satisfies N_{3d-3}.*

Problem: Implicitization of toric hypersurfaces. A central problem in geometric modeling is to find the implicit equations for a curve or surface defined by a regular or rational map. Suppose D is a divisor and $U \subseteq H^0(\mathcal{O}_X(D))$ is a basepoint free subspace with $\dim(U) = \dim(X) + 2$, such that the image of $X \subseteq \mathbb{P}(U)$ is a hypersurface. Standard methods for finding the implicit equation of X such as Gröbner bases or resultants tend to be quite slow. The best method uses an approximation complex $\mathcal{Z}$, pioneered by Busé-Jouanolou [5] and Busé-Chardin [6].

Two of the most commonly studied cases in geometric modelling are when $X = \mathbb{P}^2$ or $\mathbb{P}^1 \times \mathbb{P}^1$. Cox-Goldman-Zhang obtain results on these surfaces in [7], and Botbol-Dickenstein-Dohm explore toric surfaces in general in [2]. Botbol [1] studies higher dimensional torics. The method of approximation complexes stems from understanding the defining equations of a Rees algebra, and with suitable hypotheses on U, it can be shown that the determinant of the approximation complex $\mathcal{Z}$ is a power of the implicit equation for the hypersurface. However, in the toric setting, Botbol shows that the implicit equation is often given by a single minor of the first differential in the approximation complex. Knowing the first matrix in the approximation complex is exactly equivalent to knowing the module of first syzygies on the space of sections U, considered as an ideal in the Cox ring of X. In the very concrete case of four sections of $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(2, 1)$, [23] shows that there are only six possible numerical types of bigraded minimal free resolution, and that the implicit surface is singular along a union of lines exactly when I_U has

a linear first syzygy. It would be very interesting to have a better understanding of the multigraded syzygies of I_U and the geometry of the implicit hypersurface.

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The automorphism group of a complete rational variety with torus action of complexity one

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(joint work with Ivan Arzhantsev, Elaine Herppich, Alvaro Liendo)

Demazure [3] and Cox [2] investigated the automorphism group of a complete normal toric variety and gave a description of its roots in terms of the defining combinatorial data, i.e. the fan. Our aim is to extend these results to the more general case of a normal complete rational variety X coming with an effective torus action $T \times X \to X$ of complexity one, i.e. the dimension of T is one less than that of X.

Our approach to the automorphism group of X goes via the Cox ring $\mathcal{R}(X)$ which can be defined for any normal complete variety with finitely generated divisor class group $\mathrm{Cl}(X)$. The presence of the complexity one torus action implies that $\mathcal{R}(X)$ is of a quite special nature: generators, relations as well as the $\mathrm{Cl}(X)$-grading can be encoded in a sequence $A = a_0, \ldots, a_r$ of pairwise linearly independent vectors in $\mathbb{C}^2$ and an integral matrix

$$P \quad = \quad \begin{bmatrix} -l_0 & l_1 & \ldots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -l_0 & 0 & \ldots & l_r & 0 \\ d_0 & d_1 & \ldots & d_r & d' \end{bmatrix}$$

of size $(n+m) \times (r+s)$, where l_i are nonnegative integral vectors of length n_i, the d_i are $s \times n_i$ blocks, d' is an $s \times m$ block and the columns of P are pairwise different primitive vectors generating the column space $\mathbb{Q}^{r+s}$ as a convex cone. Conversely, the data A, P always define a Cox ring $\mathcal{R}(X) = R(A, P)$ of a complexity one T-variety X. The dimension of X equals $s+1$ and the acting torus T has $\mathbb{Z}^s$ as its character lattice. The matrix P determines the grading and the exponents occuring in the relations, whereas A is responsible for continuous aspects. For details, we refer to [4, 5].

The crucial notion for the investigation of the automorphism group $\mathrm{Aut}(X)$ are the *Demazure P-roots*; roughly speaking, these are finitely many integral linear forms u on $\mathbb{Z}^{r+s}$ satisfying a couple of linear inequalities on the columns of P. More precisely, denoting by $v_{ij}, v_k \in \mathbb{Z}^{r+s}$ the columns of P, we define them as follows.

(1) A *vertical Demazure P-root* is a tuple (u, k_0) with a linear form u on $\mathbb{Z}^{r+s}$ and an index $1 \leqslant k_0 \leqslant m$ satisfying

$$\begin{aligned} \langle u, v_{ij} \rangle &\geqslant 0 \qquad \text{for all } i, j, \\ \langle u, v_k \rangle &\geqslant 0 \qquad \text{for all } k \neq k_0, \\ \langle u, v_{k_0} \rangle &= -1. \end{aligned}$$

(2) A *horizontal Demazure P-root* is a tuple (u, i_0, i_1, C), where u is a linear form on $\mathbb{Z}^{r+s}$, $i_0 \neq i_1$ are indices with $0 \leqslant i_0, i_1 \leqslant r$, and $C = (c_0, \ldots, c_r)$

is a sequence with $1 \leqslant c_i \leqslant n_i$ such that

$$\begin{aligned}
l_{ic_i} &= 1 \qquad \text{for all } i \neq i_0, i_1,\\
\langle u, v_{ic_i}\rangle &= \begin{cases} 0, & i \neq i_0, i_1,\\ -1, & i = i_1,\end{cases}\\
\langle u, v_{ij}\rangle &\geqslant \begin{cases} l_{ij}, & i \neq i_0, i_1, \quad j \neq c_i,\\ 0, & i = i_0, i_1, \quad j \neq c_i,\\ 0, & i = i_0, \quad\ \ j = c_i,\end{cases}\\
\langle u, v_k\rangle &\geqslant 0 \qquad \text{for all } k.
\end{aligned}$$

In particular, if P is given explicitly, then the Demazure P-roots can be easily determined. Our first main result expresses the roots of the automorphism group $\mathrm{Aut}(X)$ in terms of the Demazure P-roots.

Theorem. *Let X be a nontoric normal complete rational variety with an effective torus action $T \times X \to X$ of complexity one. Then $\mathrm{Aut}(X)$ is a linear algebraic group having T as a maximal torus and the roots of $\mathrm{Aut}(X)$ with respect to T are precisely the $\mathbb{Z}^s$-parts of the Demazure P-roots.*

The basic idea of the proof is to relate the group $\mathrm{Aut}(X)$ to the group of graded automorphisms of the Cox ring. This is firstly done more generally in the more general setting of Mori dream spaces, i.e. normal complete varieties with a finitely generated Cox ring $\mathcal{R}(X)$. The grading by the divisor class group $\mathrm{Cl}(X)$ defines an action of the characteristic quasitorus $H_X = \operatorname{Spec}\mathbb{C}[\mathrm{Cl}(X)]$ on the total coordinate space $\overline{X} = \operatorname{Spec}\mathcal{R}(X)$ and X is the quotient of an open subset $\widehat{X} \subseteq \overline{X}$ by the action of H_X. The group of $\mathrm{Cl}(X)$-graded automorphisms of $\mathcal{R}(X)$ is isomorphic to the group $\mathrm{Aut}(\overline{X}, H_X)$ of H_X-equivariant automorphisms. Moreover, the group $\mathrm{Bir}_2(X)$ of birational automorphisms of X defined on an open subset of X having complement of codimension at least two plays a role.

Theorem. *Let X be a Mori dream space. Then there is a commutative diagram of morphisms of linear algebraic groups where the rows are exact sequences and the upwards inclusions are of finite index:*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & H_X & \longrightarrow & \mathrm{Aut}(\overline{X}, H_X) & \longrightarrow & \mathrm{Bir}_2(X) & \longrightarrow & 1\\
 & & \| & & \uparrow & & \uparrow & & \\
1 & \longrightarrow & H_X & \longrightarrow & \mathrm{Aut}(\widehat{X}, H_X) & \longrightarrow & \mathrm{Aut}(X) & \longrightarrow & 1
\end{array}$$

This means in particular that the unit component of $\mathrm{Aut}(X)$ coincides with that of $\mathrm{Bir}_2(X)$ which in turn is determined by $\mathrm{Aut}(\overline{X}, H_X)$, the group of graded automorphisms of the Cox ring. Coming back to rational varieties X with torus action of complexity one, the task then is a detailed study of the graded automorphism group of the rings $\mathcal{R}(X) = R(A, P)$. The key result is a purely algebraic description of the "primitive homogeneous locally nilpotent derivations" on $R(A, P)$. The

proof of the main theorem then relates the Demazure P-roots via these derivations to the roots of the automorphism group $\mathrm{Aut}(X)$.

In the second main result we describe the structure of the semisimple part $\mathrm{Aut}(X)^{\mathrm{ss}} \subseteq \mathrm{Aut}(X)$ of the automorphism group; recall that the semisimple part of a linear algebraic group is a maximal connected semisimple subgroup. We call the $\mathbb{Z}^s$-part α of a Demazure P-root *a semisimple P-root* if also $-\alpha$ is the $\mathbb{Z}^s$-part of a Demazure P-root and we denote the set of semisimple P-roots by Φ_P.

Theorem. *Let X be a nontoric normal complete rational variety with an effective torus action $T \times X \to X$ of complexity one. Then Φ_P is the root system of the semisimple part $\mathrm{Aut}(X)^{\mathrm{ss}} \subseteq \mathrm{Aut}(X)$ and we have a splitting $\Phi_P = \Phi_P^{\mathrm{vert}} \oplus \Phi_P^{\mathrm{hor}}$ with*

$$\Phi_P^{\mathrm{vert}} \;=\; \bigoplus_{\mathrm{Cl}(X)} A_{m_D-1}, \qquad\qquad \Phi_P^{\mathrm{hor}} \;\in\; \{\varnothing, A_1, A_2, A_3, A_1 \oplus A_1, B_2\},$$

where m_D is the number of invariant prime divisors in X with infinite T-isotropy that represent a given class $D \in \mathrm{Cl}(X)$. The numbers m_D as well as the possibilities for Φ_P^{hor} can be read off from the defining matrix P.

The last theorem implies, for example, that for every nontoric Mori dream surface the group $\mathrm{Aut}(X)^0$ is solvable. We apply the results moreover to almost homogeneous rational $\mathbb{C}^*$-surfaces X of Picard number one, where almost homogeneous means that $\mathrm{Aut}(X)$ has an open orbit in X. It turns out that these surfaces are always (possibly singular) del Pezzo surfaces and, up to isomorphy, there are countably many of them. In the case that X is log terminal with only one singularity, we give classifications for fixed Gorenstein index. We consider moreover varieties of dimension three that are almost homogeneous under the action of a reductive group and additionally admit an effective action of a twodimensional torus. We explicitly describe their Cox rings and list all those having a reductive automorphism group.

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Kähler-Einstein metrics on symmetric Fano T-varieties

Hendrik Süss

A Kähler metric on a complex manifold is called *Kähler-Einstein* if the corresponding Kähler form and the Ricci form of the manifold differ only by an constant factor. In this case the canonical class has to be ample, trivial or anti-ample. For the first two cases the existence and uniqueness of those metrics are important results of Aubin [1] and Yau [2]. Hence, in the case of a Calabi-Yau variety or a variety of general type we have a canonical metric associated to the corresponding complex manifold. In the case of a Fano variety the situation is much more complicated. There are examples of Fano varieties admitting such a metric and of those which do not. There are either sufficient or necessary criteria to test a variety for the existence of a Kähler-Einstein metric, but up to now no algebraic characterization of this property exists, although it is expected that it corresponds to some notion of stability [3, 4]. One sufficient criterion was given by Tian [3] in terms of so called α-invariants of the manifold X, which are known as log-canonical thresholds in the algebraic setting. For toric Fano varieties Batyrev and Selivanova developed in [5] the notion of *symmetric* toric varieties and proved the existence of Kähler-Einstein metrics on them, by using Tian's criterion. The aim of this work in progress is to generalize their result to a more general class of varieties.

Assume that a torus T of dimension acts effectively on a complete variety X. The set of points with disconnected stabilizers on X is called the *exceptional set* of the torus actions. The group of T-equivariant automorphisms $\mathrm{Aut}_T(X)$ acts naturally on the character lattice $M = \mathcal{X}(T)$. If there is no fixed point of this action except from the origin, then X is called *symmetric* with respect to the torus action. If $\dim(X) = \dim(d) - 1$ then the torus action is called to be of *complexity one*. In this case a Fano variety X has to be rational and one obtains a rational quotient map $\pi : X \dashrightarrow X/T = \mathbf{P}^1$, which is defined outside of a subset of codimension at least two. Moreover, the equivariant automorphisms descend to automorphisms of the quotient, hence there is also a natural $\mathrm{Aut}_T(X)$-action on $\mathbf{P}^1$. We are now able to state the main result.

Theorem. *Let X be a symmetric complexity-one Fano variety. If one of the following conditions is fulfilled:*

(1) *The exceptional set contains three connected components,*
(2) *the exceptional set contains two connected components and these are swapped by an element of* $\mathrm{Aut}_T(X)$*,*
(3) $\mathrm{Aut}_T(X)$ *acts without fixed points on* $\mathbf{P}^1$*,*

then X is Kähler-Einstein.

This is a generalization of the theorem by Batyrev and Selivanova. The result is obtained by calculating log-canonical thresholds using the description of algebraic torus actions via polyhedral divisors, given in [6].

Example. *Consider the quadric threefold $X = V(x_1y_1 + x_2y_2 + z^2) \subset \mathbf{P}^4$. The change of variables $x_i \mapsto y_i$ induces an involution of X. The induced automorphism of the character lattice of the acting torus is given by $u \mapsto -u$ and fixes only the trivial character. Although Q is Kähler-Einstein we are not able to apply the theorem, since the exceptional set consists only of one component, which is an open subset of $V(z) \cap X$. Nevertheless, blowing up the two torus invariant conics $V(x_i, y_i) \cap X$, $i = 1, 2$ gives two additional exceptional components with generic stabilizer $\mathbf{Z}/2\mathbf{Z}$. Moreover, the conics are obviously invariant under the involution, hence the involution lifts to an automorphism of the blowup. Now, we are in case (1) of the theorem and infer the existence of a Kähler-Einstein metric on X.*

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(Non-)Displaceable Lagrangian Toric Fibers

Miguel Abreu

(joint work with Matthew Strom Borman, Leonardo Macarini, Dusa McDuff)

1. Introduction

Let (M, ω) be a $2n$-dimensional toric symplectic manifold, with moment map $\mu : M \to \mathbb{R}^n$ and moment polytope $\Delta := \mu(M) \subset \mathbb{R}^n$ defined by

$$x \in \Delta \Leftrightarrow \ell_i(x) := \langle x, \nu_i \rangle + a_i \geqslant 0\,, \;\; i = 1, \ldots, d,$$

where d is the number of facets of Δ, each vector $\nu_i \in \mathbb{Z}^n$ is the primitive integral interior normal to the facet F_i of Δ and the a_i's are real numbers that determine $[\omega] \in H^2(M; \mathbb{R})$. Recall that (M, ω) is monotone, i.e. $[\omega] = \lambda(2\pi c_1(\omega)) \in H^2(M; \mathbb{R})$ with $\lambda \in \mathbb{R}^+$, iff $\Delta \subset \mathbb{R}^n$ can be defined as above with $a_1 = \cdots = a_d = \lambda$. Such a Δ will be called a monotone polytope and, in this case, $0 \in \Delta$.

For every $u \in \breve{\Delta} := \text{interior}(\Delta)$, we have that the fiber $T_u := \mu^{-1}(u)$ is a Lagrangian torus orbit in (M, ω). When Δ is monotone T_0 is called the centered, special or monotone torus fiber.

A natural symplectic topology question is which of these Lagrangian tori can be displaced by a Hamiltonian isotopy, i.e.

$$\text{when does there exist } \phi \in \mathrm{Ham}(M,\omega) \text{ such that } \phi(T_u) \cap T_u = \varnothing?$$

Regarding this question, I report on recent joint work with Leonardo Macarini [1] on non-displaceability and with Matthew Strom Borman and Dusa McDuff [2] on displaceability.

2. Symplectic reduction and non-displaceability

In [1] we remark how one can use simple symplectic reduction and cartesian product considerations to go from basic non-displaceability examples to much more sophisticated ones. Let us describe the symplectic reduction consideration and one of its applications.

Let $(\tilde{M},\tilde{\omega})$ be a toric symplectic manifold of dimension $2N$ with $\tilde{\mathbb{T}}$-action generated by a moment map $\tilde{\mu} : \tilde{M} \to \tilde{\Delta} \subset (\mathbb{R}^N)^*$. As before, given $u \in \mathrm{int}(\tilde{\Delta})$, let $\tilde{T}_u := \tilde{\mu}^{-1}(u)$ denote the corresponding $\tilde{\mathbb{T}}$-orbit, a Lagrangian torus in $\tilde{M}$. Let $K \subset \tilde{\mathbb{T}}$ be a subtorus of dimension $N-n$ determined by an inclusion of Lie algebras $\iota : \mathbb{R}^{N-n} \to \mathbb{R}^N$. The moment map for the induced action of K on $\tilde{M}$ is $\tilde{\mu}_K = \iota^* \circ \tilde{\mu} : \tilde{M} \to (\mathbb{R}^{N-n})^*$. Let $c \in \tilde{\mu}_K(\tilde{M}) \subset (\mathbb{R}^{N-n})^*$ be a regular value and assume that K acts freely on the level set $Z := \tilde{\mu}_K^{-1}(c) \subset \tilde{M}$. Then, the reduced space $(M := Z/K, \omega)$ is a toric symplectic manifold of dimension $2n$ with $\mathbb{T} := \tilde{\mathbb{T}}/K$-action generated by a moment map $\mu : M \to \Delta \subset (\mathbb{R}^n)^* \cong \ker(\iota^*)$ that fits in the commutative diagram

$$\begin{array}{ccc} \tilde{M} \supset Z & \xrightarrow{\tilde{\mu}} & \tilde{\Delta} \subset (\mathbb{R}^N)^* \\ {\scriptstyle\pi}\downarrow & & \uparrow \\ M & \xrightarrow{\mu} & \Delta \subset (\mathbb{R}^n)^* \end{array}$$

where π is the quotient projection and the vertical arrow on the right is the inclusion $(\mathbb{R}^n)^* \cong \ker(\iota^*) \subset (\mathbb{R}^{N-n})^*$. Note that given $T_u := \mu^{-1}(u)$, with $u \in \mathrm{int}(\Delta) \subset \mathrm{int}(\tilde{\Delta})$, we have that $\pi^{-1}(T_u) = \tilde{T}_u$.

Proposition 2.1. *Let $\psi \in \mathrm{Ham}(M,\omega)$. Then there is $\tilde{\psi} \in \mathrm{Ham}(\tilde{M},\tilde{\omega})$ such that $\tilde{\psi}(Z) = Z$ and $\pi(\tilde{\psi}(\tilde{p})) = \psi(\pi(\tilde{p}))\,,\ \forall\, \tilde{p} \in Z$. Moreover, if $\psi(T_u) \cap T_u = \varnothing$ then $\tilde{\psi}(\tilde{T}_u) \cap \tilde{T}_u = \varnothing$. Hence, if $\tilde{T}_u \subset (\tilde{M},\tilde{\omega})$ is non-displaceable, then $T_u \subset (M,\omega)$ is also non-displaceable.*

Remark 2.2. *This idea of using symplectic reduction to prove intersection properties of Lagrangian submanifolds was used by Tamarkin in* [13]*. It is also present in the work of Borman* [4] *on reduction properties of quasi-morphisms and quasi-states (see also* [5]*).*

To apply this proposition, consider the most basic non-displaceability result in this context.

Theorem 2.3 ([3, 6])**.** *For $(\mathbb{C}\mathrm{P}^n, \omega_{FS})$ we have that T_0 = Clifford torus is non-displaceable.*

The following result follows by straightforward application of Proposition 2.1.

Theorem 2.4. *Let (M^{2n}, ω) be a monotone toric symplectic manifold and let $\nu_1, \ldots, \nu_d \in \mathbb{Z}^n$ denote the primitive integral interior normals to the facets of its monotone polytope $P \subset (\mathbb{R}^n)^*$. If $\sum_{i=1}^d \nu_i = 0$ then the special torus fiber T_0 is non-displaceable.*

Proof. Since $\sum_{i=1}^d \nu_i = 0$ we have that M can be obtained as a symplectic reduction of $\mathbb{C}\mathrm{P}^{d-1}$. The monotonicity condition implies that this reduction goes through the Clifford torus. □

Remark 2.5. *The zero-sum condition can be removed by using weighted projective spaces (with work of Woodward* [14] *and forthcoming work of Cho and Poddar* [8]*), obtaining the general result in monotone case originally due to Entov-Polterovich* [9]*, Cho* [7] *and Fukaya-Oh-Ohta-Ono* [10]*.*

Remark 2.6. *With the help of another basic example, i.e. the total space of the line bundle $\mathcal{O}(-1) \to \mathbb{C}\mathrm{P}^1$, and using also a simple cartesian product consideration, one can also prove interesting non-displaceability results on certain non-monotone and/or non-Fano examples, such as:*

- *a continuum of non-displaceable torus fibers on $M = \mathbb{C}\mathrm{P}^2 \sharp 2\overline{\mathbb{C}\mathrm{P}}^2$ with a certain non-monotone symplectic form (cf. Example 10.3 in* [11]*).*
- *a particular non-displaceable torus fiber on the family of non-Fano examples given by Hirzebruch surfaces $H_k := \mathrm{P}(\mathcal{O}(-k) \oplus \mathbb{C}) \to \mathbb{C}\mathrm{P}^1$, with $2 \leqslant k \in \mathbb{N}$ (cf. Example 10.1 in* [11]*).*

3. Displaceability via extended probes

In [2] we introduce the method of extended probes, which is a way of displacing Lagrangian torus fibers in toric symplectic manifolds. This is a generalization of McDuff's original method of probes [12]. Let us briefly recall the method of probes, describe the simplest extended probes and present some applications.

Definition 3.1. *A probe P in a moment polytope $\Delta = \mu(M) \subset \mathbb{R}^n$ is a line segment in Δ based at $b_P \in \breve{F}_P$ = interior of a facet $F_P \subset \Delta$, with primitive direction vector $v_P \in \mathbb{Z}^n$ such that $\langle \nu_{F_P}, v_P \rangle = 1$. The length $\ell(P)$ of P is defined as $\ell(P) := \max\{t \in \mathbb{R}^+ \,:\, b_P + t\, v_p \in P \subset \Delta\}$. A probe P is symmetric if it exits Δ at an interior point of a facet F'_P with $\langle \nu_{F'_P}, v_P \rangle = -1$.*

Theorem 3.2 ([12])**.** *If $u = b_P + t\, v_p$ with $t < \ell(P)/2$, then the Lagrangian torus fiber $T_u = \mu^{-1}(u)$ is displaceable. Such Lagrangian torus fibers are said to be displaceable by probes.*

Remark 3.3. *When P is symmetric this displaceability result is an immediate corollary of Proposition 2.1.*

As one of the applications of this theorem, McDuff shows that for monotone toric symplectic manifolds of dimension less than or equal to six, the only Lagrangian torus fiber that cannot be displaced by probes is the special centered torus fiber T_0. Hence, in these examples the method of probes complements perfectly the known non-displaceability results.

The simplest examples where that is not the case are odd Hirzebruch surfaces H_{2k+1} for $k \geqslant 1$. where there is only one known non-displaceable fiber and a continuum of fibers that cannot be displaced by the standard probes of Theorem 3.2. One can improve on this using the simplest extended probes introduced in [2], namely symmetric extended probes.

Definition 3.4. *A symmetric extended probe $\mathcal{P}$ is formed by deflecting a probe P with a symmetric probe Q (Figure 1): $\mathcal{P} = P \cup Q \cup P' \subset \Delta$. The length $\ell(\mathcal{P})$ of $\mathcal{P}$ is defined as the sum $\ell(\mathcal{P}) := \ell(P) + \ell(P')$.*

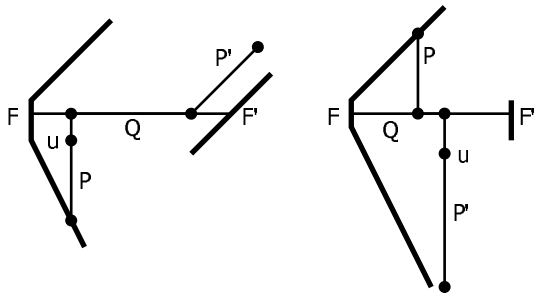

FIGURE 1. Two ways of using symmetric extended probes.

Theorem 3.5. *If*

(i) $u = b_p + tv_P \in \breve{P}$ *with* $t < \ell(\mathcal{P})/2$, *or*

(ii) $u = x' + tv_{P'} \in \breve{P}'$ *with* $\ell(P) + t < \ell(\mathcal{P})$,

then the Lagrangian torus fiber $T_u = \mu^{-1}(u)$ is displaceable (Figure 1).

This theorem can be used to prove that, in fact, H_{2k+1} has only one non-displaceable torus fiber when $k \geqslant 1$.

In [2] we consider more general notions of extended probes and use them to study displaceability of Lagrangian torus fibers on several 4-dimensional toric symplectic manifolds and orbifolds, both compact and non-compact. Although we improve quite a bit on the previously known results, in general these displaceability methods are still far from being able to perfectly complement the known non-displaceability results.

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Multiplicity free Hamiltonian manifolds

Friedrich Knop

In 1988, Thomas Delzant, [2], proved his celebrated theorem which lies at the basis of much of this conference:

Theorem. *Let T be a torus and M a compact connected Hamiltonian T-manifold with moment map $m : M \to \mathfrak{t}^*$. Assume that the T-action on M is effective and that $\dim M = 2 \dim T$. Then M is uniquely determined by its momentum image $\mathcal{P}_M := m(M) \subseteq \mathfrak{t}$. Moreover, given $\mathcal{P} \subseteq \mathfrak{t}$ there exists M with $\mathcal{P} = \mathcal{P}_M$ if and only if $\mathcal{P}$ is an $\mathcal{X}(T)$-simple convex polytope.*

(A polytope is called $\mathcal{X}(T)$-*simple*, if for all vertices $a \in \mathcal{P}$, the cone $\mathbb{R}_{\geqslant 0}(\mathcal{P} - a)$ is generated by an integral basis of the character group $\mathcal{X}(T)$.)

In the talk, we presented a generalization of Delzant's theorem to non-commutative Lie groups and even loop groups.

Let K be a compact, connected Lie group and M a connected compact Hamiltonian K-manifold with moment map $m : M \to \mathfrak{k}^*$. Let $\mathfrak{t} \subseteq \mathfrak{k}$ be a Cartan subalgebra and $\mathfrak{t}_+ \subseteq \mathfrak{t}^*$ a Weyl chamber. Then M is called *multiplicity free* if $m/K : M/K \to \mathfrak{k}^*/K$ is injective. Its *momentum polytope* is $\mathcal{P}_M := m(M) \cap \mathfrak{t}_+$ which is indeed a convex polytope by Kirwan, [4]. The *principal isotropy group* S_M is the isotropy group K_x of a generic point $x \in m^{-1}(\mathfrak{t}_+)$.

The non-commutative generalization of the uniqueness part of Delzant's theorem is:

Theorem (Knop [5]). *Let K be a compact connected Lie group and M a compact connected multiplicity free Hamiltonian manifold. Then M is uniquely determined by its momentum polytope $\mathcal{P}_M$ and its principal isotropy group S_M.*

This theorem has been conjectured by Delzant in [3] after having it verified for tori and groups of rank at most two.

The proof proceeds in two steps, a local part and a local-to-global part. For $x \in M$ let $m_+(x) \in \mathfrak{t}_+$ be the unique intersection of the coadjoint orbit $\operatorname{Ad} K\, m(x)$ with $\mathfrak{t}_+$. Then m_+ is a continuous map from M to $\mathfrak{t}_+$ with image $\mathcal{P}_M$. The local part is the statement that every $a \in \mathcal{P}_M$ has a neighborhood U such that U, $\mathcal{P}_M$ and S_M determine $m_+^{-1}(U)$ uniquely. This assertion can be shown to be equivalent to a purely invariant theoretic statement (the former "Knop conjecture"):

Theorem (Losev [7]). *Let G be a connected reductive complex algebraic group and X a smooth affine spherical G-variety. Then X is uniquely determined by the set of highest weights occurring in the coordinate ring $\mathbb{C}[X]$.*

The local-to-global argument is the following statement:

Theorem (Knop [5]). *For every open subset U of $\mathcal{P}$ define*

$$\mathcal{A}_M(U) := Aut(m_+^{-1}(U))$$

(automorphisms of Hamiltonian manifolds). Then $\mathcal{A}_M$ is a sheaf of abelian groups on $\mathcal{P}$ with

$$H^i(\mathcal{P}, \mathcal{A}_M) = 0 \text{ for all } i \geqslant 1.$$

The case $i = 1$ in conjunction with Losev's theorem implies Delzant's conjecture. Vanishing for $i = 2$ implies a generalization of the existence part of Delzant's theorem, namely a characterization of those pairs $(\mathcal{P}_M, S_M)$ which actually occur. The answer is in terms of smooth affine spherical varieties, see [5], Thm. 11.2.

Using similar methods, one can prove an extension of Delzant's conjecture to loop groups. For this assume K to be *simply connected*.

Let $\mathcal{L}K$ be the group of K-valued loops and let $\widehat{\mathcal{L}}K$ be its central extension. Let $\widehat{\mathcal{L}}_1\mathfrak{k}^* \subset \widehat{\mathcal{L}}\mathfrak{k}^*$ be the hyperplane of elements with central charge equal to 1. It can be identified with the space of K-invariant connections on the trivial bundle $K \times S^1 \to S^1$. Then Meinrenken-Woodward, [8], define the notion of a *level-1-Hamiltonian $\mathcal{L}K$-manifold* whose moment map has values in $\widehat{\mathcal{L}}_1\mathfrak{k}^*$. There is a finite dimensional version of this theory due to Alexeev-Malkin-Meinrenken, [1], the so called *K-valued moment maps* or *quasi-Hamiltonian K-manifolds*.

Let $\mathfrak{A} \subseteq \mathfrak{t}_+$ be the fundamental alcove.

Theorem (Knop). *Multiplicity free level-1-Hamiltonian $\mathcal{L}K$-manifolds (or, equivalently, quasi-Hamiltonian K-manifolds) are classified by $\mathcal{L}$-simple pairs $(\mathcal{P}, \Lambda)$ where $\mathcal{P} \subseteq \mathfrak{A}$ is a convex polytope and Λ is a discrete subgroup of $\mathfrak{t}^*$.*

Here, *$\mathcal{L}$-simplicity* is a certain explicit condition on $(\mathcal{P}, \Lambda)$ which is local on $\mathcal{P}$ (like $\mathcal{X}(T)$-simplicity above) and which involves knowledge of all smooth affine spherical varieties. The latter have been classified by Knop-Van Steirteghem

[6]. This way, it is possible to construct a plethora of new examples of quasi-Hamiltonian manifolds.

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Toric geometry and the Semple-Nash modification

Bernard Teissier

(joint work with Pedro Daniel González Pérez)

The Semple-Nash modification of an algebraic variety over a field of characteristic zero is a birational map $\nu\colon NX \to X$ such that $\nu^*\Omega^1_X$ has a locally free quotient of rank $d = \dim X$ and which is minimal for this property. It is unique up to unique X-isomorphism. Geometrically the set-theoretic fiber $|\nu^{-1}(x)|$ is the set of limit positions at x of tangent spaces to X at non singular points tending to x. In [6] Semple asks whether iterating this operation leads to a non singular model of X. The same question was asked by John Nash in the late 1960's. Nobile proved in [5] that the map ν is an isomorphism if and only if X is non singular, so that to answer positively Semple's question it suffices to prove that for some k we have $N^{k+1}X = N^kX$. The case of dimension one is easily settled positively, and Gérard Gonzalez Sprinberg proved in [4] that iterating the operation of *normalized* Semple-Nash modification (i.e., at each step, the modification followed by normalization) eventually resolves singularities of 2-dimensional normal toric varieties. This eventually led to the best result on this problem to this day, which is due to Spivakovsky (see [7]): iterating the normalized Semple-Nash modification eventually resolves singularities of normal surfaces over an algebraically closed field.

Our basic idea is that in higher dimensions it should be easier in the toric case to deal with the Semple-Nash modification without normalization. Gonzalez Sprinberg showed that the Semple-Nash modification, in characteristic zero, is isomorphic with the blowing up of the *logarithmic jacobian ideal*, which is an equivariant sheaf of ideals defined as follows:

Let k be a field and let Γ be a finitely generated subgroup of a lattice $M \simeq \mathbf{Z}^d$. Denote by $(\gamma_i)_{i\in F}$ a set of generators. The logarithmic jacobian ideal is the monomial ideal of the semigroup algebra $k[t^\Gamma]$ generated by the elements $t^{\gamma_{i_1}+\cdots+\gamma_{i_d}}$ for all $(i_1,\ldots,i_d)\subset F$ such that $\gamma_{i_1}\wedge\ldots\wedge\gamma_{i_d}\neq 0$ in $M_{\mathbf{R}}$.

Since we wanted to describe precisely blowing-ups (and not only normalized blowing-ups) and modifications we wrote down the basic theory of toric varieties without any assumption of normality or of projectivity. However Sumihiro's theorem (see [8]) on the existence of a covering by invariant affine varieties of a variety on which a torus of the same dimension acts with a dense orbit fails without the assumption of normality, and we have to set the existence of such a covering as part of the definition of a toric variety. Then an abstract toric variety has a combinatorial description: it corresponds to certain semigroups in the convex duals of the cones of a fan, which satisfy a natural gluing-up condition. This generalizes the definition of [2] which concerns toric varieties equivariantly embedded in projective space.

Given a rational strictly convex cone $\sigma\in N_{\mathbf{R}}$ and a finitely generated semigroup Γ, such that $\mathbf{Z}\Gamma = M$ and $\mathbf{R}_{\geqslant 0}\Gamma = \check{\sigma}\subset M_{\mathbf{R}}$ one has the affine toric variety $\mathrm{Spec}k[t^\Gamma]$ with its torus $T^M = \mathrm{Spec}k[t^M]$. For a face $\tau<\sigma$, the lattice $M(\tau;\Gamma)$ spanned by $\Gamma\cap\tau^\perp$ is in general a sublattice of finite index in $M(\tau)=M\cap\tau^\perp$. This corresponds to the fact that the normalization map is equivariant and its restriction to an orbit of the normalization $\mathrm{Spec}k[t^{\check{\sigma}\cap M}]$ may not be one to one. The variety $T^{\Gamma\cap\tau^\perp}=\mathrm{Spec}k[t^{\Gamma\cap\tau^\perp}]$ with torus $T^{M(\tau;\Gamma)}$ is an orbit closure in $\mathrm{Spec}k[t^\Gamma]$.

The semigroup $\Gamma_\tau=\Gamma+M(\tau;\Gamma)$ is again a finitely generated semigroup, generating the group M and with the property that the cone $\mathbf{R}_{\geqslant 0}\Gamma_\tau$ which it generates in $M_{\mathbf{R}}$ is equal to $\check{\tau}$.

A toric variety is a triple $(N,\Sigma,\mathbf{\Gamma})$ consisting of a lattice N, a fan Σ in $N_{\mathbf{R}}$ and a family of subsemigroups of the dual lattice M of N,

$$\mathbf{\Gamma}=(\Gamma_\sigma)_{\sigma\in\Sigma}$$

such that:

- The group $\mathbf{Z}\Gamma_\sigma$ generated by Γ_σ is equal to M for all $\sigma\in\Sigma$.
- The cone $\mathbf{R}_{\geqslant 0}\Gamma_\sigma$ generated by Γ_σ is equal to $\check{\sigma}$ for all $\sigma\in\Sigma$.
- Far all $\sigma\in\Sigma$ and $\tau<\sigma$, we have $\Gamma_\tau=\Gamma_\sigma+M(\tau;\Gamma_\sigma)$.

The toric variety $T^{\mathbf{\Gamma}}_\Sigma=\bigcup_{\sigma\in\Sigma}T^{\Gamma_\sigma}$ is then obtained by gluing up the affine toric varieties T^{Γ_σ} as in the normal case.

Now if we define an abstract toric variety as an irreducible separated algebraic variety X over k equipped with an algebraic action of a torus $T^M\subset X$ of the same dimension extending the action of the torus on itself by translation and covered by finitely many open affine invariant subsets, then we have:

Theorem 1. (See [3]) *An abstract toric variety is equivariantly isomorphic to a variety* $T^{\mathbf{\Gamma}}_\Sigma$.

In fact, with the natural definitions of toric morphisms, this sets up an equivalence of categories, see [3] for details.

The logarithmic jacobian ideals sheafify into an equivariant sheaf of ideals on $T^{\mathbf{\Gamma}}_\Sigma$.

Given a monomial ideal $I = (t^{n_1}, \dots, t^{n_s})$ in $k[t^\Gamma]$, it determines a piecewise linear function $\mathrm{ord}_I \colon \sigma \to \mathbf{R}$ by $\nu \mapsto \min_{m \in I} \langle \nu, m \rangle$. The cones of linearity σ_i correspond to the vertices $(m_1, \dots, m_k)$ in $\check{\sigma}$ of the Newton polyhedron of I and they are the cones of maximal dimension of a fan Σ_I with support $|\sigma|$. If to each cone $\sigma_i \subset \sigma$ we attach the semigroup $\Gamma_i = \Gamma + \langle (n_j - m_i)_{j=1,\dots,s} \rangle \subset \check{\sigma}_i \cap M$ and attach to the faces of the σ_i the semigroups corresponding to the gluing up rules explained above, we see that we have a triple $(N, \Sigma_I, \mathbf{\Gamma}_I)$ corresponding to a toric variety $T^{\mathbf{\Gamma}_I}_{\Sigma_I}$ endowed with an equivariant proper map to T^Γ. It is the blowing-up of the ideal I. This construction globalizes into the blowing up of a sheaf of equivariant ideals on a toric variety, and we can apply it to the sheaf of logarithmic jacobian ideals to obtain an analogue of the Semple-Nash modification, which is defined over any field.

The analogue in the toric case, in any characteristic, of Nobile's result is an easy combinatorial lemma: the blowing up of the logarithmic jacobian ideal is an isomorphism if and only if the toric variety $T^{\mathbf{\Gamma}}_\Sigma$ is non singular.
So iterating the logarithmic jacobian blowing up of a toric variety $T^{\mathbf{\Gamma}}_\Sigma$ produces a sequence of refinements $\Sigma^{(j)}$ of the fan Σ, with attached systems of semigroups $\mathbf{\Gamma}^{(j)}$. The goal is to prove that they stabilize: the refinement stops only when the space is non singular, as we have seen.

Ewald and Ishida have introduced in [1] the analogue in toric geometry of the Zariski-Riemann manifold in algebraic geometry: it is the space $ZR(M)$ of (additive) preorderings $\leqslant$ on M in the following sense:

- $\forall\ m, n \in M,\ m \leqslant n$ or $n \leqslant m$.
- $m \leqslant n$ and $n \leqslant p$ imply $m \leqslant p$.
- $m \leqslant n$ implies $m + p \leqslant n + p,\ \forall p \in M$,

endowed with the topology defined by the basis of open sets

$$\mathcal{U}(\theta) = \{\nu \in ZR(M) / \check{\theta} \subset L(\nu)\},$$

where θ is a rational convex cone of $N_{\mathbf{R}}$ and $L(\nu)$ is the semigroup of elements of M which are $\geqslant_\nu 0$, or non negative for the preorder ν.

With this topology, the space $ZR(M)$ is quasi compact, and it behaves like a space of valuations. In particular the preorders defined by vectors ν of the dual lattice N of M, as $m \leqslant_\nu n \iff \langle m, \nu \rangle \leqslant \langle m, \nu \rangle$ correspond to divisorial valuations. A preorder $\nu \in ZR(M)$ picks a cone in each refinement $\Sigma^{(j)}$ of Σ, and to prove that the sequence of toric varieties $(N, \Sigma^{(j)}, \mathbf{\Gamma}^{(j)})$ stabilizes, it suffices to check that for every preorder it stabilizes along the sequence of cones picked by that preorder. Then we have the following partial result:

Theorem 2. (See [3]) *The sequence* $(N, \Sigma^{(j)}, \mathbf{\Gamma}^{(j)})$ *stabilizes for any order that is lexicographic with respect to some basis of* N. *In particular it stabilizes for the "divisorial" preorders associated to vectors of* N.

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Complexity one Hamiltonian manifolds

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(joint work with Yael Karshon)

Let M be a a compact, connected n-dimensional manifold, equipped with a symplectic form $\omega \in \Omega^2(M)$. Let a torus $T \simeq (S^1)^k$ act (faithfully) on M, preserving ω. Assume that there exists a moment map $\Phi\colon M \to \mathfrak{t}^*$, where $\mathfrak{t}^*$ is the dual of the Lie algebra of T. Then the moment image $\Phi(M)$ is a convex polytope. Moreover, for any regular value $a \in M$, the **reduced space** $M_{\text{red}} := \Phi^{-1}(a)/T$ is a $2(n-k)$-dimensional symplectic orbifold. In particular, $k \leqslant n$.

If $k = n$, then (M, ω, Φ) is a **symplectic toric manifold**; these are classified.

Theorem (Delzant)**.** *The map $M \mapsto \Phi(M)$ induces a one-to-one correspondence between symplectic toric manifolds and a certain class of convex polytopes, called "Delzant polytopes."*

The next case is when $k = n - 1$. In this case, we say that (M, ω, Φ) is a **complexity one space**; moreover, M is **tall** if the the preimage $\Phi^{-1}(a)$ is not a single orbit for any $a \in \mathfrak{t}^*$. Our goal is to classify tall complexity one spaces (up to equivariant symplectomorphisms that preserve the moment map).

Note: The case $n = 2$ is due to Karshon [1], while in the algebraic case the classification was completed by Timashëv [4]

We will begin by giving several examples of complexity one spaces.

Example 1. *Given a non-negative integer g, let Σ_g be a Riemann surface of genus g. Let $M = S^2 \times \Sigma_g$, with the product symplectic structure. Finally, let S^1 act by rotations on the first factor.*

Example 2. *Suppose that $(S^1)^n$ acts on (M, ω) with moment map $J\colon M \to (\mathbb{R}^n)^*$, so that (M, ω, J) is a symplectic toric manifold. Then $(M, \omega, p \circ J)$ is a complexity one space for any rational projection $p\colon (\mathbb{R}^n)^* \to \mathfrak{t}^*$. Moreover, it is tall exactly if $p^{-1}(a) \cap J(M)$ is a segment (and not a point) for all $a \in \Phi(M)$. We will particularly focus on two examples:*

(a) *Consider the* 6*-dimensional symplectic toric manifold with*

$$J(M) = \left\{(x, y, z) \in [-3, 3] \times [-2, 2] \times [1, 4] \;\middle|\; |x| \leqslant z \text{ and } |y| \leqslant z\right\},$$

and define $p\colon \mathbb{R}^3 \to \mathbb{R}^2$ *by* $p(x, y, z) = (x, y)$*. The moment image and critical values for* $\Psi = p \circ J$ *appear in Figure 1.*

(b) *The moment image and critical values for an analogous* 8*-dimensional example appear in Figure 2.*

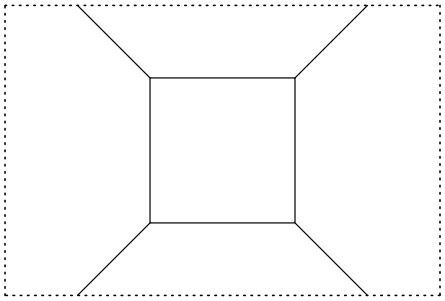

FIGURE 1. Moment image and critical values for Example 2a

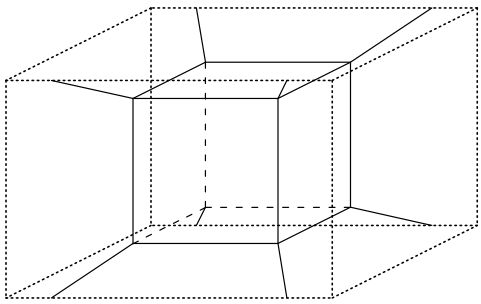

FIGURE 2. Moment image and critical values for Example 2b

We will now describe the invariants of tall complexity one spaces.

The Liouville measure on M is given by integrating the volume form $\omega^n/n!$ with respect to the symplectic orientation; the **Duistermaat-Heckman measure** assigns the value $\int_{\Phi^{-1}(A)} \omega^n/n!$ to a subset $A \subset \mathfrak{t}^*$. Note that the Duistermaat-Heckman measure determines the moment image $\Phi(M)$, because the moment image is the support of the Duistermaat-Heckman measure.

Next, let $\mathrm{Stab}(x) \subset T$ denote the stabilizer of $x \in M$, and let $\mathrm{stab}(x)$ denote its Lie algebra. If $\Phi(x)$ lies in the interior of a face F of the polytope $\Phi(M)$, then $\mathrm{stab}(x)$ contains $\left(T_{\Phi(x)}F\right)^\circ$, the annihilator to the tangent space to F at $\Phi(x)$. We say that the orbit $T \cdot x$ is **exceptional** if $\mathrm{stab}(x) \neq \left(T_{\Phi(x)}F\right)^\circ$, or if $\mathrm{Stab}(x)$ is not connected. Let $M_{\mathrm{exc}} \subset M/T$ be the set of exceptional orbits. Two sets of exception orbits M_{exc} and M'_{exc} are equivalent if there is a homeomorphism between them that preserves the moment map and takes each exceptional orbit to another exceptional orbit with the same stabilizer and isotropy representation.

Note that, in Example 1, $M_{\text{exc}} = \varnothing$, in Example 2a, M_{exc} is homotopic to S^1, and in Example 2b, it is homotopic to S^2.

The next invariant, the **genus**, is the integer from the following proposition. In Example 1, the genus is g, while in Example 2, the genus is 0.

Proposition 1. *There exists a non-negative integer g and a homeomorphism $\Psi\colon M/T \to \Sigma_g \times \Phi(M)$ such that $\pi \circ \Psi \to \Phi$, where Σ_g is a Riemann surface of genus g and $\pi\colon \Sigma_g \times \mathfrak{t}^* \to \mathfrak{t}^*$ is the natural projection.*

Finally, a **painting** is a map $f\colon M_{\text{exc}} \to \Sigma_g$ such that $f \times \Phi\colon M_{\text{exc}} \to \Sigma_g \times \mathfrak{t}^*$ is one-to-one. (In particular, if $\Phi\colon M_{\text{exc}} \to \mathfrak{t}^*$ is one-to-one, every map $f\colon M_{\text{exc}} \to \Sigma_g$ is a painting.) Moreover, two paintings are equivalent if – up to an orientation preserving diffeomorphism of Σ_g – they are isotopic through paintings. By the proposition above, there is a painting associated to each complexity one space. In Example 1, since $M_{\text{exc}} = \varnothing$, the painting is trivial. In Examples 2a and 2b, the painting is the constant map.

We can now state our main theorems [2, 3].

Theorem 1. *Two tall complexity one spaces are isomorphic exactly if they have the same (or equivalent) Duistermaat-Heckman measures, exceptional orbits, genus, and paintings.*

Theorem 2. *Given a tall complexity one space M, a natural number g, and a painting $f\colon M_{\text{exc}} \to \Sigma_g$, there exists a tall complexity one space M' with the same (or equivalent) Duistermaat-Heckman measure and exceptional orbits as M, but with genus g and painting f.*

These theorems are proved in two stages. First, we classify tall complexity one spaces over small subsets of $\mathfrak{t}^*$. Then we study how they can be glued together.

Let's apply these theorems to our examples. In Example 2a, M_{exc} is homotopic to S^1. Hence, since $\pi_1(S^2)$ is trivial, there is a unique tall complexity one space of genus 0 with the same Duistermaat-Heckman measure and exceptional orbits as Example 2a. On the other hand, if $g > 0$ then $\pi_1(\Sigma_g)$ is infinite; there are infinitely many tall complexity one space of genus g with the same Duistermaat-Heckman measure and exceptional orbits as Example 2a. In Example 2b, however, M_{exc} is homotopic to S^2. Hence, since $\pi_2(S^2)$ is infinite and $\pi_2(\Sigma_g)$ is trivial for $g > 0$, the situation is reversed. There are infinitely many tall complexity one spaces for $g = 0$ and a unique one for $g > 0$.

Note: In fact, we can describe exactly which Duistermaat-Heckman measures and exceptional orbits arise in this way; see [3].

Note: We believe that analogous theorems apply to **all** complexity one spaces, not just tall ones. However, we have not worked out all the details of the general case, which will rely on Smale's theorem on the diffeomorphisms of S^2.

Finally, I believe that I can prove the following.

Claim. *If a tall complexity one space (M, ω, Φ) admits an invariant Kähler structure, the painting is (up to equivalence) constant on each component of M_{exc}.*

In particular, at most one tall complexity one space with a given Duistermaat-Heckman measure, exceptional orbits, and genus admits an invariant Kähler structure. In most cases, none will. The proof will roughly follow the same argument as in

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Flips and antiflips of P-resolutions of CQS

Jenia Tevelev

This is a report on a joint work with Paul Hacking and Giancarlo Urzua. Motivation comes form the study of the compact moduli space of stable algebraic surfaces constructed by Kollár and Shepherd-Barron [KSB88] and Alexeev. It generalizes the moduli space of stable curves of Deligne and Mumford. Curves parametrized by the boundary points of this moduli space have nodal singularities (and generically just one node), but the situation is more complicated in dimension 2. In particular, a new feature is the presence of boundary points that correspond to irreducible surfaces with T-singularities. By definition, these singularities are cyclic quotient singularities (CQS) which admit $\mathbb{Q}$-Gorenstein smoothing. Apart from Du Val singularities (for moduli purposes, we can ignore those) T-singularities are cyclic quotient singularities

$$\frac{1}{dn^2}(1, dna-1),$$

where $0 < a < n$, $(n, a) = 1$, and $d \geqslant 1$. The parameter d is the dimension of the versal $\mathbb{Q}$-Gorenstein deformation space. If $d = 1$ then we say that the singularity is the Wahl singularity. Those are most generic T-singularities.

The following situation is quite typical: let $(X \subset \mathcal{X}) \to (0 \in \Delta)$, where $0 \in \Delta$ is a smooth curve germ, be a (non-isotrivial) $\mathbb{Q}$-Gorenstein deformation of complex smooth projective surfaces of general type, such that the special fiber X is an irreducible normal surface with only T-singularities. This family maps to a curve germ in the moduli space, and one wants to know: what is the stable limit of X? What are the smooth minimal models of the surfaces (singular and non-singular) in the moduli space around the stable limit of X?

These questions depend on running the relative minimal model program for $\mathcal{X} \to \Delta$, which typically means constructing flips (to simplify matters, let's ignore divisorial contractions in this abstract). For our purposes, we have to understand antiflips along with flips. On the "antiflip" side, we have an extremal neighborhood of flipping type

$$F^- : (C^- \subset \mathcal{X}^-) \to (Q \in \mathcal{Y}),$$

i.e. a proper birational morphism between normal 3-folds such that the canonical class $K_{\mathcal{X}^-}$ is $\mathbb{Q}$-Cartier, $\mathcal{X}^-$ is terminal, and the exceptional locus of F^- consists of a curve $C^- \subset \mathcal{X}^-$ such that $K_{\mathcal{X}^-}.C^- < 0$. We will assume that the map of special fibers $f^- \colon X^- \to Y$ has a property that $f^{-1}(Q)$ is a smooth rational curve C^- with at most two Wahl singularities on it.

On the "flip" side, we have a proper birational morphism

$$(C^+ \subset F^+) \to (Q \in \mathcal{Y})$$

where $\mathcal{X}^+$ is normal with terminal singularities, $\mathrm{Exc}(F^+) = C^+$ is a curve, and $K_{\mathcal{X}^+}$ is $\mathbb{Q}$-Cartier and F^+-ample. A flip induces a birational map $\mathcal{X}^- \dashrightarrow \mathcal{X}^+$ to which we also refer as flip. A flip exists and is is unique [M88]. The map of special fibers $X^+ \to Y$ is a so-called P-resolution. In our case the P-resolution can be called extremal, because ${f^+}^{-1}(Q)$ is a smooth rational curve C^+ with at most two Wahl singularities on it. This is very restrictive: in fact we show using a combinatorial argument that any CQS admits at most two extremal P-resolutions. Let's fix one.

By [KSB88], P-resolutions parametrize irreducible components of the formal deformation space $\mathrm{Def}(Y)$. Namely, let $\mathrm{Def}^{\mathrm{QG}}(X^+)$ denote the versal $\mathbb{Q}$-Gorenstein deformation space of X^+. For any rational surface singularity Z and its partial resolution $X \to Z$, there is an induced map $\mathrm{Def}\, X \to \mathrm{Def}\, Z$ of formal deformation spaces. In particular, we have a map $F^+ : \mathrm{Def}^{\mathrm{QG}}(X^+) \to \mathrm{Def}(Y)$. The germ $\mathrm{Def}^{\mathrm{QG}}(X^+)$ is smooth, the map F^+ is a closed embedding, and it identifies $\mathrm{Def}^{\mathrm{QG}}(X^+)$ with an irreducible component of $\mathrm{Def}(Y)$.

Consider an extremal neighborhood with one Wahl singularity. Let $\frac{n^2}{na-1} = [e_1, \dots, e_s]$ be its continued fraction. It is well-known that if $E_1, \dots, E_s$ are the exceptional divisors of the minimal resolution of X^- then $E_j^2 = -e_j$ for all j. It is not hard to see that the proper transform of C^- is a (-1)-curve intersecting only one component E_i transversally at one point. This data will be written as $[e_1, \dots, \overline{e_i}, \dots, e_s]$ so that $\frac{\Delta}{\Omega} = [e_1, \dots, e_i - 1, \dots, e_s]$, where $0 < \Omega < \Delta$ and $(Q \in Y)$ is $\frac{1}{\Delta}(1, \Omega)$. Similarly, an extremal neighborhood with two Wahl singularities can be encoded as

$$[f_1, \dots, f_{s_2}] - [e_1, \dots, e_{s_1}]$$

so that the (-1)-curve intersects F_{s_2} and E_1, and

$$\frac{\Delta}{\Omega} = [f_1, \dots, f_{s_2}, 1, e_1, \dots, e_{s_1}],$$

where $0 < \Omega < \Delta$ and $(Q \in Y)$ is $\frac{1}{\Delta}(1, \Omega)$.

On the other hand, an extremal P-resolution has an analogous data $[f_1, \dots, f_{s_2}] - c - [e_1, \dots, e_{s_1}]$, so that

$$\frac{\Delta}{\Omega} = [f_1, \dots, f_{s_2}, c, e_1, \dots, e_{s_1}]$$

where $-c$ is the self-intersection of the proper transform of C^+ in the minimal resolution of X^+, $0 < \Omega < \Delta$, and $(Q \in Y)$ is $\frac{1}{\Delta}(1, \Omega)$.

For example, let Y be the CQS $\frac{1}{11}(1, 3)$. Consider its P-resolution

$$[4] - 3 - \diamond$$

where $\diamond$ denotes a smooth point. Here the middle curve is a (-3)-curve. There are two sequences of extremal neighborhoods that antiflip this P-resolution:

$$\diamond - [\bar{2}, 5, 3] - [2, 3, \bar{2}, 2, 7, 3] - [2, 3, 2, 2, 2, \bar{2}, 5, 7, 3] - \cdots$$

and

$$[4] - [2, \bar{2}, 5, 4] - [2, 2, 3, \bar{2}, 2, 7, 4] - [2, 2, 3, 2, 2, 2, \bar{2}, 5, 7, 4] - \cdots$$

It turns out that this is a general picture, and that in fact different antiflips are connected by a flat family. Write $(0 \in D) = \mathrm{Def}^{\mathrm{QG}}(X^+)$. Let $F^+ \colon \mathcal{X}^+ \to \mathcal{Y}$ denote the universal contraction over D. Define a functor

$$G \colon (\mathrm{Schemes}/D) \to (\mathrm{Sets})$$

as follows: $G(h \colon S \to D)$ is the set of isomorphism classes of morphisms

$$F^- \colon \mathcal{X}^- \to \mathcal{Y} \times_D S$$

such that $\mathcal{X}^-/S$ is $\mathbb{Q}$-Gorenstein, and for each $s \in S$ the fiber $\mathcal{X}^-_s \to \mathcal{Y}_{h(s)}$ is an antiflip of f^+ if $h(s) = 0$ and an isomorphism if $h(s) \neq 0$.

Then G is represented by a scheme $M \to D$ together with a universal antiflip

$$F^- \colon \mathcal{X}^- \to \mathcal{Y} \times_D M.$$

The morphism $M \to D$, which is the **flipping family** of X^+, factors through $D' := \mathrm{Def}^{\mathrm{QG}}(X^+, B^+) \subset D$. The germ $(0 \in D')$ is a smooth surface. The scheme M is only locally of finite type. The morphism $M \to D'$ is a toric birational morphism, which can be described by an explicit (locally finite) two-dimensional fan, which describes solutions to a certain diophantine equation of the Pell type.

As a corollary, $\mathbb{Q}$-Gorenstein deformations of X^+ in $\mathrm{Def}^{\mathrm{QG}}(X^+)$ outside of this toric surface have no terminal antiflip. It is a very interesting question do determine if it has a canonical antiflip.

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Quantum Kirwan map for toric orbifolds

Christopher Woodward
(joint work with Eduardo Gonzalez)

Let G be a complex reductive group and X a smooth projectively-embedded G-variety. If G acts locally freely on the semistable locus in X then the geometric invariant theory quotient $X/\!\!/G$ constructed by Mumford is a smooth proper Deligne-Mumford stack with projective coarse moduli space. In Kirwan [14], the natural map $H_G(X) \to H(X/\!\!/G)$ from the equivariant cohomology of X with rational coefficients to the cohomology of $X/\!\!/G$ was shown to be surjective. This sometimes allows the computation of a presentation of $X/\!\!/G$, by computing the kernel of the Kirwan map. For example, if G is a torus acting on a vector space X with weights contained in half-space, then $X/\!\!/G$ is a toric stack and the kernel is the Stanley-Reisner ideal; this recovers the presentation of the cohomology by Danilov-Jurkiewicz [5, 12, 13].

In this report we describe a quantum version of this map, that is, a generalization to quantum cohomology, and discuss its application to presenting the orbifold quantum cohomology of proper smooth toric Deligne-Mumford stacks. We denote by $QH_G(X) = H_G(X) \otimes \Lambda_X^G$ the equivariant quantum cohomology of X as defined by Givental [7], given by the equivariant cohomology tensored with a Novikov ring Λ_X^G, and by $QH(X/\!\!/G)$ the orbifold quantum cohomology of the quotient as defined by Abramovich-Graber-Vistoli [1]. Each of these has the structure of a Frobenius manifold; in particular, on each tangent space there is a product whose structure coefficients are given by sums of Gromov-Witten invariants. There is then a quantum version of Kirwan's map

$$\kappa_X^G : QH_G(X) \to QH(X/\!\!/G),$$

a formal (only the Taylor coefficients are convergent) non-linear map with the following properties:

(1) The linearization $D_\alpha \kappa_X^G : T_\alpha QH_G(X) \to T_{\kappa_X^G(\alpha)} QH(X/\!\!/G)$ is a homomorphism, for any $\alpha \in QH_G(X)$.
(2) The leading order term (set $q = 0$) is the classical Kirwan map onto the untwisted sector $H(X/\!\!/G) \otimes \Lambda_X^G \subset QH(X/\!\!/G)$
(3) The quantum Kirwan map intertwines the graph Gromov-Witten potential $\tau_{X/G}$ of $X/\!\!/G$ with the gauged Gromov-Witten potential τ_X^G of X, so that there is a commutative diagram

(1)
$$\begin{array}{ccc} QH_G(X) & \xrightarrow{\kappa_X^G} & QH(X/\!\!/G) \\ & {\scriptstyle \tau_X^G} \searrow \quad \swarrow {\scriptstyle \tau_{X/G}} & \\ & \Lambda_X^G & \end{array}$$

(4) The quantum Kirwan map is defined by virtual enumeration of *affine gauged maps*, representable morphisms from weighted projective lines

$\mathbb{P}(1,r), r > 0$ to the quotient stack X/G mapping the stacky point at infinity $\mathbb{P}(r)$ to the git quotient $X/\!\!/G$.

More precisely, there is a proper Deligne-Mumford stack $\overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X)$ of stable n-marked affine gauged maps equipped with a relatively perfect obstruction theory and evaluation maps

$$\mathrm{ev} \times \mathrm{ev}_\infty : \overline{\mathcal{M}}^G_{n,1}(\mathbb{C}, X) \to (X/G)^n \times X/\!\!/G$$

and using the Behrend-Fantechi machinery [3] one may define

$$\kappa^G_X(\alpha) = \sum_{n \geqslant 0, d \in H^G_2(X,\mathbb{Q})} (q^d/n!)\, \mathrm{ev}_{\infty,*}\, \mathrm{ev}^*(\alpha, \ldots, \alpha).$$

The gauged Gromov-Witten potential is defined by virtual enumeration of *Mundet-semistable gauged maps*, morphisms from $\mathbb{P}^1$ to the quotient stack X/G satisfying a certain semistability condition generalizing semistability for vector bundles. In the case G is a torus acting on a vector space, the gauged potential is closely related to Givental's I-function, and the diagram above has some overlap with the "mirror theorems" of Givental [7], Lian-Liu-Yau [15], Iritani [10] and others. However, in the formulation above, the "mirror map" is defined geometrically, not as the solution to an algebraic equation. The properness of the moduli stack of affine gauged maps and Mundet semistable maps is a combination of results of Ott [16], Venugopalan [19], and Ziltener [20].

In joint work with E. Gonzalez, we further study the quantum Kirwan map in the case that G is a torus acting on a vector space X, with weights contained in half-space so that $X/\!\!/G$ is proper and, by assumption, smooth. We compute a presentation for the shifted quantum cohomology on the toric stack $X/\!\!/G$, by showing that

(1) $D_0\kappa^G_X$ is surjective (a quantum analog if Kirwan surjectivity)
(2) The kernel contains the quantum Stanley-Reisner ideal QSR^G_X in Batyrev [2]
(3) After a suitable completion, as in Fukaya et al [6], the map

$$QH_G(X)/QSR^G_X \to QH(X/\!\!/G)$$

is an isomorphism.

Many cases of this result were known previously: Batyrev [2] and Givental [8] for the case of semi-Fano toric varieties, Iritani [10] for certain other toric varieties, Fukaya et al [6] for toric varieties in general using open-closed Gromov-Witten theory, and Coates-Corti-Lee-Tseng [4] for the case of weighted projective spaces. Our proof has the interesting features that it does not use the classical result, and does not use the open-closed Gromov-Witten invariants in Fukaya et al [6].

To prove quantum Kirwan surjectivity, one notes that for X, G as above with $\dim(X) = k$ with weights $\beta_1, \ldots, \beta_k$, an affine gauged map of homology class $d \in H^G_2(X, \mathbb{Q})$ is specified by a morphism $u = (u_1, \ldots, u_k) : \mathbb{A} \to X$ satisfying

(1) the degree of u_j is at most (d, β_j)

(2) the leading coefficients for components j such that $(d, \beta_j) \in \mathbb{Z}$ define a semistable point for the action of G.

From this description one sees that $D_0\kappa_X^G$ contains the fundamental class in each twisted sector, and is surjective. To see the Batyrev relations, one uses that the gauged potential is a hypergeometric function of the type considered by Gelfand-Kapranov-Zelevinsky, and commutativity of the diagram (1) implies that the Batyrev relations are in the kernel of $D_0\kappa_X^G$ [18]. Finally, to see that these are all the relations, one notes that in the case that $X/\!\!/G$ is semi-Fano $QH_G(X)/QSR_X^G$ is exactly the same dimension as $QH(X/\!\!/G)$ by Kouchnirenko's theorem, as noted by Iritani [11, 3.10]. In general, (after completion as in Fukaya et al [6]) one can reduce to the semi-Fano case by varying the symplectic class by $c_1^G(X) \in H_2^G(X, \mathbb{Q})$, which is a version of the minimal model program for toric orbifolds in Reid [17]. Details will appear elsewhere.

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Lagrangian Floer thoery and Mirror symmetry of compact toric manifolds

Kaoru Ono
(joint work with K. Fukaya, Y.-G. Oh, H. Ohta)

I reported some results based on our joint works [3, 4]. In general, there are obstructions to defining Floer cohomology of Lagrangian submanifolds. The trouble comes from "bubbling-off" of holomorphic discs. In order to describe obstructions and how to rectify the coboundary operator when it is possible, we introduced the filtered A_∞-algebra for each relatively spin Lagrangian submanifold using moduli spaces of bordered stable maps of genus 0 with boundary marked points. Our theory is constructed over the Novikov ring

$$\Lambda_0 = \{\sum_i a_i T^{\lambda_i} | a_i \in \mathbf{C}, \lambda_i \in \mathbf{R}, \lambda_i \to +\infty\}.$$

By taking the minimal exponent with non-vanishing coefficient a_i, we define an additive valuation $\mathfrak{v}_T : \Lambda_0 \to \mathbf{R} \cup \{+\infty\}$. Denote the field of fractions by Λ and the maximal ideal $\{\mathfrak{v}_T > 0\}$ by Λ_+. The de Rham version of the filtered A_∞-algebra associated to a relatively spin Lagrangian submanifold L in a closed symplectic manifold (X, ω) is a "quantum deformation" of the de Rham algebra with k-ary operations $\mathfrak{m}_k = \sum_{\beta \in H_2(X,L;\mathbf{Z})} \mathfrak{m}_{k,\beta} T^{\int_\beta \omega}$, $k = 0, 1, 2, , \ldots$. Here $\mathfrak{m}_{k,\beta}$ is the chain level k-ary intersection operation coming from the stable compactification of moduli space of holomorphic discs with $k+1$ boundary marked points. The Maurer-Cartan equation is the following:

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) = 0.$$

If b satisfies the Maurer-Cartan equation, we call b a Maurer-Cartan element or a bounding cochain. If the left hand side of the Maurer-Cartan equation is proportional to the unit 1_L, the constant function 1 on L, we call b a weak Maurer-Cartan element and define $\mathfrak{PO}^L(b)$ by

$$\sum_{k=0}^{\infty} \mathfrak{m}_k(b, \ldots, b) = \mathfrak{PO}^L(b) 1_L.$$

Denote by $\mathcal{M}_{\mathrm{weak}}(L)$ the space of weak Maurer-Cartan elements and call $\mathfrak{PO}^L : \mathcal{M}(L) \to \Lambda_+$ the potential function of L. The following theorem is proved in [2].

Theorem A. If there exist $b_i \in \mathcal{M}_{\mathrm{weak}}(L_i)$, $i = 0, 1$, such that $\mathfrak{PO}^{L_0}(b_0) = \mathfrak{PO}^{L_1}(b_1)$, we can rectify the original Floer coboundary operator using b_i.

The cohomology $H^*(X; \Lambda)$ of the symplectic manifold (X, ω) carries a family of quantum product structures parametrized by itself $H^*(X; \Lambda_0)$, which is a part of

the Frobenius manifold structure. Similar to this case, the filtered A_∞-structure mentioned above can be also deformed by a cycle $\mathfrak{b}$ in X with coefficients in Λ_0. We call such a deformation $\{\mathfrak{m}_k^{\mathfrak{b}}\}$ the bulk deformation by $\mathfrak{b}$. Hence we have the potential function $\mathfrak{PO}^{L,\mathfrak{b}} : \mathcal{M}_{\mathrm{weak},\mathfrak{b}}(L) \to \Lambda_+$ and a direct analog of Theorem A.

In the case of compact toric Kähler manifold (X, J, ω) of complex dimension n and a Lagrangian torus fiber L of the moment map, we can use the subspace of T^n-invariant forms instead of all differential forms on L. When we consider bulk deformations, we consider T^n-invariant cylces. Let $\pi : X \to \mathbf{R}^n$ be the moment map and denote by P its image. Then there exist primitive integral vectors v_i and real numbers c_i such that

$$P = \{u \in \mathbf{R}^n | \langle v_i, u\rangle - c_i \geqslant 0, i = 1, \dots, m\}.$$

For $u \in Int(P)$, we set $L(u) = \pi^{-1}(u)$, which is a free T^n-orbit. Thanks to T^n-equivariance, we find that

$$H^1(L(u); \Lambda_0) \subset \mathcal{M}_{\mathrm{weak},\mathfrak{b}}(L(u)).$$

We study the restriction of the potential function to $H^1(L(u); \Lambda_0)$ and call it simply the potential function. Pick a basis $\{\mathbf{e}_i\}$ of $H^1(L(u); \mathbf{Z})$ and set the coordinates $x_1, \dots, x_n$. The leading order part of the potential function $\mathfrak{PO}^{L(u),\mathfrak{b}}$ is written as the sum of contributions from "meridian classes" $\beta_j \in H(X, L(u); \mathbf{Z})$ around irreducible toric divisors and have the same form as Landau-Ginzburg superpotential, though we have appropriate powers of T in coefficients . If $b = \sum x_i \mathbf{e}_i$ is a critical point of $\mathfrak{PO}^{L(u),\mathfrak{b}}$, we find that the corresponding Floer cohomology deformed by $\mathfrak{b}, b$ is isomorphic to the ordinary cohomology of $L(u)$. This follows from the definition of $\mathfrak{m}_1^{\mathfrak{b},b}$ and the fact that the ordinary cohomology of the torus is generated by elements of degree 1. In the case of toric Fano manifolds, the potential function is eqaul to its leading order part. However, there appear higher order terms, in general.

We introduce variables $y_i(u) = e^{x_i}$ depending on u and $y_i = T^{c_i} y_i(u)$. It turns out that $\mathfrak{PO}^{L(u),\mathfrak{b}}(T^{-c_i} y_i)$ as a function of y_i does not depend on u. We write it $\mathfrak{PO}^{X,\mathfrak{b}}(y_i)$, which is called the potential function of the toric manifold X with bulk $\mathfrak{b}$. If $\mathfrak{PO}^{X,\mathfrak{b}}$ has a critical point $\mathbf{y} = (y_1, \dots, y_n) \in (\Lambda \backslash \{0\})^n$ such that $u_{\mathbf{y}} = (\mathbf{v}_T(y_1), \dots \mathbf{v}_T(y_n)) \in Int(P)$, we have non-vanishing Floer cohomology of $L(u_{\mathbf{y}})$ with bulk $\mathfrak{b}$ and weak Maurer-Cartan element $b_{\mathbf{y}} = \sum (x_{\mathbf{y}})_i \mathbf{e}_i$. Here $(x_{\mathbf{y}})_i = \log(T^{-\mathfrak{v}_T(y_i)} y_i) \in \Lambda_0$.

For Fano toric manifolds, the quantum cohomology ring is isomorphic to the Jacobian ring of the Landau-Ginzburg superpotential (Batyrev, Givental). Since the potential function may not be a Laurent polynomial but an infinite series, we need to introduce an appropriate extension of the Laurent polynomial ring. Roughly speaking, we consider the ring of infinite series $\sum_I a_I y^I$, with coefficients $a_I \in \Lambda$, such that $\sum_I a_I \mathbf{y}^I$ converges in Λ_0 as long as $\mathfrak{v}_T(\mathbf{y}) \in Int(P)$. Then we define the Jacobian ring $\mathfrak{Jac}(\mathfrak{PO}^{X,\mathfrak{b}})$ of $\mathfrak{PO}^{X,\mathfrak{b}}$ by the quotient of the above ring by the closure of the ideal generated by first order partial derivatives of $\mathfrak{PO}^{X,\mathfrak{b}}$.

By considering the variation of potential functions $\mathfrak{PO}^{X,\mathfrak{b}}$ parametrized by $H^*(X;\Lambda_0)$, we have the following theorem [3]:

Theorem B. There exists a ring isomorphism $ks_{\mathfrak{b}} : QH^*_{\mathfrak{b}}(X;\Lambda_0) \to \mathfrak{Jac}(\mathfrak{PO}^{X,\mathfrak{b}})$.

In fact, this isomorphism also intertwines pairings on both sides and is expected to be the isomorphism between the Frobenius manifold structure on the quantum cohomology and flat structures, in the sense of K. Saito, for $\mathfrak{PO}^{X,\mathfrak{b}}$, once the latter is understood properly.

We set

$$Crit(\mathfrak{PO}^{X,\mathfrak{b}}) = Hom_{\Lambda-\mathrm{alg}}(\mathfrak{Jac}(\mathfrak{PO}^{X.\mathfrak{b}}) \otimes_{\Lambda_0} \Lambda, \Lambda).$$

This set is considered as the set of idempotents, which are units in the irreducible components of the Jacobian ring to the direct sum of local Artinian rings. We denote by $e_{\mathbf{y}}$ the idempotent of $QH^*_{\mathfrak{b}}(X;\Lambda)$ corresponding to $\mathbf{y} \in Crit(\mathfrak{PO}^{X,\mathfrak{b}})$. We can show that $Crit(\mathfrak{PO}^{X,\mathfrak{b}}) \subset \pi^{-1}(Int(P))$ and the existence of Lagrangian torus fibers with non-vanishing Floer cohomology. Such Lagrangian torus fibers are not displaceable by Hamiltonian diffeomorphisms. If all critical points are non-degenrates, i.e., the Hessian is non-singular at each critical points, we find that $\#Crit(\mathfrak{PO}^{X,\mathfrak{b}}) = \text{rank } H^*(X)$

M. Abouzaid has obtained a criterion for a collection of objects in Fukaya category split generates Fukaya category. Theorem B can be used to verify the criterion adjusted to our case.

Theorem C. [1] The objects corresponding to $Crit(\mathfrak{PO}^{X,\mathfrak{b}})$ split generates the Fukaya category of the compact toric Kaähler manifold (X, J, ω) with bulk deformation by $\mathfrak{b}$.

Using the spectral invariants of Hamiltonian diffeomorphisms, Entov and Polterovich developed the theory of Calabi quasi-morphisms and partial symplectic quasi-states. They constructed a partial symplectic quasi-state for each idempotent of the quantum cohomology ring. In the case of compact toric Kaähler manifolds, they showed the existence of non-displaceable Lagrangian torus fibers. They also introduced the notion of heavyness/superheavyness with respect to a partial sympletic quasi-state ζ for the study of non-displaceablity of a subset.

In [4], we showed the following theorem and gave a relation between Entov-Polterovich's theory and our theory.

Theorem D. For each $\mathbf{y} \in Crit(\mathfrak{PO}^{X,\mathfrak{b}})$, the Lagrangian torus fiber $L(u_{\mathbf{y}})$ is superheavy with respect to the symplectic partial quasi-state associated to the idempotent of $QH^*_{\mathfrak{b}}(X;\Lambda)$ corresponding to $\mathbf{y}$.

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Integrable systems, toric degenerations and Okounkov bodies

Kiumars Kaveh
(joint work with Megumi Harada)

A (completely) integrable system on a symplectic manifold is a Hamiltonian system which admits a maximal number of first integrals (also called 'conservation laws'). A first integral is a function which is constant along the Hamiltonian flow; when there are a maximal number of such, then one can describe the integral curves of the Hamiltonian vector field implicitly by setting the first integrals equal to constants. In this sense an integrable system is very well-behaved. For a modern overview of this vast subject, see [2] and its extensive bibliography. The theory of integrable systems in symplectic geometry is rather dominated by specific examples (e.g. 'spinning top', 'Calogero-Moser system', 'Toda lattice'). The main contribution of this work, summarized in Theorem 1 below, is a construction of an integrable system on (an open dense subset of) a variety X under only very mild hypotheses. Our result therefore substantially contributes to the set of known examples, with a corresponding expansion of the possible applications of integrable systems theory to other research areas.

We begin with a definition. For details see e.g. [3]. Let (X, ω) be a symplectic manifold of real dimension $2n$. Let $\{f_1, f_2, \ldots, f_n\}$ be functions on X.

Definition. *The functions $\{f_1, \ldots, f_n\}$ form an* ***integrable system*** *on X if they pairwise* Poisson-commute*, i.e. $\{f_i, f_j\} = 0$ for all i, j, and if they are* functionally independent*, i.e. their derivatives $df_1, \ldots, df_n$ are linearly independent almost everywhere on X.*

We recall two examples which may be familiar to researchers in algebraic geometry.

Example. *A (smooth projective) toric variety X is a symplectic manifold, equipped with the pullback of the standard Fubini-Study form on projective space. The (compact) torus action on X is in fact Hamiltonian in the sense of symplectic geometry and its moment map image is precisely the polytope corresponding to X. The torus has real dimension $n = \frac{1}{2}\dim_{\mathbb{R}}(X)$, and its n components form an integrable system on X.*

Example. *Let $X = \mathrm{GL}(n, \mathbb{C})/B$ be the flag variety of nested subspaces in $\mathbb{C}^n$. For λ a regular highest weight, consider the usual Plücker embedding $X \hookrightarrow \mathbb{P}(V_\lambda)$ where V_λ denotes the irreducible representation of $\mathrm{GL}(n, \mathbb{C})$ with highest weight λ. Equip X with the Kostant-Kirillov-Souriau symplectic form coming from its identification with the coadjoint orbit $\mathcal{O}_\lambda$ of $U(n, \mathbb{C})$ which meets the positive Weyl chamber at precisely λ. Then Guillemin-Sternberg build an integrable system on X by viewing the coadjoint orbit $\mathcal{O}_\lambda$ as a subset of hermitian $n \times n$ matrices and*

taking eigenvalues (listed in increasing order) of the upper-left $k \times k$ submatrices for all $1 \leqslant k \leqslant n-1$. This is the Guillemin-Sternberg/Gel'fand-Cetlin integrable system on the flag variety. (See [4] *for details.)*

More generally, suppose now X is a projective variety and $\mathcal{L}$ a very ample line bundle on X with ring of sections

$$R(\mathcal{L}) = \bigoplus_k H^0(X, \mathcal{L}^{\otimes k}).$$

Let $n = \dim_{\mathbb{C}}(X)$. Pick ν a valuation with values in $\mathbb{Z}^n$ (e.g. corresponding to some choice of flag of subvarieties) and let $\Delta(X, \mathcal{L}, \nu)$ denote the corresponding Okounkov body. Denote by $S := S(X, \mathcal{L}, \nu) \subset \mathbb{Z}^n \times \mathbb{Z}_{\geqslant 0}$ the value semigroup of ν on the algebra of sections $R(\mathcal{L})$.

Theorem 1. *In the setting above, suppose that S is a finitely generated semigroup (and hence $\Delta(X, \mathcal{L}, \nu)$ is a rational polytope). Then there exist $f_1, \ldots, f_n$ functions on X such that*

- *the f_i are continuous on X and differentiable on an open dense subset U of X,*
- *the f_i pairwise Poisson-commute on U,*
- *the image of X under $\mu := (f_1, \ldots, f_n) : X \to \mathbb{R}^n$ is precisely the Okounkov body $\Delta(X, \mathcal{L}, \nu)$.*

We also show that the integrable systems constructed in this way behave well with respect to GIT/symplectic quotients. Some important examples of Theorem 1 include: (1) flag varieties of general reductive algebraic groups, (2) weight varieties (which are GIT quotients of flag varieties) and (3) spherical varieties. The moment images of the integrable systems in (1) (respectively (2)) are the string polytopes of Littelmann-Berenstein-Zelevinsky (respectively their corresponding slices).

Remark. *Among other things, our theorem addresses a question posed to us by Julius Ross and by Steve Zelditch: does there exist, in general, a 'reasonable' map from a variety X to its Okounkov body? At least under the technical assumption that the value semigroup S is finitely generated, our theorem suggests that the answer is yes.*

We now we briefly sketch the idea of our proof. The essential ingredient is the toric degeneration from X to the (not necessarily normal) toric variety X_0 corresponding to the semigroup $S(X, \mathcal{L}, \nu)$ (see [1] and [7]). The normalization of X_0 is the toric variety associated to the polytope $\Delta(X, \mathcal{L}, \nu)$. Let $f : \mathcal{X} \to \mathbb{C}$ denote the flat family with special fiber $f^{-1}(0) \cong X_0$ and $f^{-1}(t) = X_t \cong X$ for $t \neq 0$. Since toric varieties are integrable systems (see example above), the idea is to "pull back" the integrable system on X_0 to one on X. To accomplish this we use the so-called 'gradient Hamiltonian vector field' (first defined by Ruan and also used by Nishinou-Nohara-Ueda, cf. [6, 5]) on $\mathcal{X}$, where we think of $\mathcal{X}$ as a symplectic space by embedding it into an appropriate weighted projective space. The main technicality which must be overcome to make this sketch rigorous is to appropriately deal with the singular points of $\mathcal{X}$ such that the f_i are continuous

on all of $\mathcal{X}$ (not just on smooth points). It turns out that, in order to deal with this issue, we need a subtle generalization of the famous Łojasiewicz inequality.

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Noncommutative toric geometry

Alastair Craw

(joint work with A. Quintero Vélez)

Noncommutative toric geometry is the study of noncommutative algebras that arise from collections of reflexive sheaves on affine toric varieties. As with the classical study of toric geometry, the goal is to exploit the rich interplay between algebra, combinatorics and geometry. The main results presented in this talk, which introduce a noncommutative analogue of the cellular resolutions of Bayer–Sturmfels [1] appeared recently in the paper [5].

The input data is a Gorenstein toric variety $X = \operatorname{Spec}\mathbb{C}[\sigma^\vee \cap M]$ of dimension n, together with the choice of a collection $\mathscr{E} := (E_0, E_1, \ldots, E_r)$ of distinct rank one reflexive sheaves on X. The main objects of study are

- the algebra $A = \operatorname{End}_{\mathscr{O}_X}(\bigoplus_{0 \leqslant i \leqslant r} E_i)$ via a quiver Q with potential W;
- the CW-complex Δ in a real n-torus call a 'higher quiver';
- partial resolutions $Y_\theta \to X$ obtained by variation of GIT quotient.

Many authors have studied noncommutative algebras via quivers with potential, especially for algebras of global dimension three. Here the algebra need not be the Jacobian algebra, leading us to new families of examples in dimension greater than three. Toric algebras and higher superpotentials have also been studied by Bocklandt [2], Bocklandt–Schedler–Wemyss [3] and Broomhead [4].

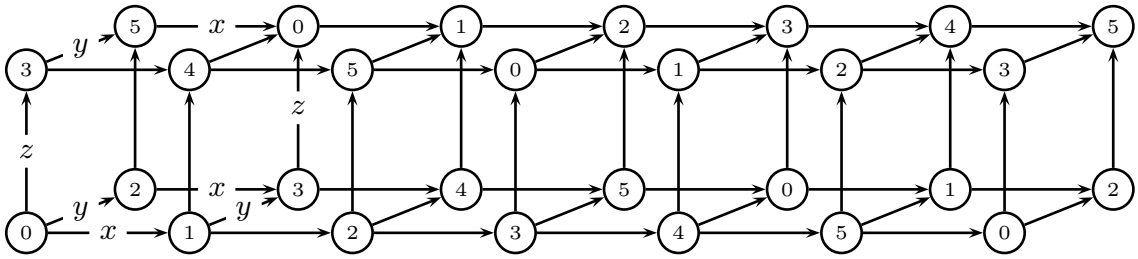
To illustrate the main idea we present a fundamental example. Let $X = \mathbb{C}^n/G$ for a finite abelian subgroup $G \subset \mathrm{SL}(n, \mathbb{C})$, and consider $\mathscr{E} = (E_\rho : \rho \in G^\vee)$. Given the sequence

$$0 \longrightarrow M \longrightarrow \mathbb{Z}^d \xrightarrow{\deg} \mathrm{Cl}(X) \longrightarrow 0, \tag{1}$$

we obtain a real n-torus $(S^1)^n = (M \otimes_{\mathbb{Z}} \mathbb{R})/M$. For $\rho \in G^\vee$, let $\Delta(\rho)$ denote the union of all M-translates of the unit cube $\{\ell + (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n | 0 \leqslant \alpha_i \leqslant 1\}$ for any lattice point $\ell \in \deg^{-1}(E_\rho)$. The *Higher McKay quiver* is the CW complex

$$\Delta := \bigcup_{\rho \in G^\vee} \Delta(\rho)$$

in $(S^1)^n$. We write Δ_k for the set of k-cells in Δ, and we orient each 1-cell in Δ so that it points in the positive direction in $\mathbb{R}^n$. To illustrate Δ, consider the action of $\mathbb{Z}/6$ generated by the diagonal matrix $\operatorname{diag}(\varepsilon, \varepsilon^2\varepsilon^3)$ for $\varepsilon = e^{2\pi i/6}$:



The induced quiver $Q = (\Delta_0, \Delta_1)$ with vertices the 0-cells and arrows the oriented 1-cells is the McKay quiver. The boundary of each 2-cell comprises a pair of paths $p^\pm$ that share the same head and tail, and we obtain a two-sided ideal of relations in the path algebra $\mathbb{C}Q$ by setting $J := (p^+ - p^- \in \mathbb{C}Q : \exists\ \eta \in \Delta_2$ such that $p^\pm$ determine the boundary of $\eta)$, and we have

$$A := \operatorname{End}_{\mathscr{O}_X}\left(\bigoplus_{0 \leqslant i \leqslant r} E_i\right) \cong \mathbb{C}[x_1, \ldots, x_n] * G \cong \mathbb{C}Q/J.$$

Let $\{e_i \in A : i \in \Delta_0\}$ denote the idempotents in A, and let $\varepsilon \colon \Delta \times \Delta \to \{0, \pm 1\}$ be an incidence function on Δ.

Theorem 0.1. *The minimal projective (A, A)-bimodule resolution of A is the* cellular resolution *determined by Δ in the following sense: it is of the form $P^\bullet \to A$ where for $0 \leqslant k \leqslant n$ we have*

$$P_k := \bigoplus_{\eta \in \Gamma_k} A e_{\mathsf{h}(\eta)} \otimes [\eta] \otimes e_{\mathsf{t}(\eta)} A.$$

and $d_k\big(1 \otimes [\eta] \otimes 1\big) = \sum_{\operatorname{cod}(\eta', \eta) = 1} \varepsilon(\eta, \eta') \cdot \overleftarrow{\partial}_{\eta'} \eta \otimes [\eta'] \otimes \overrightarrow{\partial}_{\eta'} \eta$, where the elements $\overleftarrow{\partial}_{\eta'} \eta$, $\overrightarrow{\partial}_{\eta'} \eta \in A$ measure the difference between η and the codimension-one cell η'.

Put simply, the higher quiver Δ encodes refined information about A, just as the fan of a toric variety encodes refined geometric information about the variety.

Returning to the general case, consider $\mathscr{E}$ on X and set $A := \operatorname{End}_{\mathscr{O}_X}(\bigoplus_{0 \leqslant i \leqslant r} E_i)$. The *quiver of sections* Q of $\mathscr{E}$ has vertex set $Q_0 = \{0, 1, \ldots, r\}$ and arrows from i to j defined by indecomposable, torus-invariant sections of $\operatorname{Hom}(E_i, E_j)$. As with the McKay quiver, Q can be realised as an infinite, M-periodic quiver in $\mathbb{R}^d$ with basis the torus-invariant divisors in X. Choosing the standard inner product on $\mathbb{R}^d$, we consider the image of Q under the orthogonal projection $f \colon \mathbb{R}^d \to M \otimes_{\mathbb{Z}} \mathbb{R}$ and obtain an abstract quiver (arrows may a priori intersect) in the real n-torus

$(S^1)^n = (M \otimes_{\mathbb{Z}} \mathbb{R})/M$. Nevertheless, a problem remains: how to fill in 2-cells, 3-cells and so on when we know only the vertices and edges?

The solution is to introduce the potential. A cycle p in Q is *anticanonical* if it arises from the divisor $\sum_{\rho\in\sigma(1)} D_\rho$ in $\mathrm{Hom}(E_i, E_i) \cong H^0(\mathscr{O}_X) \cong H^0(\omega_X)$ for some i, where the second isomorphism holds since X is Gorenstein. The potential is

$$W = \sum_{p \text{ anticanonical cycle}} p.$$

One can define the partial derivative $\partial_q W$ with respect to any path q. The paths q for which $\partial_q W$ is the sum of precisely two paths determine a two-sided ideal

$$J_W := (p^+ - p^- \in \mathbb{C}Q \ : \ \exists\, q \text{ such that } \partial_q W = p^+ - p^- = 0),$$

and we say that A is a *consistent* toric algebra iff $A \cong \mathbb{C}Q/J_W$. Note that there are no signs in W; rather, we add a single sign in defining the generators $p^+ - p^-$ (which is possible precisely because our generators are binomials). The idea is to build Δ by first defining the CW-complexes $\Delta(i)$ for $0 \leqslant i \leqslant r$ in terms of anticanonical cycles that pass through vertex $i \in Q$. Roughly, we set

$$\Delta(i) = M\text{-translates of } \left\{ \begin{array}{c} \text{CW-complex determined by the images under } f \\ \text{of anticanonical cycles from } \ell \mapsto \ell + (1, \ldots, 1) \end{array} \right\}.$$

In practice one must first understand the orthogonal projection f in order to check whether Δ is well-defined. The main result of [5] does this for several families, thereby extending Theorem 0.1 beyond the orbifold case as follows.

Theorem 0.2. *Let A be a consistent toric algebra that is either:*

(i) *of dimension 3, arising from a consistent dimer model on a 2-torus; or*
(ii) *of dimension 4, arising from a tilting bundle on a smooth toric Fano 3-fold.*

Then the higher quiver $\Delta \subseteq (S^1)^n$ exists, and the minimal projective (A, A)-bimodule resolution of A is the cellular resolution as in Theorem 0.1.

More generally, we conjecture that the statement of Theorem 0.2 holds for every consistent toric algebra of global dimension equal to $n = \dim X$.

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Operational K-theory for toric varieties

Dave Anderson
(joint work with Sam Payne)

The equivariant singular and Chow cohomology rings of a smooth toric variety are well understood, as is the Grothendieck ring of equivariant algebraic vector bundles. Specifically, these rings are computed as the rings of piecewise polynomials or piecewise exponential functions on the fan associated to the toric variety. The primary goal of the work described in this talk is to extend these descriptions to singular varieties.

Let $M \cong \mathbb{Z}^n$ be the character group of the dense torus T, and let $N = \mathrm{Hom}(M, \mathbb{Z})$. Let Δ be a fan in $N_{\mathbb{R}}$ consisting of rational polyhedral cones, and write $|\Delta| \subseteq N_{\mathbb{R}}$ for its support. For each cone σ in the fan, there is a quotient $M_\sigma = M/(\sigma^\perp \cap M)$, together with natural maps $M_\sigma \to M_\tau$ whenever $\sigma \supseteq \tau$. The ring of *piecewise polynomial functions* on Δ is

$$PP^*(\Delta) = \{\text{continuous } f\colon |\Delta| \to \mathbb{R} \text{ s.t. } f|_\sigma \text{ lies in } \mathrm{Sym}^* M_\sigma \text{ for all } \sigma \in \Delta\}.$$

Analogously, the ring of *piecewise exponential functions* (or *piecewise Laurent polynomials*) is

$$PLP(\Delta) = \{\text{continuous } f\colon |\Delta| \to \mathbb{R} \text{ s.t. } f|_\sigma \text{ lies in } \mathbb{Z}[M_\sigma] \text{ for all } \sigma \in \Delta\},$$

where $\mathbb{Z}[M] \cong \mathbb{Z}[e^{\pm t_1}, \ldots, e^{\pm t_n}]$ is the character group, interpreted as exponential functions on $N_{\mathbb{R}}$.

When the toric variety $X = X(\Delta)$ is smooth, these combinatorially defined rings have nice geometric interpretations:

Theorem 1 (cf. [1]). *The equivariant (singular or Chow) cohomology of a smooth toric variety $X = X(\Delta)$ is*

$$H_T^* X = A_T^* X = PP^*(\Delta).$$

Theorem 2 (cf. [2, 11]). *The equivariant K-theory ring of a smooth toric variety $X = X(\Delta)$ is*

$$K_T^\circ X = PLP(\Delta).$$

Both of these statements are false for singular toric varieties in general. However, since the right-hand sides are canonically defined rings associated to any toric variety, one is led to seek a geometric interpretation that works for singular varieties. An extension of Theorem 1 was given by Payne:

Theorem 3 (cf. [8]). *The equivariant operational Chow cohomology of any toric variety $X = X(\Delta)$ is*

$$A_T^* X = PP^*(\Delta).$$

Here the left-hand side comes from the operational bivariant Chow theory of Fulton and MacPherson [3, §9]. Our first main result extends Theorem 2, by introducing an (equivariant) operational K-theory.

Theorem 4 (Anderson-Payne). *The equivariant operational K-theory of any toric variety $X = X(\Delta)$ is*

$$\mathrm{op}K_T^\circ X = PLP(\Delta).$$

Similar results have been obtained by Ray and Williams [9] for the K-theory of equivariant topological vector bundles of weighted projective spaces, under some restrictions on the weights.

As with operational Chow cohomology, the operational K-ring $\mathrm{op}K_T^\circ(X)$ is defined so that it acts on a corresponding "homology" theory. For any variety Y with an action of T, let $K_\circ^T(Y)$ be the K-theory of equivariant coherent sheaves on Y. This plays the role of a homology theory—it is covariant for equivariant proper maps. An element of $\mathrm{op}K_T^\circ(X)$ is a collection $c = (c_g)$ of endomorphisms $c_g \colon K_\circ^T(Y) \to K_\circ^T(Y)$, one for each equivariant map $g \colon Y \to X$. These collections are required to satisfy some compatibility axioms, modelled on the classical cap product.

Despite its apparently unwieldy definition, we show that operational K-theory is in some respects better behaved than the K-theory of vector bundles. For example, motivated by similar theorems for Chow cohomology [4, 10], we prove:

Theorem 5 (Anderson-Payne). *Let X be any complete T-*linear variety*. Then there is a natural isomorphism*

$$\mathrm{op}K_T^\circ(X) = \mathrm{Hom}_{R(T)}(K_\circ^T(X), R(T)).$$

(Here $R(T) \cong \mathbb{Z}[M]$ is the representation ring of T.)

The notion of a T-linear variety is based on a definition of Totaro [10], and includes all toric varieties, as well as spherical varieties and Schubert varieties.

In the special case where T is trivial, Theorem 5 implies that $\mathrm{op}K^\circ(X)$ is a finitely generated and torsion-free $\mathbb{Z}$-module. This stands in contrast to ordinary K-theory of vector bundles—even for simplicial projective toric threefolds, $K^\circ(X)$ may be uncountably generated [6].

Although their statements only involve Grothendieck groups, the proofs of both Theorems 4 and 5 invoke higher algebraic K-theory. A key part of the proof of Theorem 4 is an equivariant version of a result of Gillet [5]: when $X' \to X$ is an *equivariant envelope*, the sequence

$$K_\circ^T(X' \times_X X') \to K_\circ^T(X') \to K^T(X) \to 0$$

is exact. With this in hand, formal methods used by Kimura [7] in the Chow case reduce the problem to the smooth case, where the theorem is known.

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The Topology of Toric Origami Manifolds

Tara Holm

(joint work with Ana Rita Pires)

In the past 30 years, there has been a flurry of research on the topology of compact symplectic manifolds that are equipped with Hamiltonian group actions. A key tool is the **momentum map**, whose components are perfect Morse-Bott functions with critical set the fixed points of the group action. Important classes of examples include **toric symplectic manifolds** and **generalized flag varieties**. Compact toric symplectic manifolds are in one-to-one correspondence with **Delzant polytopes**. The cohomology rings of both toric manifolds and flag varieties are concentrated in even degrees. The cohomology ring of a toric variety may be described in terms of generators and relations by the **Jurkiewicz-Danilov Theorem**. The equivariant cohomology of a compact toric symplectic manifold is the **Stanley-Reisner ring** of the corresponding Delzant polytope. The equivariant cohomology of both toric manifolds and flag varieties injects as a subring of the equivariant cohomology of the fixed point set, with image given by the **GKM description**.

There have been a number of generalizations of toric varieties that enjoy the same cohomological properties. Building on work of Hattori and Masuda [HtMs, Ms], Masuda and Panov defined a **torus manifold** to be a $2n$-dimensional closed connected orientable smooth manifold M equipped with an effective smooth action of an n-dimensional torus $\mathbb{T} = (\mathbb{S}^1)^n$ with non-empty fixed set [MsPn]. They

characterize those torus manifolds whose equivariant cohomology ring has the Stanley-Reisner description, and determine when it has the GKM description.

Folded symplectic forms were introduced by Eliashberg and Martinet [E, Mr]. They are closed 2-forms that are non-degenerate except along a hypersurface $Z \subset M$, where the form has 1-dimensional kernel. As in the symplectic case, these admit a **Darboux theorem**. Cannas da Silva proved that any manifold with a stable almost complex structure is folded symplectic [C]. In particular, every compact orientable 4-manifold is folded symplectic, and the form can be chosen to be in any cohomology class. The folded form is **origami** if the null-foliation on Z is fibrating, with compact, connected, oriented fibers. Cannas da Silva, Guillemin and Woodward; and Cannas da Silva, Guillemin and Pires have studied the geometry of origami manifolds [CGW, CGPr].

We may define Hamiltonian group actions on folded symplectic manifolds. A Hamiltonian torus action is **toric** if $\dim(\mathbb{T}) = \frac{1}{2}\dim(M)$. Chris Lee has made a systematic study of toric 4-folds in his thesis [L]. Cannas da Silva, Guillemin and Pires proved that compact toric origami manifolds are classified by **origami templates** $(\mathcal{P}, \mathcal{F})$, where $\mathcal{P}$ is a collection of Delzant polytopes and $\mathcal{F}$ a collection of facets in the polytopes in $\mathcal{P}$. The image of the toric origami manifold under the momentum map is the superposition of the polytopes in the collection $\mathcal{P}$, as shown on the left in the figure below. In contrast to the symplectic case, this need not be convex.

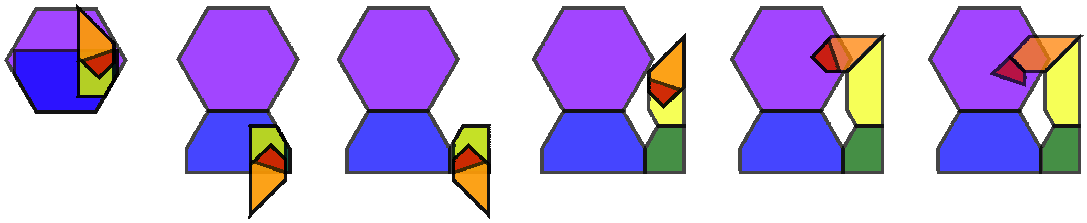

FIGURE 1: When the origami template is **acyclic**, we may unfold the template to see the combinatorics of $(\mathcal{P}, \mathcal{F})$ better.

Toric origami manifolds need not be orientable. For example, the real projective plane and the Klein bottle are 2-dimensional toric origami manifolds. Origami templates for some 4-dimensional manifolds are shown in Figure 2. When they are orientable, toric origami manifolds provide many examples of torus manifolds. If the origami template is acyclic and the fold is coörientable, then the toric origami manifold is orientable.

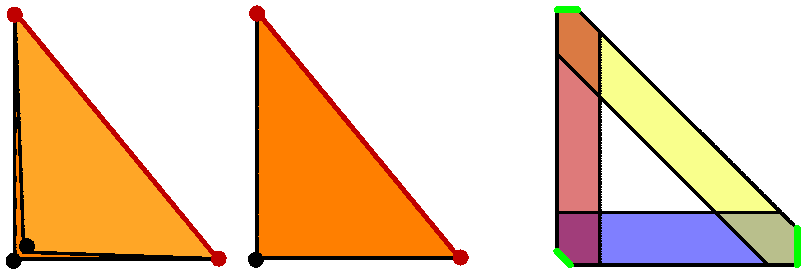

FIGURE 2: Origami templates for S^4 and $\mathbb{R}P^4$, and a non-acyclic template.

A torus manifold M is **locally standard** if every point in M has an invariant neighbourhood U weakly equivariantly diffeomorphic to an open subset $W \subset \mathbb{C}^n$ invariant under the standard $\mathbb{T}^n$-action on $\mathbb{C}^n$.

Lemma 1. *Suppose that $(M, Z, \omega, \Phi, \mathbb{T})$ is a toric origami manifold with coöriented folding hypersurface. Then M is locally standard.*

This allows us to deduce a wide range of facts about the ordinary and equivariant cohomology of M, using Masuda and Panov's work on torus manifolds. We may also prove some of these results directly, using topological tools familiar from symplectic geometry. We provide a straight-forward topological proof of the following, similar in spirit to arguments in [HsK].

Theorem 2. *Let $\mathbb{T} \circlearrowright M$ be a compact toric origami manifold with acyclic origami template and coöriented folding hypersurface. Then the cohomology $H^*(M; \mathbb{Q})$ is concentrated in even degrees.*

This immediately implies that the Leray-Serre spectral sequence that computes the equivariant cohomology of M must collapse at the E_2-term. We may apply results of Franz and Puppe [FPp] to deduce that the equivariant cohomology may be described in terms of the **one-skeleton**. We can then give a GKM description of the equivariant cohomology. We expect the Theorem 2 to hold with integer coefficients, with the same conclusions about equivariant cohomology with $\mathbb{Z}$ coefficients.

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Higher order toric projective duality

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(joint work with Alicia Dickenstein and Ragni Piene)

Higher order dual varieties can be considered as generalizations of the classical notion of discriminant. Let $A \subset \mathbb{Z}^n$ be a finite subset of lattice points. The *discriminant* of A, Δ_A, is a homogeneous polynomial in $N+1 = |A|$ variables with the following property:

$$\Delta_A(c_0, \ldots, c_N) = 0$$
$$\Leftrightarrow$$
$$s(x_1, \ldots, x_n) = \textstyle\sum_{a \in A} c_a x^a \text{ has at least one multiple root in } (\mathbb{C}^*)^n.$$

The discriminant Δ_A is in fact the defining equation of an irreducible codimension one irreducible sub variety of the dual projective space $\mathbb{P}^{N\,\vee}_{\mathbb{C}}$, via the following general notion of duality.

Definition. Let $i : X \hookrightarrow \mathbb{P}^N$ be an embedding of an algebraic variety. The dual variety is defined as:

$$X_i^\vee = \overline{\{H \in \mathbb{P}^{N\,\vee} \text{ s.t. } H \text{ is tangent to } X \text{ at some smooth point } x \in X_{sm}\}}$$

The expected codimension of $X_i^\vee$ is one and, when this is the case, the defining homogeneous polynomial Δ_i is the discriminant of i. The polynomial Δ_A is therefore well defined when the dual variety $X_{i_A}^\vee$ of an associated toric embedding $i_A : X_A \hookrightarrow \mathbb{P}^{|A|-1}$ has the expected codimension one and $\Delta_A = \Delta_{i_A}$.

When the codimension of the dual variety is higher than one the embedding is said to be *dually defective*. Classifying the exceptions (i.e. defective embeddings) and find a formula for Δ_A constitutes an active area of algebraic geometry and combinatorics. Defective embeddings have been studied and classified for example in [8, 7]. More recently a number of results on the A-discriminants have appeared:

- The case $N = n+2$ has beed characterized in [4].
- The case $N = n+3, n+4$ has been studied in [1]
- When X_A is non singular a characterization can be found in [6].
- The case when X_A is $\mathbb{Q}$-factorial has been studied in [2].
- A description of the tropical dual variety can be found in [5].

In order to give a generalization involving multiple roots of higher multiplicity we need to formally define the concept of "higher tangency" for an hyperplane. Let $\mathcal{L} := i^*(\mathcal{O}_{\mathbb{P}^N}(1))$. The vector space $\mathcal{L}/\mathfrak{m}_x^{k+1}\mathcal{L}$ is the fibre at $x \in X_{sm}$ of the k-th principal parts (or jet) sheaf $\mathcal{P}^k_X(\mathcal{L})$, which has generic rank $\binom{n+k}{n}$. It we identify $H^0(\mathbb{P}^N, \mathcal{O}(1)) \otimes \mathcal{O}_X \simeq \mathcal{O}_X^{N+1}$ there is a natural map (of coherent sheaves) called the k-th jet map:

$$j_k : \mathcal{O}_X^{N+1} \to \mathcal{P}^k_X(\mathcal{L})$$

which is given fiberwise by the linear map

$$j_{k,x} : H^0(\mathbb{P}^N, \mathcal{O}(1)) \to H^0(X, \mathcal{L}) \to H^0(X, \mathcal{L}/\mathfrak{m}_x^{k+1}\mathcal{L})$$

induced by the map of $\mathcal{O}_X$-modules $\mathcal{L} \to \mathcal{L}/\mathfrak{m}_x^{k+1}\mathcal{L}$. So if $s \in H^0(X, \mathcal{L})$ then $j_{k,x}(s)$ is the order k truncation of the Taylor series expansion of s with respect to the local coordinates $x_1, \ldots, x_n$ and itt can be written as

$$j_{k,x}(s) = (s(x), \frac{\partial s}{\partial x_1}(x), \ldots, \frac{\partial s}{\partial x_n}(x)), \frac{1}{2}\frac{\partial^2 s}{\partial x_1^2}(x), \ldots, \frac{1}{2}\frac{\partial^2 s}{\partial x_n^2}(x)), \ldots).$$

The space $\mathbb{P}(image(j_{1,x})) = \mathbb{P}(\mathcal{P}^1_X(\mathcal{L})_x) = \mathbb{T}_{X,x} \cong \mathbb{P}^n$ is the embedded tangent space at the point x. More generally, the linear space $\mathbb{P}(image(j_{k,x})) = \mathbb{T}^k_{X,x}$ is called the *k-th osculating space* at x.

Definition We say that a hyperplane H is tangent to X to order k at a smooth point x if $\mathbb{T}^k_{X,x} \subseteq H$. The k-th dual variety is

$$X^{(k)} := \overline{\{H \in \mathbb{P}^{N\vee} \mid H \supseteq \mathbb{T}^k_{X,x} \text{ for some } x \in X_{\mathrm{sm}}\}}.$$

In particular, $X^{(1)} = X^\vee$. Alternatively, one can define $X^{(k)}$ as the closure of the image of the map

$$\gamma_k : \mathbb{P}((\operatorname{Ker} j_k)^\vee|_{X_{k-\mathrm{cst}}}) \to \mathbb{P}^{N\vee}, \tag{1}$$

where $X_{k-\mathrm{cst}}$ denotes the open set of X where the rank of j_k is constant. Note that $X^{(k)} \subseteq X^{(k-1)}$. Moreover, $X^{(2)}$ is contained in the singular locus of $X^\vee$, since a necessary condition for a point $H \in X^\vee$ to be smooth, is that the intersection $H \cap X$ has a singular point of multiplicity 2: if $H \supseteq \mathbb{T}^k_{X,x}$ for $k \geqslant 2$, then $H \cap X$ has a singular point of multiplicity $\geqslant k+1$.

When X is non singular $\mathcal{P}^k_X(\mathcal{L})$ is a vector bundle and when j_k is surjective $\operatorname{Ker} j_k$ is also locally free. This allows convenient Chern class computations.

Definition. We say that the embedding $i\colon X \hookrightarrow \mathbb{P}^m$ is *k-jet spanned* at a smooth point $x \in X$ if the k-th osculating space to X at x has the maximal dimension, $\binom{n+k}{k}-1$, or, equivalently, the map $j_{k,x}$ is surjective. We say that ι is k-jet spanned if it is k-jet spanned at all smooth points $x \in X$.

Lemma [9, 3]. Assume X is a smooth variety of dimension n, and that the embedding $i\colon X \hookrightarrow \mathbb{P}^N$ is k-jet spanned. Then

(1) the embedding ι is k-defective if and only if $c_n(\mathcal{P}^k_X(\mathcal{L})) = 0$;
(2) if i is not k-defective, then $\deg X^{(k)} = deg(\gamma_k)c_n(\mathcal{P}^k_X(\mathcal{L}))$;
(3) if i is generically $(k+1)$-jet spanned, then the embedding is not k-defective;
(4) if i is $(k+1)$-jet ample, then $\deg(\gamma_k) = 1$, and thus $\deg_k X^{(k)} = \deg X^{(k)}$.

The concept of jet ampleness is in general stronger than jet spannedness, but for non singular toric varieties the two notions coincide. In toric geometry many geometrical properties have been related to the associated polytope. This is also the case for higher duality.

Theorem [3] Let i be a 2-jet spanned toric embedding of a smooth threefold X, corresponding to a lattice polytope P of dimension three. Then $\dim(X^{(2)}) =$

$|P \cap \mathbb{Z}^3| - 7$ unless if $(X, \mathcal{L}) = (\mathbb{P}^3, \mathcal{O}(2))$, in which case $P = 2\Delta_3$ and the second dual variety $X^{(2)}$ is empty. Moreover:

1) $\deg X^{(2)} = 120$ if $(X, \mathcal{L}) = (\mathbb{P}^3, \mathcal{O}(3))$.
2) $\deg X^{(2)} = 6(8(a+b+c) - 7)$ if $(X, \mathcal{L}) = (\mathbb{P}(\mathcal{O}_{\mathbb{P}^1}(a), \mathcal{O}_{\mathbb{P}^1}(b), \mathcal{O}_{\mathbb{P}^1}(c)), 2\xi)$, where $a, b, c \geqslant 1$ and ξ denotes the tautological line bundle.
3) In all other cases,

$$\deg(\gamma_2)(\deg X^{(2)}) = 62Vol(P) - 57F + 28E - 8V + 58Vol(P^\circ) + 51F_1 + 20E_1$$

where $Vol(P)$, F, E (resp. $Vol(P^\circ)$, F_1, E_1) denote the (lattice) volume, area of facets, length of edges of P (resp. of the convex hull of the interior lattice points of P), and V is the number of vertices of P.

The tropical higher order dual variety can be defined by extending the $(n+1) \times (N+1)$ matrix

$$\overline{A} = \begin{pmatrix} 1 & \dots & 1 \\ & A & \end{pmatrix}$$

By considering higher order derivatives of the monomial x^a, for $a \in A$ one can define a $\binom{n+k}{k} \times (N+1)$ matrix A^k and show that $X^{(k)} = \overline{\cup_{y \in Ker(A^k)} Orb(y)}$ where the action is the $(\mathbb{C}^*)^n$ action induced on $\mathbb{P}^N$ by the columns of A.

Theorem [3] The tropicalization of the cone over $X_A^{(k)}$, $trop(Y_A^{(k)}) \subset \mathbb{R}^{N+1}$ is equal to the Minkowski sum

$$trop(Y_A^{(k)}) = \text{ Rowspan}(\overline{A}) + trop(\text{Rowspan}(A^{(k)})).$$

Its image $\pi(trop(Y_A^{(k)}))$ in $\mathbb{R}^{N+1}/\sim$ gives the tropicalization of the k-th dual variety $X_A^{(k)}$.

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Open Gromov-Witten invariants on toric manifolds

Siu-Cheong Lau
(joint work with K.W. Chan, N.C. Leung, H.H. Tseng)

Let X be a compact toric manifold of complex dimension n and $q \in H^2(X, \mathbb{C})$ be a complexified Kähler class of X. When $-K_X$ is numerically effective, we extract the open Gromov-Witten invariants of X from its mirror map. This gives an open analogue of closed-string mirror symmetry discovered by Candelas-de la Ossa-Green-Parkes [1]. Namely, under mirror symmetry, the computation of open Gromov-Witten invariants is transformed to a PDE problem of solving Picard-Fuchs equations.

The mirror of (X, q) is defined to be a certain holomorphic function W_q on $(\mathbb{C}^\times)^n$ called the superpotential[1]. Closed-string mirror symmetry states that the deformation of W_q encodes Gromov-Witten invariants of X. More precisely, it states that there is an isomorphism

$$QH^*(X, q) \cong \mathrm{Jac}(W_q)$$

as Frobenius algebras, where $QH^*(X, q)$ denotes the small quantum cohomology ring of (X, q) and

$$\mathrm{Jac}(W_q) := \frac{\mathbb{C}[z_1^{\pm 1}, \ldots, z_n^{\pm 1}]}{\left\langle z_1 \frac{\partial W_q}{\partial z_1}, \ldots, z_n \frac{\partial W_q}{\partial z_n} \right\rangle}$$

is the Jacobian ring of W_q.

Based on physical arguments, Hori-Vafa [6] gave a recipe to write down a Laurent polynomial W_q° from the fan configuration of X. It turns out that W_q° only records the 'leading order terms' and is not equal to W_q in general. The difference $W_q - W_q^\circ$ is called instanton corrections.

Traditionally, the instanton corrections are written down from the PDE approach. Namely, one writes down a Picard-Fuchs system using the fan configuration of X, and solves it explicitly for the 'mirror map' $\check{q}(q)$. Then define

$$W_q^{\mathrm{PF}} := W^\circ_{\check{q}(q)}.$$

When the anti-canonical line bundle $-K_X$ is numerically effective, W_q^{PF} fits into the mirror symmetry framework mentioned above, namely

$$QH^*(X, q) \cong \mathrm{Jac}(W_q^{\mathrm{PF}})$$

as Frobenius algebras [5, 7].

On the other hand, the instanton corrections are realized by Fukaya-Oh-Ohta-Ono [4] using open Gromov-Witten invariants as follows. Let $\mathbf{T} \subset X$ be a Lagrangian toric fiber and $\pi_2(X, \mathbf{T})$ be the set of homotopy classes of maps $(\Delta, \partial\Delta) \to (X, \mathbf{T})$, where Δ denotes the closed unit disk. For $\beta \in \pi_2(X, \mathbf{T})$,

[1]More precisely W is a formal series over the Novikov field. Here we assume it is convergent in some neighborhood of $q = 0$, and thus defines a holomorphic function on $(\mathbb{C}^\times)^n$.

the moduli space $\mathcal{M}_1(\beta)$ of stable disks representing β and its virtual fundamental class $[\mathcal{M}_1(\beta)] \in H_n(\mathbf{T})$ are defined by use of Kuranishi structures. The one-pointed Gromov-Witten invariant associated to β is defined as

$$n_\beta := \int_{[\mathcal{M}_1(\beta)]} \mathrm{ev}^*[\mathrm{pt}]$$

where $[\mathrm{pt}] \in H^n(\mathbf{T})$ is the point class and $\mathrm{ev} : \mathcal{M}_1(\beta) \to \mathbf{T}$ is the evaluation map. Then

$$W_q^{\mathrm{LF}} := \sum_{\beta \in \pi_2(X,\mathbf{T})} n_\beta Z_\beta$$

gives another definition of the instanton-corrected superpotential, where Z_β is an explicitly written monomial for each β. Notice that the above formal sum involves infinitely many terms in general, and is well-defined over the Novikov field. Fukaya-Oh-Ohta-Ono [3] proved that

$$QH^*(X, q) \cong \mathrm{Jac}(W_q^{\mathrm{LF}})$$

as Frobenius algebras.

While W_q^{PF} and W_q^{LF} originates from totally different approaches, they lead to the same mirror symmetry statements. The following conjecture is made in a joint work with Chan, Leung and Tseng [2]:

Conjecture 1 ([2])**.** *Let X be a toric manifold with $-K_X$ numerically effective, and let W^{PF} and W^{LF} be the superpotentials in the mirror as introduced above. Then*

$$W^{PF} = W^{LF}. \tag{1}$$

Conjecture 1 can be proved under the technical assumption that W^{LF} converges analytically (instead of just being a formal sum):

Theorem 2 ([2])**.** *Let X be a toric manifold with $-K_X$ numerically effective, and let W^{PF} and W^{LF} be the superpotentials in the mirror as explained above. Then*

$$W^{PF} = W^{LF}$$

provided that each coefficient of W^{LF} converges in an open neighborhood around $q = 0$.

The above technical assumption on the convergence of W^{LF} is satisfied when $\dim X = 2$, or when X is of the form $\mathbb{P}(K_S \oplus \mathcal{O}_Y)$ for some toric Fano manifold Y.

The function W^{LF} is a generating function of open Gromov-Witten invariants n_β (and thus can be regarded as an object in the 'A-side'), whereas W^{PF} arises from solving Picard-Fuchs equations (and so is an object in the 'B-side'). Using this equality, the task of computing the open Gromov-Witten invariants is transformed to solving Picard-Fuchs equations which has been known to experts. Thus the above equality gives a mirror symmetry method to compute open Gromov-Witten invariants.

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Classifying Toric Degenerations of Fano Threefolds

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(joint work with Jan Arthur Christophersen)

Let V be a smooth Fano variety over $\mathbb{C}$ with very ample anticanonical divisor. Consider the following

Problem. *Classify all toric Fano varieties with at most Gorenstein singularities to which V degenerates in its anticanonical embedding.*

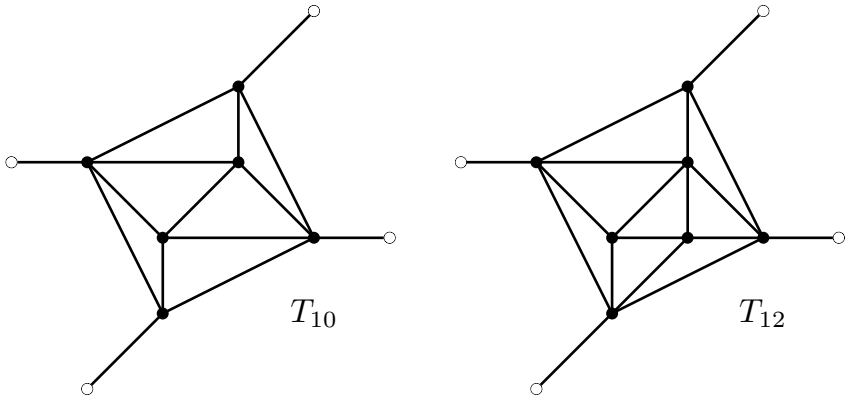
Before addressing this problem, we first discuss how to construct many (but not all) toric degenerations of a given Fano variety V. Let $\mathcal{K}$ be a simplicial complex, and $A_{\mathcal{K}}$ the corresponding Stanley-Reisner ring. This graded ring gives rise to a projective Stanley-Reisner scheme $\mathbb{P}(\mathcal{K}) = \operatorname{Proj} A_{\mathcal{K}}$. We call $\mathcal{K}$ *unobstructed* if $T^2_{A_{\mathcal{K}}}$ vanishes, where $T^2_{A_{\mathcal{K}}}$ is the second cotangent cohomology of the ring $A_{\mathcal{K}}$. We call $\mathcal{K}$ *Fano* if it is the join $\mathcal{K} = T * \Delta_k$ of a triangulated sphere T and simplex Δ_k. Note that if T is unobstructed, then so is $\mathcal{K}$.

Let $\mathcal{H}_V$ denote the Hilbert scheme parametrizing subvarieties of $\mathbb{P}(|-K_V|)$ with the same Hilbert polynomial as V. The variety V corresponds to a point $[V] \in \mathcal{H}_d$, and it follows from standard vanishing results that this point is smooth, and thus lies on a single irreducible component $C_V \subset \mathcal{H}_V$. If we assume that $[V]$ is a general point on C_V, we then have the following:

Proposition 1 ([1, cf. Proposition 2.3])**.** *Let V be a Fano variety as above, and $\mathcal{K}$ an unobstructed Fano simplicial complex such that V has an embedded degeneration to $\mathbb{P}(\mathcal{K})$. Let P be any reflexive polytope with corresponding toric Fano variety X. If P has a regular unimodular triangulation of the form $\mathcal{K}$, then V has an embedded degeneration to X.*

Thus, in order to construct toric degenerations of the smooth Fano variety V, we would like to find an unobstructed Fano simplicial complex $\mathcal{K}$ such that V degenerates to $\mathbb{P}(\mathcal{K})$. This can be done for rank one index one Fano threefolds of

degree $4 \leqslant d \leqslant 16$, that is, those with $\mathrm{Pic}(V) = \langle -K_V \rangle$ and $4 \leqslant (-K_V)^3 \leqslant 16$. Indeed, let $T_4 = \partial\Delta_3$, the boundary complex of the tetrahedron, $T_6 = \partial\Delta_2 * \partial\Delta_1$, the bipyramid over the boundary of a triangle, and for even $8 \leqslant d \leqslant 16$ let T_d be the unique triangulation of the two-sphere on $d/2+2$ vertices such that the degree of each vertex is either 4 or 5. Note that T_d may be constructed from T_{d-2} via an edge subdivision. By [4], these triangulations are all unobstructed.



Theorem 2 ([1, cf. Corollary 3.3])**.** *For even $4 \leqslant d \leqslant 16$, let V_d be a general rank one index one Fano threefold of degree d. Then V_d has an embedded degeneration to $\mathbb{P}(T_d * \Delta_0)$.*

Combining this with Proposition 1, we are able to construct many toric degenerations of V_d. In fact, by using some slightly more elaborate arguments we are able to return to our original problem and *completely classify* all degenerations of smooth Fano threefolds of degree $\leqslant 12$ to Gorenstein toric Fano varieties, see [2, Theorem 1.1]. In addition to the above construction, an important ingredient in this classification is local obstruction calculus, which we carry out with the help of the computer package `VersalDeformations` [3].

We are also interested in Fano varieties of higher dimension, and thus in unobstructed triangulations of S^n for $n \geqslant 3$. Let $\mathcal{A}_n$ denote the boundary complex of the dual of the n-associahedron: faces of $\mathcal{A}_n$ correspond to triangulations of the n-gon. For example, $\mathcal{A}_4 = \partial\Delta_1$, and $\mathcal{A}_5$ is the boundary complex of the 5-gon.

Theorem 3 ([1, cf. Theorem 5.3])**.** *The simplicial complex $\mathcal{A}_n$ is unobstructed for all $n \geqslant 4$.*

By [5], the Grassmannian $G(2,n)$ degenerates to $\mathbb{P}(\mathcal{A}_n * \Delta_{n-1})$. Thus, we may use the above theorem to find many new toric degenerations of $G(2,n)$ and linear sections thereof.

The sequence T_d of unobstructed simplicial complexes generalizes to higher dimensions:

Theorem 4 ([1, cf. Theorem 6.2])**.** *Let $n > r \geqslant 4$. Then there is a sequence of unobstructed triangulated spheres $\mathcal{K}_0 = \mathcal{A}_r * \mathcal{A}_{n-r+s}$, $\mathcal{K}_1, \ldots, \mathcal{K}_{(n-r)(r-3)} = \mathcal{A}_n$ such that $\mathcal{K}_i$ is an edge subdivision of $\mathcal{K}_{i-1}$.*

Example. *Taking $n = 6$ and $r = 4$, we recover the sequence T_{10}, T_{12}, T_{14}. Indeed, $T_{10} = \mathcal{A}_5 * \mathcal{A}_4$, and $T_{14} = \mathcal{A}_6$. Taking $n = 5$ and $r = 4$, this extends to include $T_8 = \mathcal{A}_4 * \mathcal{A}_4 * \mathcal{A}_4$.*

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(Almost) Lagrangian fibre bundles

Reyer Sjamaar

Let B be a connected n-manifold. An *almost Lagrangian fibre bundle* over B is a triple $\mathcal{L} = (M, \omega, \pi)$, where M is a $2n$-manifold, ω is a nondegenerate 2-form on M, called the *almost symplectic form*, which has the property that $d\omega = \pi^*\eta$ for some closed 3-form η on B, and $\pi\colon M \to B$ is a locally trivial fibre bundle with the property that ω restricts to 0 on every fibre of π. The 3-form η is uniquely determined by ω and is called the *twisting form* of $\mathcal{L}$. For simplicity we will furthermore assume that the fibres of π are compact and connected. We call $\mathcal{L}$ a *Lagrangian* fibre bundle if $\eta = 0$, i.e. ω is a symplectic form.

Example. *Let $p\colon B \to \mathbf{R}^n$ be a local diffeomorphism and let β be a 2-form on B. Let $\mathbf{T}$ be the circle $\mathbf{R}/\mathbf{Z}$. The* angle form *on $\mathbf{T}$ is dq, where q is the coordinate on $\mathbf{R}$. Let M be the product $B \times \mathbf{T}^n$, equipped with the nondegenerate 2-form $\omega = \sum_{j=1}^n dp_j \wedge dq_j + \pi^*\beta$, where $\pi\colon M \to B$ is the projection onto the first factor. The functions p_j are the* action variables *and the (multivalued) functions q_j are the* angle variables*. The triple $\mathcal{L} = (M, \omega, \pi)$ is an almost Lagrangian fibre bundle with twisting form $\eta = d\beta$.*

Almost Lagrangian fibre bundles arise in nonholonomic mechanics and their basic structure is to a large extent analogous to that of Lagrangian fibre bundles. To begin with, a result of Fassò and Sansonetto [6] asserts the existence of *local action-angle variables* on any almost Lagrangian fibre bundle $\mathcal{L}$. This means that every point in B has an open neighbourhood U such that the restriction of $\mathcal{L}$ to U is of the kind described in the above example.

The global structure of $\mathcal{L}$ can be analysed as done by Duistermaat [4] in the Lagrangian case. Duistermaat's results were refined by Dazord and Delzant [2];

see also Zung [13] and my obituary notice [12]. One notes that every 1-form α on B produces a vector field $v(\alpha)$ on M:

$$\Omega^1(B) \xrightarrow{\pi^*} \Omega^1(M) \xrightarrow{\omega^\sharp} \mathcal{X}(M), \qquad \alpha \longmapsto \pi^*(\alpha) \longmapsto \omega^\sharp\pi^*(\alpha) = v(\alpha).$$

This vector field is tangent to the fibres of π and we denote its time 1 flow by $\phi(\alpha)$. Similarly, every cotangent vector $\alpha \in T_b^*B$ produces a vector field $v_b(\alpha)$ on the fibre $\pi^{-1}(b)$, whose time 1 flow we call $\phi_b(\alpha)$. The map $\phi_b\colon T_b^*B \times \pi^{-1}(b) \to \pi^{-1}(b)$ so defined is a transitive action of the abelian Lie group T_b^*B, and the kernel of the action is the *period lattice* P_b at b. Collecting these actions for all $b \in B$ we get an action

$$T^*B \times_B M \to M$$

with kernel the lattice bundle $P = \coprod_{b\in B} P_b$. Here we view the cotangent bundle T^*B as a bundle of Lie groups over B, and each fibre of this bundle of groups is acting on the corresponding fibre of M. To obtain an effective action we divide by the kernel. The quotient $T = T^*B/P$ is a bundle of tori over B and M is a *T-torsor* in the sense that the action

$$T \times_B M \to M$$

of T on the bundle M is simply transitive on each fibre. Moreover, the subbundle P is a Lagrangian submanifold of T^*B and the cotangent symplectic form on T^*B descends to the quotient bundle of tori T.

The existence of the Lagrangian lattice bundle P means that the base B of our almost Lagrangian fibre bundle $\mathcal{L}$ is equipped with a *tropical affine structure*, i.e. an atlas whose transition maps are in the tropical affine group $\mathbf{G} = \mathbf{GL}(n,\mathbf{Z}) \ltimes \mathbf{R}^n$. (See e.g. [7, Chapter 1].) This atlas gives rise to the *monodromy class* $\mu(P) \in H^1(B,\mathbf{G})$, which depends not on $\mathcal{L}$, but only on the lattice bundle P. The monodromy $\mu(P)$ is trivial if and only if $\mathcal{L}$ admits global action variables, that is to say the tropical affine structure on B is pulled back from $\mathbf{R}^n$ by a local diffeomorphism $p\colon B \to \mathbf{R}^n$.

Just as a circle bundle on a space X is characterized up to isomorphism by its Chern class in $H^1(X,\mathbf{U}(1)) \cong H^2(X,\mathbf{Z})$, a T-torsor M on B is characterized up to isomorphism by its Chern class $c(M) \in H^1(B,T) \cong H^2(B,P)$. (See e.g. [8, Ch. 5].) The following theorem says that every T-torsor possesses a nondegenerate form ω which turns it into an almost Lagrangian fibre bundle, and that the de Rham class of the corresponding twisting form is determined by the Chern class of the torsor.

Theorem. *Every T-torsor $\pi\colon M \to B$ possesses a compatible almost symplectic form ω. Its twisting form $\eta \in Z^3(B)$ satisfies $[\eta] = d_{P,*}c(M)$.*

Here "Z^k" denotes closed k-forms and $d_{P,*}\colon H^2(B,P) \to H^3(B,\mathbf{R})$ is the map in cohomology induced by a differential operator d_P, which is defined as follows: the exterior derivative $d\colon \Omega^1(B) \to Z^2(B)$ kills sections of P (because P is Lagrangian) and so descends to an operator $d_P\colon \Gamma(B,T) \to Z^2(B)$.

Any 2-form σ on B acts on $\mathcal{L} = (M,\omega,\pi)$ by the formula $\sigma\cdot\mathcal{L} = (M,\omega+\pi^*\sigma,\pi)$. This action changes the twisting form by the exact 3-form $d\beta$.

Theorem. *The compatible almost symplectic form on a T-torsor $\pi\colon M \to B$ is unique up to the action of $\Omega^2(B)$.*

The theory of (almost) Lagrangian fibre bundles invites comparison with symplectic toric geometry. The fundamental theorem of symplectic toric geometry, due to Delzant [3], states that a compact symplectic manifold equipped with a completely integrable Hamiltonian torus action is determined up to isomorphism by its moment polytope.

Given a suitable ("Delzant") polytope, one can recover the corresponding symplectic manifold as a symplectic quotient of a linear action of a larger torus on a symplectic vector space. This observation appears to be due to Audin [1]; see also Guillemin [9, Ch.frm[o]–] and Duistermaat and Pelayo [5].

An alternative construction of the Delzant space corresponding to a polytope can be found in [11, § 3.4]: one starts with the cotangent bundle of the torus and performs symplectic cuts (in the sense of Lerman [10]) along all the facets of the polytope. One can replace the cotangent bundle of the torus by an almost Lagrangian fibre bundle and perform a similar cutting process, but this remains to be explored.

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Topological toric manifolds

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(joint work with Hiroaki Ishida, Yukiko Fukukawa)

In this report, a toric manifold is a compact non-singular toric variety. As is well-known, there is a bijective correspondence between toric manifolds and complete non-singular fans. I reported a topological generalization of this classical fact, worked jointly with H. Ishida and Y. Fukukawa ([5]).

Previous to us, two topological analogues of a toric manifold have been introduced and a theory similar to toric geometry is developed for them using topological technique. One topological analogue is what is now called a *quasitoric manifold*[1] introduced by Davis-Januszkiewicz [2] around 1990 and the other is a *torus manifold* introduced by Masuda [6][2] and Hattori-Masuda [3] around 2000.

A quasitoric manifold is a closed smooth manifold M of even dimension, say $2n$, with an effective smooth action of $(S^1)^n$, such that M is locally equivariantly diffeomorphic to a representation space of $(S^1)^n$ and the orbit space $M/(S^1)^n$ is a simple convex polytope. A projective toric manifold with the restricted action of the compact torus is a quasitoric manifold but there are many quasitoric manifolds which do not arise this way. For example $\mathbb{C}P^2\#\mathbb{C}P^2$ with a smooth action of $(S^1)^2$ is quasitoric but not toric because it does not allow a complex (even almost complex) structure, as is well-known. Davis-Januszkiewicz [2] show that quasitoric manifolds M are classified in terms of pairs (Q, v) where Q is a simple convex polytope identified with the orbit space $M/(S^1)^n$ and v is a function on the facets of Q with values in $\mathbb{Z}^n$ satisfying a certain unimodularity condition.

A torus manifold is a closed smooth manifold M of even dimension, say $2n$, with an effective smooth action of $(S^1)^n$ having a fixed point. An orientation datum called an omniorientation is often incorporated in the definition of a torus manifold. The action of $(S^1)^n$ on a toric or quasitoric manifold has a fixed point, so they are torus manifolds. A typical and simple example of a torus manifold which is neither toric nor quasitoric is $2n$-sphere S^{2n} with a natural smooth action of $(S^1)^n$ for $n \geqslant 2$. The orbit space $S^{2n}/(S^1)^n$ is contractible but there are many torus manifolds whose orbit spaces by the torus action are not contractible unlike in the case of toric or quasitoric manifolds. Although the family of torus manifolds is much larger than that of toric or quasitoric manifolds, one can associate a combinatorial object $\Delta(M)$ called a multi-fan to an omnioriented torus manifold M. Roughly speaking, a multi-fan is also a collection of cones but cones may overlap unlike ordinary fans. When M arises from a toric manifold, the multi-fan $\Delta(M)$ agrees with the ordinary fan associated with M. In general, the multi-fan $\Delta(M)$ does not determine M, but it contains a lot of geometrical information on

[1]Davis-Januszkiewicz [2] uses the terminology *toric manifold* but it was already used in algebraic geometry as the meaning of smooth toric variety, so Buchstaber-Panov [1] started using the word *quasitoric manifold*.

[2]In [6], the notion of *unitary toric manifold* is introduced. It is a torus manifold with invariant unitary (or weakly almost complex) structure.

M, e.g. genera of M such as signature, Hirzebruch T_y (or χ_y) genus and elliptic genus can be described in terms of $\Delta(M)$.

In the talk, I explained a third topological analogue of a toric manifold introduced in [5], which we believe is the correct topological analogue. Remember that a toric manifold of complex dimension n is a compact smooth algebraic variety with an effective algebraic action of $(\mathbb{C}^*)^n$ having an open dense orbit. It is known that a toric manifold is covered by finitely many invariant open subsets each equivariantly and algebraically isomorphic to a direct sum of complex one-dimensional *algebraic* representation spaces of $(\mathbb{C}^*)^n$. Based on this observation we define our topological analogue of a toric manifold as follows.

Definition. We say that a closed smooth manifold X of dimension $2n$ with an effective *smooth* action of $(\mathbb{C}^*)^n$ having an open dense orbit is a (compact) *topological toric manifold* if it is covered by finitely many invariant open subsets each equivariantly diffeomorphic to a direct sum of complex one-dimensional *smooth* representation spaces of $(\mathbb{C}^*)^n$.

We remark that there are many more smooth representations of $(\mathbb{C}^*)^n$ than algebraic ones. This stems from the fact that since $\mathbb{C}^* = \mathbb{R}_{>0} \times S^1$ as smooth groups, any *smooth* endomorphism of $\mathbb{C}^*$ is of the form

$$g \mapsto |g|^{b+\sqrt{-1}c}\Big(\frac{g}{|g|}\Big)^v \quad \text{with } (b+\sqrt{-1}c, v) \in \mathbb{C} \times \mathbb{Z} \tag{1}$$

and this endomorphism is algebraic if and only if $b = v$ and $c = 0$. Therefore the group $\mathrm{Hom}(\mathbb{C}^*, \mathbb{C}^*)$ of *smooth* endomorphisms of $\mathbb{C}^*$ is isomorphic to $\mathbb{C} \times \mathbb{Z}$ while the group $\mathrm{Hom}_{alg}(\mathbb{C}^*, \mathbb{C}^*)$ of *algebraic* endomorphisms of $\mathbb{C}^*$ is isomorphic to $\mathbb{Z}$. This implies that topological toric manifolds are much more abundant than toric manifolds.

Nevertheless, topological toric manifolds have similar topological properties to toric manifolds. For instance, the orbit space of a topological toric manifold X by the restricted compact torus action is a manifold with corners whose faces (even the orbit space itself) are all contractible and any intersection of faces is connected unless it is empty, so the orbit space looks like a simple polytope. This implies that the cohomology ring $H^*(X;\mathbb{Z})$ of X is generated by degree two elements as a ring like the toric or quasitoric case.

As a combinatorial counterpart to a topological toric manifold, we introduced in [5] the notion of a *topological fan* generalizing the notion of a simplicial fan in toric geometry. A simplicial fan of dimension n is a collection of simplicial cones in $\mathbb{R}^n$ satisfying certain conditions. It can be regarded as a pair (Σ, v) of an abstract simplicial complex Σ and a map $v\colon \Sigma^{(1)} \to \mathbb{Z}^n$, where Σ is the underlying simplicial complex of the fan, $\Sigma^{(1)}$ is the set of vertices in Σ which correspond to one-dimensional cones in the fan, and v assigns primitive integral vectors lying on the one-dimensional cones. We note that the target group $\mathbb{Z}^n$ of the map v should actually be regarded as the group $\mathrm{Hom}_{alg}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ of *algebraic* homomorphisms from $\mathbb{C}^*$ to $(\mathbb{C}^*)^n$.

We define a topological fan of dimension n to be a pair $\Delta = (\Sigma, \beta)$ of an abstract simplicial complex Σ and a map $\beta\colon \Sigma^{(1)} \to \mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ satisfying certain conditions, where $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ denotes the group of *smooth* homomorphisms from $\mathbb{C}^*$ to $(\mathbb{C}^*)^n$. We may think of a topological fan as a collection of cones in $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ by forming cones $\mathcal{C}$ using the Σ and β. As observed in (1), $\mathrm{Hom}(\mathbb{C}^*, (\mathbb{C}^*)^n)$ is isomorphic to $\mathbb{C}^n \times \mathbb{Z}^n$, so we may regard β as a map to $\mathbb{C}^n \times \mathbb{Z}^n$ and write $\beta = (b + \sqrt{-1}c, v)$ accordingly. Then an ordinary simplicial fan is a topological fan with $b = v$ and $c = 0$. Cones obtained from the pair (Σ, b), which are the projected images of the cones $\mathcal{C}$ on the real part of the first factor of $\mathbb{C}^n \times \mathbb{Z}^n$, do not overlap and define an ordinary simplicial fan over $\mathbb{R}$ while cones formed from the pair (Σ, v), which are the projected images of $\mathcal{C}$ on the second factor of $\mathbb{C}^n \times \mathbb{Z}^n$, may overlap and define a multi-fan. A topological fan Δ is called *complete* if the ordinary fan (Σ, b) is complete, and *non-singular* if the multi-fan (Σ, v) is non-singular, i.e. if $\{v(\{i\})\}_{i \in I}$ is a part of a $\mathbb{Z}$-basis of $\mathbb{Z}^n$ for any $I \in \Sigma$.

Theorem ([5]). There is a bijective correspondence between

$$\{\text{Omnioriented topological toric manifolds of dimension } 2n\}$$

and

$$\{\text{Complete non-singular topological fans of dimension } n\}$$

generalizing the well-known bijection in toric geometry.

Remark. Recently, Zung-Minh [8] note that the above theorem can be recovered from their viewpoint. See [4], [7] for further development on topological toric manifolds.

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Combinatorial questions related to stringy E-polynomials of Gorenstein polytopes

Benjamin Nill
(joint work with Jan Schepers)

1. Motivation

Over the last three decades mirror symmetry has spurred interest in finding the Hodge numbers of irreducible Calabi-Yau manifolds (over $\mathbb{C}$). The 'Hodge diamond' of a Calabi-Yau threefold is completely described by knowing $(h^{1,1}, h^{2,1})$. So far ten thousands of these pairs have been found [5], however, all of them in the range of $h^{1,1} + h^{2,1} \leqslant 502$. Therefore, the main question attributed to Yau is still open:

Are there only finitely many Hodge numbers of n-dimensional irreducible Calabi-Yau manifolds?

2. The Batyrev-Borisov construction and Gorenstein polytopes

The vast amount of examples of CY-folds are *CICY*: (resolutions of) complete intersection Calabi-Yau varieties in toric varieties associated to *reflexive polytopes* $\Delta \subset \mathbb{R}^d$, see [6, 7]. Reflexive polytopes are lattice polytopes that appear as dual pairs [1]. In 1994 Batyrev realized that anticanonical hypersurfaces in these toric varieties are Calabi-Yau, and they can be crepantly resolved (for $d \leqslant 4$). Moreover, he proved that Hodge numbers for Calabi-Yau manifolds constructed in this way by Δ and its dual Δ^* have mirror-symmetric Hodge numbers. In 1996 Batyrev and Borisov generalized these results to CICYs [2]. A CICY Y of codimension r in a Gorenstein toric Fano variety X_Δ is associated to a Minkowski decomposition of a reflexive polytope $\Delta = \Delta_1 + \cdots + \Delta_r$ into lattice polytopes. Batyrev and Borisov showed that the *stringy E-polynomial* of Y

$$E_{st}(Y) := \sum_{p,q} (-1)^{p+q}\, {h_{st}}^{p,q}(Y)\, u^p v^q$$

can be computed in a purely combinatorial way. (Here, ${h_{st}}^{p,q}(Y)$ denote the *stringy Hodge numbers* of the possibly singular Y, see [1, 2]).

In 2001 Borisov and Mavlyutov simplified this formula as follows. Let $P = \Delta_1 * \cdots * \Delta_r$ be the *Cayley polytope* of $\Delta_1, \ldots, \Delta_r$, see [3]. Then P is a $(d+r-1)$-dimensional *Gorenstein polytope of index r*, i.e., rP is a reflexive polytope (up to a translation). Gorenstein polytopes also appear as dual pairs P, P^*, see e.g. [3]. In [4] it is shown that $E_{st}(Y)$ equals

$$E_{st}(P,t) := \frac{1}{(uv)^r} \sum_{\varnothing \leqslant F \leqslant P} (-u)^{\dim F + 1}\, \widetilde{S}(F, u^{-1}v)\, \widetilde{S}(F^*, uv),$$

where F^* is the face of P^* corresponding to F. The occuring $\widetilde{S}$-polynomials will be described in the next section. Notice that this formula makes sense for *any* Gorenstein polytope! In fact, in [3] Batyrev and the first author conjecture that

these stringy E-functions of Gorenstein polytopes should share all the properties of stringy E-polynomials of Calabi-Yau manifolds. In particular:

Theorem 2.1 ([8])**.** *Stringy E-functions of Gorenstein polytopes are polynomials.*

Is it still open, whether the degree of $E_{st}(P,t)$ equals the expected degree $\dim(P)+1-2r$. Coming back to the original motivation, Question 4.21 in [3] asks whether there are (up to multiples) only finitely many stringy E-polynomials of Gorenstein polytopes with fixed number $\dim(P)+1-2r$. This would give an affirmative answer of Yau's conjecture for all CYCIs.

3. A closer look at $\widetilde{S}$-polynomials

Let P be any lattice polytope of dimension d. The polynomial $\widetilde{S}(P,t) \in \mathbb{Z}[t]$ mixes information about lattice points with the combinatorics of the polytope. In [4] it is defined as

$$\widetilde{S}(P,t) := \sum_{\varnothing \leqslant F \leqslant P} (-1)^{d-\dim F} h^*_F(t)\; g_{(F,P]*}(t),$$

where we sum over all faces F of P and where $[F,P]$ denotes the interval in the Eulerian poset of faces of P. Here, the h^*-polynomial is a transform of the famous Ehrhart polynomial counting lattice points in multiples [3], while the toric g-polynomial is associated to posets of faces, see [10]. (Remark: for reasons which will become below we use here Stanley's notation for the g-polynomial which differs from the one used in [3, 8].) Borisov and Mavlyutov [4] showed the following non-obvious result using algebro-geometric reasoning:

Theorem 3.1 ([4])**.** *$\widetilde{S}$-polynomials of lattice polytopes have non-negative coefficients.*

Is there a combinatorial proof of this result?

If $P=\sigma$ is just a simplex, then in [3] it is observed that $\widetilde{S}(\sigma,t)$ equals

$$l^*_\sigma(t) := \sum_{k\in\mathbb{N}} \#\left(\text{interior lattice points of height } k \text{ in } \Pi(S\times 1)\right) t^k,$$

where Π denotes the parallelepiped spanned. By convention, $l^*_\varnothing(t)=1$.

Let $\mathcal{T}$ be a regular lattice triangulation of P. Here is our observation:

$$\widetilde{S}(P,t) = \sum_{\sigma\in\mathcal{T}} l^*_\sigma(t)\; l_{P,\mathcal{T},\sigma}(t),$$

where

$$l_{P,\mathcal{T},\sigma}(t) := \sum_{\sigma\leqslant F\leqslant P} (-1)^{d-\dim(F)}\; h_{\mathrm{lk}_{\mathcal{T}|_F}(\sigma)}(t)\; g_{(F,P]*}(t).$$

Conjecture: $l_{P,\mathcal{T},\sigma}(t)$ has non-negative coefficients.

This would give a canonical way to write $\widetilde{S}(P,t)$ as a sum of nonnegative polynomials clearly separating the lattice data from the combinatorial information. Some final remarks:

(1) For $\sigma = \varnothing$, $l_{P,\mathcal{T},\varnothing}(t)$ equals precisely Stanley's *local h-polynomials*, see [10]:

$$\sum_{\varnothing \leqslant F \leqslant P} (-1)^{d-\dim(F)}\, h_{\mathcal{T}|_F}(t)\, g_{(F,P]*}(t).$$

One of the main results in [10] is the proof of their non-negativity.

(2) The previous definition looks similar to the definition of the $\widetilde{S}$-polynomials. Indeed, the $\widetilde{S}$-polynomials are exactly the Ehrhart analogues of local h-polynomials as defined by Stanley in [10]! He conjectured in Conjecture 7.14 in [10] also their non-negativity which was later proved by Karu [11] in 2008 – five years after the (at that time unnoticed) proof of Borisov and Mavlyutov.

(3) Karu proved in [11, Corollary 1.2] in our notation

$$h^*_P(t) = \sum_{\varnothing \leqslant F \leqslant P} \widetilde{S}(F,t)\, g_{[F,P)}(t),$$

a result proved independently by the second author in [9, Prop.2.9].

(4) As an immediate corollary of these observations we can prove an (unpublished) conjecture of Stapledon: If $\mathcal{T}$ is unimodular, then $\widetilde{S}(P,t)$ is unimodal. Namely, since $l^*_\sigma = 0$ for $\sigma \neq \varnothing$ in this case, $\widetilde{S}(P,t)$ equals $l_{P,\mathcal{T},\varnothing}$ which is unimodal by [10, Thm.7.9].

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Autoequivalences of toric surfaces

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(joint work with Nathan Broomhead)

The derived category $D^b(X)$ of a smooth, projective variety X can be interpreted in two ways: On the one hand, it provides a natural homological structure sitting above K-theory $K(X)$ or cohomology $H^*(X, \mathbb{Q})$. On the other hand, derived categories give rise to an interesting equivalence relation for varieties X and Y, by way of $D^b(X) \cong D^b(Y)$; this is called 'Fourier-Mukai partners' or 'derived equivalence'. From both points of view, it is natural to study derived symmetries of X, i.e. autoequivalences of the derived category $D^b(X)$.

The geometric automorphisms $\mathrm{Aut}(X)$ appear as autoequivalences via $f \mapsto Rf_*$. Furthermore, every line bundle $L \in \mathrm{Pic}(X)$ gives rise to the line bundle twist $L \otimes \cdot$, which is an exact autoequivalence of $\mathrm{Coh}(X)$ and hence an element of $\mathrm{Aut}(D^b(X))$. Furthermore, the shift functor $[1]$ is an obvious autoequivalence of $D^b(X)$. These types of symmetries are available on any variety and are thus called 'standard'; they combine to the subgroup

$$A(X) := (\mathrm{Pic}(X) \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1].$$

Certain varieties possess a larger class of derived symmetries. Here are some known results for surfaces (always smooth and projective over an algebraically closed field; we write $\mathbb{C}$): If X is a del Pezzo surface, i.e. ω_X^{-1} ample, or has ample ω_X, a condition stronger than being of general type, then $\mathrm{Aut}(D^b(X)) = A(X)$ by a famous result of Bondal and Orlov [1]. By contrast, abelian surfaces always have autoequivalences beyond the standard ones by Mukai's classical [8] and the same holds for K3 surfaces due to the existence of spherical twists [10].

We describe the autoequivalences of toric surfaces. As the anti-canonical bundle ω_X^{-1} of a toric surface X is big, we expect rather few non-standard derived symmetries. They may exist, however, and part of our result is that they will be related to spherical twists coming from -2-curves. Assume that $C \subset X$ is a -2-curve, i.e. a smooth, rational curve with $C^2 = -2$. Then the structure sheaf of C has the following properties:

$$\mathcal{O}_C \otimes \omega_X = \mathcal{O}_C,\ \mathrm{Hom}(\mathcal{O}_C, \mathcal{O}_C) = \mathbb{C},\ \mathrm{Ext}^1(\mathcal{O}_C, \mathcal{O}_C) = 0,\ \mathrm{Ext}^2(\mathcal{O}_C, \mathcal{O}_C) = \mathbb{C},$$

where the first equality follows from adjunction, the second from C being connected, the third from C being rigid and the last from Serre duality. An object with these properties is called spherical [10] and gives rise to an autoequivalence of $D^b(X)$, the *spherical twist* $\mathsf{T}_{\mathcal{O}_C}$, defined by distinguished triangles coming from the cones of the natural evaluation maps

$$\mathrm{Hom}^\bullet(\mathcal{O}_C, A) \to \mathcal{O}_C \to A \to \mathsf{T}_{\mathcal{O}_C}(A)$$

for any $A \in D^b(X)$. This actually is a functor; see for example [5, §8.1]. It should be thought of a categorical lift of the reflection in $K(X)$ along the class $[\mathcal{O}_C]$ of Euler-square $\chi(\mathcal{O}_C, \mathcal{O}_C) = 2$. This interpretation is supported by the easily checked properties $\mathsf{T}_{\mathcal{O}_C}(\mathcal{O}_C) = \mathcal{O}_C[1]$ and $\mathsf{T}_{\mathcal{O}_C}(A) = A$ whenever $\mathrm{Hom}^\bullet(\mathcal{O}_C, A) =$

0. Note that line bundles of any degree on C, considered as torsion sheaves on X, are spherical; in particular, $\mathcal{O}_C(-1)$ is.

After introducing some notation

$$\begin{aligned} \Delta(X) &:= \{C \subset X \text{ irreducible } (-2)\text{-curve}\}, && \text{a possibly infinite set;} \\ \mathrm{Pic}_\Delta(X) &:= \langle \mathcal{O}_X(C) \mid C \in \Delta \rangle, && \text{as a subgroup of } \mathrm{Pic}(X); \\ B(X) &:= \langle \mathsf{T}_{\mathcal{O}_C(i)} \mid C \in \Delta(X), i = 0, -1 \rangle, && \text{as a subgroup of } \mathrm{Aut}(D^b(X)). \end{aligned}$$

we are ready to present our first main result:

Theorem 1. *Let X be a smooth, projective surface and consider the conditions*

(1) The anti-canonical bundle is big.

(2) The -2-curves on X form disjoint chains of type A.

(3) $\mathrm{Pic}(X) \cong \mathrm{Pic}_\Delta(X) \oplus P$ where P is an $\mathrm{Aut}(X)$-invariant complement.

If X satisfies (1) and (2) then $\mathrm{Aut}(D^b(X))$ is generated by $\mathrm{Pic}(X)$, $\mathrm{Aut}(X)$, $\mathbb{Z}[1]$ and $B(X)$. If X satisfies (1)–(3) then there is the following decomposition of $\mathrm{Aut}(D^b(X))$

$$\mathrm{Aut}(D^b(X)) = B(X) \rtimes (P \rtimes \mathrm{Aut}(X)) \times \mathbb{Z}[1].$$

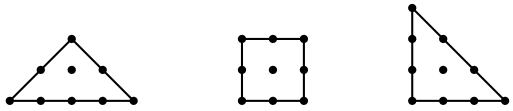
The application to toric surfaces will be given in a moment. We point out that this is one of very few cases where the autoequivalence group of a variety can be fully described — apart from the minimal case of ample $\omega_X^{\pm 1}$ already mentioned above, the structure is known only for abelian varieties, a result of Orlov [9].

Let us state that condition (1) is needed to invoke Kawamata's [7]. Condition (2) allows to draw on [6] of Ishii and Uehara where they also show the general relation $\mathsf{T}_{\mathcal{O}_C}\mathsf{T}_{\mathcal{O}_C(1)}(A) = \mathcal{O}_X(C) \otimes L$. This leads to $\mathrm{Pic}(X) \cap B(X) = \mathrm{Pic}_\Delta(X)$, and [6] also prove $\mathrm{Aut}(X) \cap B(X) = 1$. These properties hint at the semi-direct product decomposition for $\mathrm{Aut}(D^b(X))$ of the theorem but we need condition (3) in order to make it work.

Turning to toric surfaces, conditions (1) and (2) are well-known in toric geometry. Regarding (3), recall that toric varieties were introduced by Demazure [3] who completely described their automorphism groups. We only need a corollary of this achievement: $\mathrm{Aut}(X)$ is generated by its identity component together with $\mathrm{Aut}(\Sigma(X))$, the group of automorphisms of a fan Σ for X.

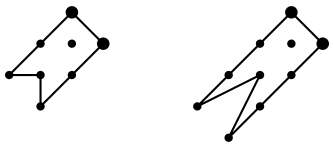
Theorem 2. *If X is a smooth, projective, toric surface, then the conditions (1) and (2) of Theorem 1 are satisfied. All but three such surfaces admit a splitting $\mathrm{Pic}_\Delta(X) \subset \mathrm{Pic}(X)$. An $\mathrm{Aut}(X)$-invariant complement exists if and only if an $\mathrm{Aut}(\Sigma(X))$-invariant complement exists.*

We describe the three exceptions mentioned in the theorem by drawing the lattice generators for the rays in their fans as polygons:



We follow with a series of examples where invariant splittings do exist: This is the case for 'generic' toric surfaces, i.e. when the group of fan automorphisms is

trivial. For a more interesting case, suppose $\mathrm{Aut}(\Sigma(X)) = \mathbb{Z}/2$ and that the action exchanges two rays which do not correspond to -2-curves, and whose generators form a $\mathbb{Z}$-basis for N. T series of toric surfaces given by fans over the following polygons satisfy these conditions:



Excluding the two marked curves, the remaining torus invariant divisors form a basis for $\mathrm{Pic}(X)$, and the subset of these divisors which are not -2-curves generate an $\mathrm{Aut}(\Sigma(X))$-invariant complement to $\mathrm{Pic}_\Delta(X)$.

For the following example, computer algebra was used to make sure that no invariant complement exists. This shows that condition (3) of Theorem 1 is necessary.

Finally, we mention that [2, §5] deals with further surfaces to which Theorem 1 applies. These are certain complexity one surfaces, i.e. having a $\mathbb{C}^*$ action but not necessarily toric.

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After the hike on Wednesday evening there was a session for younger researchers to talk about their work in the form of a ten minute talk followed by five minutes of discussion. This is the collection of their abstracts.

On the Isotropy of Stacky Polytopes

Rebecca Goldin
(joint work with Megumi Harada, David Johannsen, and Derek Krepski)

A *stacky polytope* (introduced in [1] following [2]) is a triple (N, Δ, β), where N is a rank d finitely generated abelian group, $\Delta \subset (N \times \mathbb{R})^* \cong \mathbb{R}^d$ is a rational simple polytope with n facets, and β is a map $\mathbb{Z}^n \to N$ with finite cokernel, sending basis vectors e_i to vectors normal (not nec. primitive) to facets of Δ. To a stacky polytope, one can associate a toric Deligne-Mumford stack $\mathcal{X}$ realized as a symplectic reduction $[\mathbb{C}^n /\!/ G]$ of $\mathbb{C}^n$ by a linear action of a compact (not nec. connected) abelian Lie group G.

We characterized the combinatorial conditions under which an associated toric DM stack $\mathcal{X}$ is a global quotient of a manifold by a finite group. At every vertex p of the polytope, let V_p the integral span of $\{\beta(e_i)\}$ of the normal vectors *associated to the facets touching* p. Then $\mathcal{X}$ is a global quotient if an only if V_p equals the $\mathbb{Z}$-span of *all* the normal vectors $\{\beta(e_i)\}_{i=1,\dots,n}$. This generalizes a condition known when N is free.

References

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Lower bounds for Gromov width of $U(n)$ and $SO(n)$ coadjoint orbits.

Milena Pabiniak

Gromov width of a symplectic manifold (M^{2m}, ω) is

$$Gromov(M^{2m}, \omega) = \sup\{\pi r^2 \mid (B^{2m}(r), \omega_{standard}) \hookrightarrow (M^{2m}, \omega) \text{ symplectically } \}.$$

Let $G = U(n)$ or $SO(n)$, and T be its maximal torus. The coadjoint orbit, $\mathcal{O}_\lambda$, through $\lambda \in \mathfrak{t}^*$ is a symplectic manifold, with Kostant-Kirillov form ω_{KK}. We use the torus action coming from the Gelfand-Tsetlin system to construct symplectic embeddings of balls and prove

$$Gromov(\mathcal{O}_\lambda, \omega_{KK}) \geqslant \min\{\langle \alpha_j^\vee, \lambda \rangle; \alpha_j \text{ a coroot}, \langle \alpha_j^\vee, \lambda \rangle > 0\}.$$

In many known cases the Gromov width is given by the above formula (complex Grassmannians [KT], complete flag manifolds satisfying some additional integrality conditions [Z]). It is also an upper bound in the case of regular orbits of any compact connected Lie group (with additional integrality condition; [Z]).

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Twist functors

Andreas Hochenegger

(joint work with David Ploog)

In this talk, I reported on a work in progress together with David Ploog.

The starting point was the following observation. Let $\pi\colon \tilde{\mathcal{F}}_2 \to \mathcal{F}_2$ a blowup of the second Hirzebruch surface in a point on the unique (-2)-curve $C \subset \mathcal{F}_2$. We can associate to this curve C an autoequivalence of $\mathcal{D}^b(\mathcal{F}_2)$, the so-called *spherical twist* T_C as introduced by Seidel and Thomas in [1]. This notion is not stable under blowups. Namely, the twist functor T_F associated to $F = \pi^*C$ is no longer an equivalence.

But there is a decomposition $\mathcal{D}^b(\tilde{\mathcal{F}}_2) = \langle \mathcal{Q}_F, \mathcal{D}_F \rangle$ such that T_F becomes an equivalence after restriction to the two parts. Moreover, this decomposition coincides with $\mathcal{D}^b(\tilde{\mathcal{F}}_2) = \langle \mathcal{O}_E(-1), \mathcal{D}^b(\mathcal{F}_2) \rangle$, where E is the exceptional divisor.

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Torus actions on complex manifolds

Hiroaki Ishida

Whenever a compact torus $G = (S^1)^m$ acts on a connected smooth manifold M effectively, it follows from the slice theorem that each orbit $G \cdot x$ satisfies that $\dim_{\mathbb{R}} G \cdot x \geqslant 2m - \dim_{\mathbb{R}} M$.

Suppose that M is a compact connected complex manifold of complex dimension n and that $G = (S^1)^m$-action on M preserves the complex structure on M. If the equality above holds for some $x \in M$, then we can find a G-equivariant principal $\mathbb{C}^{m-n}$-bundle $\mathbb{C}^{m-n} \to M_\Delta \to M$, where M_Δ is a non-singular toric variety of complex dimension m. In case $m = n$, this result is nothing but the main theorem in [1] which states that a compact connected complex manifold M of complex dimension n with an effective $(S^1)^n$-action preserving the complex structure on M and having a fixed point is actually a complete nonsingular toric variety.

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Partially ample line bundles on toric varieties

Nathan Broomhead

(joint work with Artie Prendergast-Smith)

The notion of ampleness is fundamental to algebraic geometry. There is a cohomological characterizsation (Serre): a line bundle is ample if a sufficiently high power of it kills cohomology of any given coherent sheaf in positive degrees. A natural extension is the idea of a q-ample line bundle for $q \in \mathbb{N}$: those for which a sufficiently high power kills cohomology in degrees above q. It remains an open problem to give a good description of the (not necessarily convex) cone of q-ample line bundles on a variety. In [1] we give such a description in the case of projective toric varieties. We use results of [2] and a toric method of calculating cohomology.

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GKM-sheaves and equivariant cohomology

Thomas Baird

In [2] Goresky, Kottwitz and MacPherson observed that for a large class of interesting T-spaces X (e.g. toric manifolds and flag varieties), the equivariant cohomology is encoded in a graph Γ_X (now called *GKM-graphs*). In recent work [1] this framework was expanded to assign GKM data to a class of T-spaces that includes all compact, smooth T-manifolds. To a T-manifold we define a *GKM-hypergraph* equipped with *GKM-sheaf* F_X such that $H^*_T(X) \cong H^0(F_X)$ if X is equivariantly formal.

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Losev-Manin moduli spaces and toric varieties associated to root systems

Mark Blume

Losev and Manin introduced fine moduli spaces $\overline{L}_n$ of stable n-pointed chains of projective lines. The moduli space $\overline{L}_n$ is isomorphic to the toric variety $X(A_{n-1})$ associated with the root system A_{n-1}. In general, a root system R of rank n defines an n-dimensional smooth projective toric variety $X(R)$ associated with its fan of Weyl chambers. We discuss the relation between $\overline{L}_n$ and $X(A_{n-1})$, and generalisations of the Losev-Manin moduli spaces for the other families of classical root systems. Further we consider moduli stacks of pointed chains of projective lines related to the Losev-Manin moduli spaces that coincide with certain toric stacks described in terms of the Cartan matrices of root systems of type A.

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