Introduction to Computational Stochastic PDEs, CUP, 2014

Here are the typos/errors that we know about in the first edition. Detailed corrections to MATLAB codes are given on-line[1]. Let us know if you find anymore.
12th September 2024

Chapter 1. **p17** In Definition 1.60, the Hilbert–Schmidt norm should be defined as

$$\|L\|_{\mathrm{HS}(U,H)} \coloneqq \left(\sum_{j=1}^{\infty} \|L\phi_j\|^2\right)^{1/2},$$

(the norm on the right is the one in H not in U).

p21 Lemma 1.78 (Dini's lemma) requires additionally the assumption that f is continuous (to guarantee that $f(x) - f_n(x) \geq \epsilon$ for the limit point x). In the application of Dini's lemma for the proof of Theorem 1.80 (Mercer's theorem), this holds true for $g(x) = G(x, x)$.

Chapter 2. **p42** Figure 2.1 is misprinted.

p56 In the first displayed equation of the proof, the first term in the integrand should be $p(x)(e(x)')^2$ (the $e(x)$ has one too many dashes).

p79 The right-hand side of (2.109) should be $c_1|\hat{u} - I_h\hat{u}|^2_{H^2(\Delta^*)}$.

Assumption 2.64 in general can only be verified for constant boundary data g; H^2-regularity results usually include an extra term on the right-hands side to account for $g \neq 0$ – see (Renardy and Rogers, 2004, Theorem 8.53) or (McLean, 2000, Theorem 4.10).

Chapter 3. **p116** `meshgrid` is misused in Example 3.40 and Algorithm 3.6. See `exa_3.40.m`.

p132 Following the comment on p487 below, the last line of the proof should be

$$\|u(t_n) - \tilde{u}_n\| \leq E\ \exp(Ln\Delta t).$$

Chapter 4. **p159** Nensen's inequality → Jensen's inequality (in proof of T4.58 (iii)).

p179 Bayes' theorem is due to the Reverend Thomas Bayes and the apostrophe is written after and not, as in Exercise 4.11, before the s. In the same exercise, $p_{X,Y}$ is incorrectly defined and it should be

$$\mathbb{P}(X = x_k, Y = y_j) = P_{X,Y}(k, j)$$

(interchange j and k).

Chapter 5. **p185** In the second displayed equation on the left-hand side, delete the comma

p203 By "If the process is Gaussian" in Theorem 5.29, we mean X is Gaussian in the sense of Definition 4.38, which immediately implies that the ξ_j are Gaussian. It can be shown that, if all the finite-dimensional distributions are Gaussian, and sample paths $X \in L^2(\mathfrak{T})$, then X is also Gaussian in the sense of Definition 4.38. See Theorem 2 of Rajput, B. S. & Cambanis, S. Gaussian processes and Gaussian measures. Ann. Math. Stat. 43, 1944–1952.

p204 The uniform convergence in (5.39) is a consequence of Mercer's Theorem (Theorem 1.80). In particular, write $X(t) - X_J(t) = X(t) - \mu(t) - \sum_{j=1}^{J} \sqrt{\nu_j}\phi_j(t)\xi_j$. Then, $\mathbb{E}(X(t) - X_J(t))^2 = C(t,t) - \sum_{j=1}^{J} \nu_j|\phi_j(t)|^2 \geq 0$ from the KL expansion (as ξ_j has mean zero and unit variance). This converges to zero as $J \to \infty$ uniformly in t by Mercer's Theorem (or directly by Lemma 1.78 as C and ϕ_j are continuous).

Chapter 6. **p235** In Algorithm 6.3, the normalisation by variance in $\Delta\nu$ in the increments $\Delta\mathcal{W}_j$ is missing. To account for this, as $f(\nu_j)\,\Delta\nu_j = (\ell/2\pi) \times 2(\pi/\ell)/(J-1) = \frac{1}{J-1}$, the last line should be `Z=Z*sqrt(1/(J-1))`. See also `quad_sinc.m`.

[1] `https://github.com/tonyshardlow/picspde`

p239 In Equation (6.34), the outer sum over k has not been carried over from the previous line and should read

$$\tilde{Z}_R(t_n) = \sum_{k=0}^{N-1} e^{\mathrm{i}2\pi kn/N} \sum_{m=0}^{M-1} e^{\mathrm{i}(-R+m\Delta\nu)\,t_n} \sqrt{f(\nu_{kM+m})}\, \Delta \mathcal{W}_{kM+m}.$$

p279 The first two entries in the second column of the matrix V displayed above Example 7.41 should be $\tilde{u}_{2n_1+1}$ and $\tilde{u}_{2n_1+2}$, not $\tilde{u}_{2n_2+1}$ and $\tilde{u}_{2n_2+2}$.

Chapter 7. **p291** In Algorithm 7.10 (`turn_band_wm.m`), `f` should be an even function of `s` (missing modulus).

p305 In Lemma 7.67, the quantity $M_0(u)$ is infinite as stated and cannot be used for proving Theorem 7.68. We show here how to prove Theorem 7.68 using the Garsia–Rodemich-Rumsey (GRR) inequality for $D = (0, 1)$. For an alternative approach, see M. Talagrand, 'Lower Bounds for Stochastic Processes', Springer, 2021 (e.g., Section 2.5 Continuity of Gaussian Processes).

Let $\Psi(t) = \exp(t^2)$, and $p(t) = t^q$, for $t \in [0, 1]$.

(Note: GRR applies to a wider class of Ψ and p, which we do not describe here — see Garsia, Rodemich, and Rumsey (1971) for details. GRR can also be developed for dimension $d > 1$ — see Garsia, 'Continuity properties of Gaussian processes with multidimensional time parameter', Berkeley Symp. on Math. Statist. and Prob., 1972).

Lemma (Garsia–Rodemich–Rumsey)**.** *Let f be a continuous function on $[0, 1]$ and suppose that*

$$M_0(f) = \int_0^1 \int_0^1 \Psi\left(\frac{|f(s) - f(t)|}{p(s-t)}\right) ds\, dt < \infty.$$

Then, for all $s, t \in [0, 1]$,

$$|f(s) - f(t)| \le \int_0^{|s-t|} \Psi^{-1}\left(\frac{4M_0(f)}{r^2}\right) dp(r).$$

Lemma (Lemma 7.67 revised for $D = (0, 1)$)**.** *Let $D = (0, 1)$. Fix $q, \epsilon > 0$. For a constant C_ϵ depending only on ϵ and q, we have, for any $u \in C(\overline{D})$,*

$$|u(x) - u(y)| \le \left(C_\epsilon + \sqrt{\log 4M_0}\right) |x-y|^{q-\epsilon}, \qquad x, y \in D, \tag{1}$$

where we assume

$$M_0(u) \coloneqq \int_0^1 \int_0^1 \exp\left(\frac{u(x) - u(y)}{|x-y|^q}\right)^2 dx\, dy < \infty. \tag{2}$$

Proof. Then,

$$M_0(u) = \int_0^1 \int_0^1 \exp\left(\frac{u(x) - u(y)}{|x-y|^q}\right)^2 dx\, dy = \int_0^1 \int_0^1 \Psi\left(\frac{|u(x) - u(y)|}{p(x-y)}\right) dx\, dy$$

for $\Psi(t) = \exp(t^2)$ and $p(t) = t^q$. $M_0(u)$ is finite by assumption and the GRR lemma applies. Note that $\Psi(t) \ge 1$ for $t \ge 0$ and so $M_0(u) \ge 1$. Further, $\Psi^{-1}(t) = \sqrt{\log(t)}$ and $p'(t) = q\, t^{q-1}$ for all $t \ge 1$. For any $\epsilon > 0$, there exists C_ϵ such that, for any $r \in (0, 1)$,

$$\begin{aligned}
\sqrt{\log(4M_0(u)/r^2)} &= \sqrt{\log(4M_0(u)) + 2|\log r|} \\
&\le \sqrt{\log(4M_0(u))} + \frac{1}{\log 4M_0(u)} |\log r| \\
&\le \sqrt{\log(4M_0(u))} + C_\epsilon r^{-\epsilon}.
\end{aligned}$$

(as $\log 4M_0(u) > 1$ and $\sqrt{a+b} \le \sqrt{a} + b/2\sqrt{a}$ for $a, b > 0$). Then, from GRR, for any $x, y \in D = (0, 1)$,

$$\begin{aligned}
|u(x) - u(y)| &\le \int_0^{|x-y|} \sqrt{\log\left(\frac{4M_0(u)}{r^2}\right)}\, q\, r^{q-1}\, dr \\
&\le \sqrt{\log 4M_0(u)} |x-y|^q + C_\epsilon \frac{q}{q-\epsilon} |x-y|^{q-\epsilon}.
\end{aligned}$$

For a possibly different C_ϵ, this completes the proof of (1). □

This version of Lemma 7.67 can be used in the proof of Theorem 7.68.

Theorem (Theorem 7.68 revised)**.** *Let D be a bounded domain and $\{u(\boldsymbol{x})\colon \boldsymbol{x} \in \overline{D}\}$ be a mean-zero Gaussian random field such that, for some $L, s > 0$,*

$$\mathbb{E}\big[|u(\boldsymbol{x}) - u(\boldsymbol{y})|^2\big] \le L\|\boldsymbol{x} - \boldsymbol{y}\|_2^s, \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \overline{D}. \tag{3}$$

For any $p \ge 1$, there exists a random variable K such that $e^K \in L^p(\Omega)$ and

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| \le K(\omega)\|\boldsymbol{x} - \boldsymbol{y}\|_2^{(s-\epsilon)/2}, \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \overline{D}, \quad a.s. \tag{4}$$

Proof. We can find a truncated Karhunen–Loève expansion $u_J(\boldsymbol{x})$ such that $u_J(\cdot, \omega) \in C(\overline{D})$ for all $\omega \in \Omega$. Fix $p \ge 1$ and let

$$\Delta_J(\boldsymbol{x}, \boldsymbol{y}) := \frac{u_J(\boldsymbol{x}) - u_J(\boldsymbol{y})}{\sqrt{8\,L\|\boldsymbol{x} - \boldsymbol{y}\|_2^s}}, \qquad \boldsymbol{x}, \boldsymbol{y} \in \overline{D}.$$

The random variables in the Karhunen–Loève expansion are *iid* with mean zero and hence $\mathbb{E}\big[\Delta_{J+1} \,\big|\, \Delta_J\big] = \Delta_J$. Further, the function $t \mapsto \exp(t^2)$ is convex and Jensen's inequality implies that $\mathbb{E}\Big[e^{\Delta_{J+1}^2} \,\Big|\, \Delta_J\Big] \ge e^{\Delta_J^2}$. That is, $e^{\Delta_J^2}$ is a submartingale and Doob's submartingale inequality gives

$$\mathbb{E}\left[\sup_{1 \le j \le J} e^{2\Delta_j(\boldsymbol{x},\boldsymbol{y})^2}\right] \le 4\mathbb{E}\Big[e^{2\Delta_J(\boldsymbol{x},\boldsymbol{y})^2}\Big].$$

Note that $\mathbb{E}\big[\Delta_J(\boldsymbol{x}, \boldsymbol{y})^2\big] \le L\|\boldsymbol{x} - \boldsymbol{y}\|_2^s / 8\,L\|\boldsymbol{x} - \boldsymbol{y}\|_2^s \le 1/8$ for every $\boldsymbol{x}, \boldsymbol{y}$. Consequently, $\Delta_J \sim \mathrm{N}(0, \sigma^2)$ where the variance satisfies $\sigma^2 \le 1/8$. As $1/2\sigma^2 - 2 \ge 1/4\sigma^2$,

$$\mathbb{E}\Big[e^{2\Delta_J(\boldsymbol{x},\boldsymbol{y})^2}\Big] = \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{2x^2} e^{-x^2/2\sigma^2}\, dx \le \frac{1}{\sqrt{2\pi\sigma^2}} \int_{\mathbb{R}} e^{-x^2/4\sigma^2}\, dx = \sqrt{2}.$$

We conclude that

$$\mathbb{E}\left[\sup_{1 \le j \le J} e^{2\Delta_j(\boldsymbol{x},\boldsymbol{y})^2}\right] \le \sqrt{2}, \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \overline{D}.$$

Let

$$M_1(u) := \int_D \int_D \sup_{J \in \mathbb{N}} e^{2\Delta_J(\boldsymbol{x},\boldsymbol{y})^2}\, d\boldsymbol{x}\, d\boldsymbol{y}.$$

As D is bounded, $M_1 \in L^1(\Omega)$. Therefore, by Lemma 7.67 (revised), we have

$$|u_J(\boldsymbol{x}, \omega) - u_J(\boldsymbol{y}, \omega)| \le K(\omega)\|\boldsymbol{x} - \boldsymbol{y}\|_2^{(s-\epsilon)/2}, \qquad \forall \boldsymbol{x}, \boldsymbol{y} \in \overline{D}.$$

for $K(\omega) = \sqrt{8L}(\sqrt{\log 4M_1(u)} + C_\epsilon)$. We may write $\exp(pK) = \phi(M_1)$ where

$$\phi(t) = \exp(p\sqrt{8L}(\sqrt{\log 4t} + C_\epsilon)).$$

The function ϕ is concave and so, by Jensen's inequality, $\mathbb{E}[\exp(pK)] = \mathbb{E}[\phi(M_1)] \le \phi(\mathbb{E}[M_1]) < \infty$. We conclude that $e^K \in L^p(\Omega)$.

As $u_J(\boldsymbol{x}) \to u(\boldsymbol{x})$ as $J \to \infty$ almost surely, this gives (**??**). □

Chapter 8.

Chapter 9.

Chapter 10. **p442–469** `meshgrid` is misused in Examples 10.12 and 10.40 and in Algorithms 10.5 and 10.10. See `exa_10.40.m`.

Appendix.

p487 The discrete Gronwall inequality (Lemma A.14) is incorrect. It should either be

Lemma 1. *Consider $z_n \geq 0$ such that $z_n \leq a + bz_{n-1}$ for $n = 1, 2, \ldots$ and $a, b \geq 0$. If $b = 1$, then $z_n \leq z_0 + na$. If $b \neq 1$, then*

$$z_n \leq b^n z_0 + \frac{a}{1-b}(1 - b^n).$$

or

Lemma 2. *Consider $z_n \geq 0$ such that*

$$z_n \leq a + b \sum_{k=0}^{n-1} z_k, \quad \text{for } n = 0, 1, 2 \ldots \tag{*}$$

and constants $a, b \geq 0$. Then, $z_n \leq a(1+b)^n \leq a \exp(bn)$.

The first lemma is (Stuart and Humphries, 1997, Theorem 1.1.12).

To prove the second lemma, notice it is true for $n = 0$. Assume it is true for $z_0, \ldots, z_{n-1}$. Then,

$$\begin{aligned}
z_n &\leq a + b \sum_{k=0}^{n-1} z_k \quad \text{by (*)} \\
&\leq a + b \sum_{k=0}^{n-1} a(1+b)^k \quad \text{by induction assumption} \\
&= a + ab \frac{1-(1+b)^n}{1-(1+b)} \quad \text{by geometric summation formula} \\
&\leq a + ab \frac{1-(1+b)^n}{-b} = a + ab((1+b)^n - 1) = a(1+b)^n.
\end{aligned}$$

Therefore, $z_n \leq a(1+b)^n$ for all $n = 0, 1, 2, \ldots$ by induction. Finally, $z_n \leq a \exp(bn)$ as $1 + x \leq \exp(x)$ for $x \geq 0$. The second lemma can be used in the proof of Theorems 3.55 and 10.34.