Stochastic Perturbations of the Allen–Cahn equation

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Abstract

Consider the Allen-Cahn equation with small diffusion ϵ^2 perturbed by a space time white noise of intensity σ. In the limit, $\sigma/\epsilon^2 \to 0$, solutions converge to the noise free problem in the L_2 norm. Under these conditions, asymptotic results for the evolution of phase boundaries in the deterministic setting are extended, to describe the behaviour of the stochastic Allen-Cahn PDE by a system of stochastic differential equations. Computations are described, which support the asymptotic derivation.

Key Words dynamics of phase-boundaries, stochastic partial differential equations, Asymptotics.

AMS Subject Classifications 60H15, 74N20, 45M05.

1 Introduction

Consider the Itô stochastic partial differential equation

$$\begin{aligned} du =& \Big[\epsilon^2 u_{xx} + f(u)\Big]\, dt + \sigma\, dW(t),\\ u_x =& 0 \quad \text{at } x = 0, 1, \qquad u = g \text{ at } t = 0. \end{aligned} \tag{1.1}$$

where $\epsilon \ll 1$, the system is gradient $f = -\nabla F(u)$, and W is a space–time white noise. Thus, if e_i is an orthonormal basis for $L_2(0,1)$ and β_i are IID standard Brownian motions then

$$W(t) = \sum e_i \beta_i(t).$$

A full introduction to space-time white noise and the theory of stochastic PDEs is given by [4]. The potential F will be a double well potential having wells of equal depth and minima at $s_\pm$. We have in mind particularly

$$F(u) = \frac{1}{8}(1-u^2)^2, \quad f(u) := -\nabla F(u) = \frac{1}{2}(u - u^3), \tag{1.2}$$

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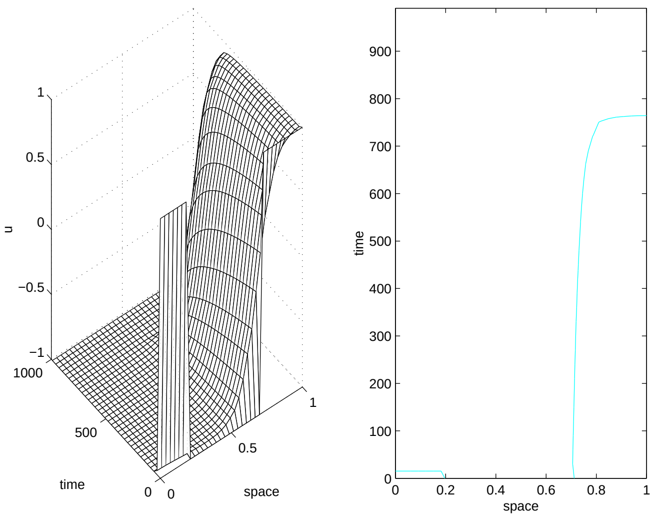

Figure 1: Solution of stochastic Allen–Cahn: $\epsilon = 0.08$, $\sigma = 0.00$: surface and contour plots. Computed $\Delta t = 0.005$ and $\Delta x = 0.008$

which when substituted in (1.1) yields the Allen–Cahn equation and where $s_{\pm} = \pm 1$.

Figures 1–5 show typical solutions of the Allen–Cahn equation with small noise and with Neumann conditions. The right hand figure gives a tracking of the interface position, defined as the contour $u = 0$. The solutions were computed using the backward Euler finite difference scheme described in [11]; in the figures Δt denotes time step and Δx denotes the grid spacing of the discretisation. The initial condition consists of three regions, two taking value $+1$ and the third taking value -1. For the unperturbed equation $\sigma = 0$, eventually the two inner interfaces disappear, leaving a single region where the solution is approximately -1 away from the boundary. With homogeneous Dirichlet boundary conditions, there would be a boundary layer, where the solutions changes rapidly at the boundary, to satisfy the boundary condition.

There are many results for the equation in case $\sigma = 0$. The equation was originally written down as a model of the evolution of the alignments in crystals [1]. Chafee-Infante [3] study the equation on a bounded domain as a bifurcation problem in the limit $\epsilon \to 0$. The equation is shown to have only two stable equilibria for ϵ sufficiently small, corresponding to solutions of only one phase. New equilibria are created as $\epsilon \to 0$, but all are unstable. The equation exhibits meta stability, meaning that solutions quickly move to a state where u takes values near the minima of F except at interfacial layers of width ϵ. These states are not equilibria, but do persist for exponentially long amounts of time. The evolution of the meta stable states has been described as an ODE in the positions of the interface by a number of authors [12, 6, 2].

The effect of perturbations on the Allen-Cahn equation has been studied previously by Laforgue-O'Malley [8, 7] and Reyna-Ward [10, 13]. These papers discuss small deterministic perturbations of the operator and indicate that metastability is very sensitive to perturbation. The present work tackles stochastic perturbations, but brings out a similar result, that the exponential drift responsible for the metastability may be dominated by noise.

To accurately describe the nature of the ODE approximation to (1.1) with $\sigma = 0$,

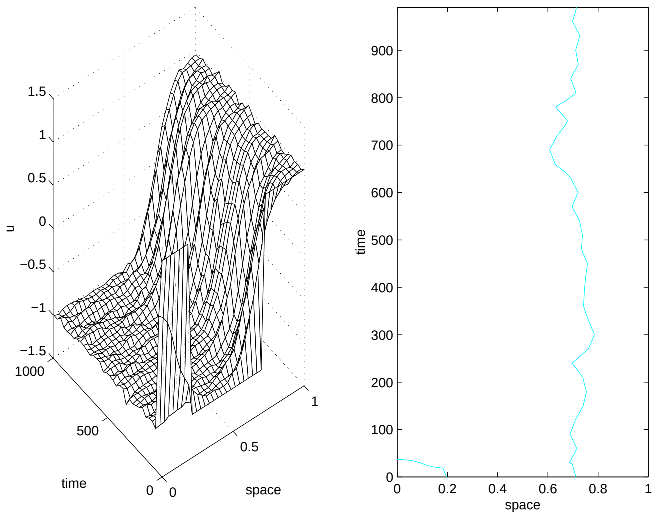

Figure 2: Solution of stochastic Allen–Cahn: $\epsilon = 0.08$, $\sigma = 0.015$: surface and contour plots. Computed with $\Delta t = 0.005$ and $\Delta x = 0.008$. The ratio $\sigma/\sqrt{\epsilon} = 0.05$.

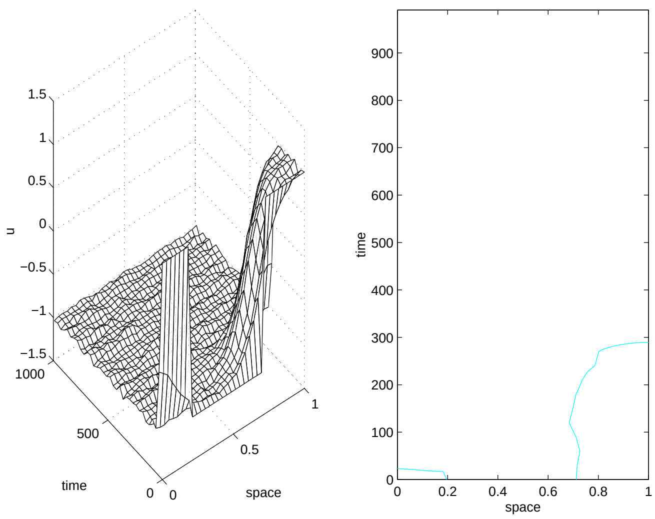

Figure 3: as in Figure 2, except a different realisation of the noise.

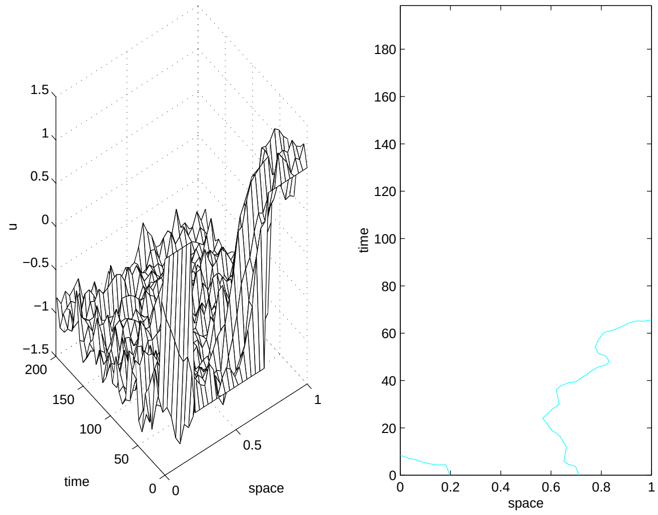

Figure 4: Solution of stochastic Allen–Cahn: $\epsilon = 0.08$, $\sigma = 0.1$: surface and contour plots. Computed $\Delta t = 0.005$ and $\Delta x = 0.008$. The ratio $\sigma/\sqrt{\epsilon} = 0.35$.

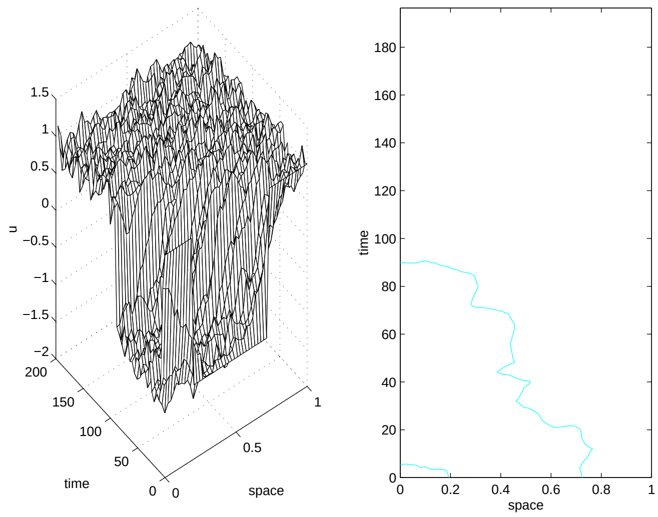

Figure 5: Solution of stochastic Allen–Cahn: as in Figure 4 but a different realisation.

introduce U, the solution to the free space problem

$$U_{xx} + f(U) = 0, \qquad U(\pm\infty) = s_\pm, \qquad U(0) = 0. \tag{1.3}$$

Let $\mathbf{h} = (h_1, \ldots, h_N)$ denote the positions of the interfaces. Let $\alpha_i = (-1)^i \alpha_0$, where $\alpha_0 = \pm 1$ indicates whether $u(0) \approx s_\pm$. Ward [12] uses the following approximation to solutions u of (1.1) when ϵ is small

$$u^h = C_0 + \sum_{i=1}^{N} \left\{ U\Big(\frac{\alpha_i(x - h_i)}{\epsilon}\Big) - C_i \right\}, \quad C_i = \begin{cases} s_+, & \alpha_i = 1; \\ s_-, & \alpha_i = -1; \end{cases} \tag{1.4}$$

This is not the only way to define an approximation u^h, see for example [2] for a slightly different approach.

For convenience, fix $h_0 = 0$ and $h_{N+1} = 1$ as the positions of the homogeneous Neumann boundaries. The ODE describing the evolution of $\mathbf{h}$ is

$$\frac{dh_i}{dt} = \frac{2\epsilon}{\|U'\|^2} \Big[\mu_{i+1} e^{-\sigma_{i+1}(1+\delta_{i,N})\epsilon^{-1}\ell_{i+1}} - \mu_i e^{-\sigma_i(1+\delta_{i,1})\epsilon^{-1}\ell_i} \Big], \quad i = 1, \ldots, N \tag{1.5}$$

where $\ell_i := h_i - h_{i-1}$ denotes the distance between interfaces; $\delta_{i,j}$ is the Kronecker delta function; μ_i and σ_i are positive constants described later in terms of F (in case F given by (1.2), $\mu_i = 4$, $\|U'\|^2 = 2/3$, $\sigma_i = 1$). This equation holds upto the time of collapse of an interface (when $h_{i+1} - h_i \le \epsilon$, some i) upto exponentially small terms. This result has been established rigorously in [2].

In this paper, the above results are extended somewhat to include the case where $\sigma > 0$. Equation (1.1) is well posed for all time; its existence and uniqueness properties are described in [5]. The simplest case is when f is globally Lipschitz from $L_2(0,1)$ to itself, in which case a mild solution exists taking values in $L_2(0,1)$. In §2, we show rigorously for Dirichlet boundary conditions that in this case the basic structure of the problem is preserved when $\sigma \ll \epsilon^{1/2}$; in particular, for initial data in $L_2(0,1)$ and all $T > 0$, there exists K such that

$$\mathbf{E}\|u_{\epsilon,0}(t) - u_{\epsilon,\sigma}(t)\|^2 \le K\sigma^2/\epsilon, \qquad 0 \le t \le T, \tag{1.6}$$

where $u_{\epsilon,\sigma}$ is the solution of (1.1) with diffusion coefficient ϵ^2 and noise intensity σ. (The function $f(u) = u - u^3$ is not Lipschitz as required, but experiments indicate the same phenomena hold). When $\sigma \gg \epsilon^{1/2}$, the noise dominates the solution, a consequence of space–time white noise having being ill posed in $L_2(0,1)$ (viz., $\mathbf{E}\|W(t)\|^2 = \infty$).

An SDE is formally derived in §3 to account for the motion of the interfaces when $\sigma \ll \epsilon^{1/2}$. The interface positions $\mathbf{h}$ are defined as being the minimiser of $\|V\|$ where $V := u - u^h$ over $h \in \mathbf{R}^N$ with $h_{i+1} - h_i \ge \epsilon$ and $N = 1, 2, \ldots$. The SDE is

$$\begin{aligned} dh_i =& \frac{\epsilon}{\|U'\|^2} \frac{1}{1 - A_i} \Big[\mu_{i+1} e^{-\sigma_{i+1}(1+\delta_{i,N})\epsilon^{-1}\ell_{i+1}} - \mu_i e^{-\sigma_i(1+\delta_{i,1})\epsilon^{-1}\ell_i} \Big]\, dt \\ &+ \frac{\sigma\epsilon^{1/2}}{\|U'\|}\, d\beta_i(t) + \mathcal{O}(\|V\|^2)\ dt, \end{aligned}$$

where $A_i := C\epsilon^{-1/2}\langle \mathbf{e}_i, V\rangle$, for a constant C and a unit vector $\mathbf{e}_i$ (to be defined later), and $\beta_i(\cdot)$ are IID standard Brownian motions. The equation may become singular even

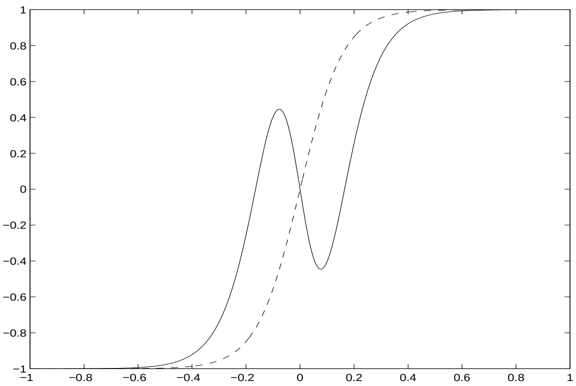

Figure 6: Plot of U (dashed) and $U + \langle V, \mathbf{e}_i\rangle \mathbf{e}_i$ (solid) when $h_i = 0$, $A_i = 1$, $F(u) = (1-u^2)^2/8$, and $\epsilon = 0.08$.

when V is order $\epsilon^{1/2}$, as is expressed by the term $1/(1-A_i)$. The term A_i is large when V has a considerable component in $U''((x-h_i)/\epsilon)$, which essentially describes the direction of a branching interface as depicted in Figure 6. When the equation makes sense, the term A_i has a negligible effect on the dynamics of $\mathbf{h}$ as it is multiplied by exponentially small terms.

The precise relation between the Brownian motions β_i and the white noise W is described in §3. However, when β_i and W are considered independent, we expect that the trajectories of the interfaces $\mathbf{h}$ given by the stochastic PDE and stochastic ODE should converge weakly as V becomes small. By (1.6), for an initial condition $u_0 = u^{\mathbf{h}}$, $\mathbf{E}\|V\|^2$ is order σ^2/ϵ. Thus, let $\tilde{h}$ be a solution of

$$dh_i = \frac{\epsilon}{\|U'\|^2}\Big[\mu_{i+1}e^{-\sigma_{i+1}(1+\delta_{i,N})\epsilon^{-1}\ell_{i+1}} - \mu_i e^{-\sigma_i(1+\delta_{i,1})\epsilon^{-1}\ell_i}\Big]\,dt + \frac{\sigma\epsilon^{1/2}}{\|U'\|}\,d\beta_i(t), \tag{1.7}$$

for initial condition $\mathbf{h} = \mathbf{h}_0$ (that is, we neglect $1/(1-A_i)$ and the error $\mathcal{O}(\|V\|^2)$). Let $\mathbf{h}$ minimise $\|u - u^{\mathbf{h}}\|$ where u solves (1.2) with $u_0 = u^{\mathbf{h}_0}$. We would like

$$\mathbf{E}G(\tilde{\mathbf{h}}(t)) - \mathbf{E}G(\mathbf{h}(t)) \to 0 \qquad \text{as } \sigma/\epsilon^2, \epsilon \to 0 \tag{1.8}$$

for smooth test functionals $G\colon \mathbf{R}^N \to \mathbf{R}$ where the expectation is taken over all $\tilde{\mathbf{h}}$ (resp., $\mathbf{h}$) which have dimension N at time t.

The last section of this paper, §4, covers numerical experiments that support (1.8). The experiments compute the mean and variance of the deviation of the interface position from its initial position for the asymptotic SDE (1.7) and for (1.1) for a single interface initial condition. Thus, we take the first steps to examine (1.8) for $g(x) = (x - h_0)$ and $g(x) = (x - \hat{x})^2$, where $\hat{x} = \mathbf{E}h$. The computations indicate agreement between the two dynamical systems for $\sigma/\epsilon^{1/2} = 0.1$ and 0.035.

2 Finite time limits as $\sigma \to 0$

The finite time limits in σ and ϵ of the stochastic Allen–Cahn equations (1.1) with homogeneous Dirichlet boundary conditions are studied. Throughout this section we take the nonlinearity f to be globally Lipschitz from $L_2(0,1)$ to itself. Denote the solution of the Allen–Cahn equation (1.1) by $u_{\epsilon,\sigma}$, and let $u_\epsilon := u_{\epsilon,0}$, the solution to the noise free problem.

The space–time white noise W may be considered in terms of its Fourier expansion. If e_i is a complete orthonormal system for $L_2(0,1)$, and β_i is a sequence of independent standard Brownian motions, the process $W(\cdot)$ may be thought of as

$$W(t) = \sum_{i=1}^{\infty} e_i \beta_i(t).$$

It is clear that $W(\cdot)$ does not converge in $L_2(0,1)$. However, stochastic integrals can be defined with respect to an operator that smoothes the process $W(\cdot)$ [4]. This is made explicit by the Itô isometry. The Itô isometry in infinite dimensions states that, for a linear operator Φ mapping H to H,

$$\mathbf{E}\Big\| \int_0^t \Phi(s)\, dW(s) \Big\|^2 = \int_0^t \|\Phi(s)\|_{HS}^2 \, ds. \tag{2.1}$$

($\|\cdot\|_{HS}$ is the Hilbert-Schmidt norm, see [4]). It will be important to estimate this quantity when $\Phi(s) = e^{-\epsilon^2 A(t-s)}$.

Lemma 2.1 *For all $t > 0$, there exists $C_t > 1$ such that*

$$\frac{C_t^{-1}}{\epsilon} \le \int_0^t \|e^{-\epsilon^2 A(t-s)}\|_{HS}^2 \, ds \le \frac{C_t}{\epsilon}, \qquad 0 < \epsilon \le 1.$$

Proof This result is proved for A with homogeneous Dirichlet conditions.

(lower bound) When A is defined with Dirichlet conditions, its eigenvalues are $k^2\pi^2$ for $k = 1, 2, \ldots$. Hence, from (2.1),

$$\begin{aligned}
\int_0^t \|e^{-\epsilon^2 A(t-s)}\|_{HS}^2 \, ds &= \sum_{k=1}^{\infty} \frac{1}{2\epsilon^2 k^2 \pi^2}(1 - e^{-2\epsilon^2 k^2 t \pi^2}) \\
&\ge \sum_{k=\lfloor 1/\epsilon \rfloor}^{\infty} \frac{1}{2\epsilon^2 k^2 \pi^2}(1 - e^{-2t\pi^2}) \\
&\ge \frac{1}{2\epsilon^2 \pi^2}(1 - e^{-2t\pi^2}) \int_{\lfloor 1/\epsilon \rfloor}^{\infty} \frac{1}{s^2}\, ds \\
&\ge \frac{1}{2\epsilon \pi^2}(1 - e^{-2t\pi^2}).
\end{aligned}$$

(upper bound) For all $t > 0$, there exists a constant K_t so that

$$(1 - e^{-2\lambda t \pi^2}) \le K_t \lambda, \qquad \text{for } 0 \le \lambda \le 1.$$

Hence,

$$\begin{aligned}
\int_0^t \|e^{-\epsilon^2 A(t-s)}\|_{HS}^2\, ds &\le \sum_{k=1}^{\lfloor 1/\epsilon\rfloor} \frac{1}{2\epsilon^2 k^2\pi^2}(1-e^{-2\epsilon^2 k^2 t\pi^2}) \\
&\quad + \sum_{k=1+\lfloor 1/\epsilon\rfloor}^{\infty} \frac{1}{2\epsilon^2 k^2\pi^2}(1-e^{-2\epsilon^2 k^2 t\pi^2}) \\
&\le \sum_{k=1}^{\lfloor 1/\epsilon\rfloor} \frac{K_t\epsilon^2 k^2}{2\pi^2\epsilon^2 k^2} + \sum_{k=\lfloor 1/\epsilon\rfloor+1}^{\infty} \frac{1}{2\epsilon^2 k^2\pi^2} \\
&\le \frac{K_t}{2\pi^2\epsilon} + \frac{1}{2\epsilon^2\pi^2}\sum_{k=1+\lfloor 1/\epsilon\rfloor}^{\infty}\frac{1}{k^2} \\
&\le \frac{K_t}{2\pi^2\epsilon} + \frac{\epsilon}{2\epsilon^2\pi^2},
\end{aligned}$$

as required. o

Lemma 2.2 *Consider $t>0$; for a constant C_γ depending on γ,*

$$\|A^{-\gamma}(I-e^{-At})\| \le C_\gamma\, t^\gamma, \qquad 0<\gamma\le 1;$$

$$\|A^\gamma e^{-At}\| \le C_\gamma\, t^{-\gamma}, \qquad \gamma>0.$$

Proof This is a standard result on fractional powers of sectorial operators [9]. o

Theorem 2.3 *Fix $T>0$. There are three limits as $\epsilon,\sigma\to 0$:*

(i) In the limit $\epsilon,\sigma\to 0$ with $\sigma/\epsilon^{1/2}\to 0$,

$$\mathbf{E}\sup_{0\le t\le T}\|u_{\epsilon,\sigma}(t)-u_\epsilon(t)\|^2\to 0;$$

(ii) Suppose further that f is globally Lipschitz from $H^{-r}(0,1)$ to $H^{-r}(0,1)$. For each $r>1/2$, there exists a process $u(\cdot)$ taking values in $H^{-r}(0,1)$ such that in the limit $\epsilon,\sigma\to 0$ with $\sigma/\epsilon^{1/2}\to\nu$,

$$\mathbf{E}\sup_{0\le t\le T}\|u_{\epsilon,\sigma}(t)-u(t)\|^2_{H^{-r}(0,1)}\to 0;$$

(iii) In the limit $\sigma/\epsilon^{1/2}\to\infty$,

$$\mathbf{E}\sup_{0\le t\le T}\|u_{\epsilon,\sigma}(t)\|^2\to\infty.$$

Proof (i) Clearly,

$$u_{\epsilon,\sigma}(t) - u_\epsilon(t) = \int_0^t e^{-\epsilon^2 A(t-s)} \Big[f(u_{\epsilon,\sigma}(s)) - f(u_\epsilon(s)) \Big]\, ds$$
$$+ \sigma \int_0^t e^{-\epsilon^2 A(t-s)}\, dW(s).$$

The stochastic integral may be bounded as follows: by the Itô Isometry (2.1) and for $0 \le t \le T$,

$$\mathbf{E}\| \int_0^t e^{-\epsilon^2 A(t-s)}\, dW(s)\|^2 = \int_0^t \|e^{-\epsilon^2 A(t-s)}\|_{HS}^2\, ds$$

(by Lemma 2.1)

$$\le \frac{C_T}{\epsilon}.$$

Therefore, denoting the Lipschitz constant of f by K, we have for $0 \le t \le T$,

$$\Big(\mathbf{E}\|u_{\epsilon,\sigma}(t) - u_\epsilon(t)\|^2)^{1/2} \le \int_0^t K(\mathbf{E}\|u_{\epsilon,\sigma}(s) - u_\epsilon(s)\|^2)^{1/2}\, ds + C_T^{1/2} \frac{\sigma}{\epsilon^{1/2}}.$$

By applying Gronwall's lemma, we have proved

$$\Big(\mathbf{E} \sup_{0\le t\le T} \|u_{\epsilon,\sigma}(t) - u_\epsilon(t)\|^2\Big)^{1/2} \le \frac{\sigma}{\epsilon^{1/2}} e^{Kt} C_T^{1/2} \to 0, \qquad \text{as } \sigma/\epsilon^{1/2} \to 0. \tag{2.2}$$

(ii) Consider a sequence (σ_n, ϵ_n) with $\sigma_n \to 0$ and $\sigma_n/\epsilon_n^{1/2} \to \nu$ as $n \to \infty$. For $n, m \in \mathbf{N}$, the Variation of Constants formula gives

$$u_{\epsilon_n,\sigma_n}(t) - u_{\epsilon_m,\sigma_m}(t) = \int_0^t (e^{-\epsilon_n^2 A(t-s)} - e^{-\epsilon_m^2 A(t-s)}) f(u_{\epsilon_n,\sigma_n}(s))\, ds$$
$$+ \int_0^t e^{-\epsilon_m^2 A(t-s)} \Big[f(u_{\epsilon_n,\sigma_n}(s)) - f(u_{\epsilon_m,\sigma_m}(s)) \Big]\, ds$$
$$+ (\sigma_n - \sigma_m) \int_0^t e^{-\epsilon_n^2 A(t-s)}\, dW(s)$$
$$+ \sigma_m \int_0^t (e^{-\epsilon_n^2 A(t-s)} - e^{-\epsilon_m^2 A(t-s)})\, dW(s).$$

Each term can be bounded in $H^{-2r}(0,1)$ for $r \ge 1/4$. Recall that in the Dirichlet case that $\|\cdot\|_{-r} := \|A^{-r}\cdot\|$ is equivalent to the $H^{-2r}(0,1)$ norm. This norm is used here to gain the necessary inequalities.

Consider the first term: By Lemma 2.2 (without loss take $\epsilon_m > \epsilon_n$),

$$\Big(\mathbf{E}\| \int_0^t (e^{-\epsilon_n^2 A(t-s)} - e^{-\epsilon_m^2 A(t-s)}) f(u_{\epsilon_n,\sigma_n}(s))\, ds\|_{-r}^2\Big)^{1/2}$$
$$\le \int_0^t \|A^{-r}(I - e^{-(\epsilon_m^2 - \epsilon_n^2)A(t-s)})\| \cdot \|e^{-\epsilon_n^2 A(t-s)}\| \cdot (\mathbf{E}\|f(u_{\epsilon_n,\sigma_n})\|^2)^{1/2}\, ds$$

(by Lemma 2.2)

$$\leq C \int_0^t (t-s)^r (\epsilon_m^2 - \epsilon_n^2)^r K (\mathbf{E}\|u_{\epsilon_n,\sigma_n}\|^2)^{1/2}\, ds.$$

By (2.2), $\mathbf{E}\|u_{\epsilon_n,\sigma_n}\|^2$ may be bounded uniformly in limits $\epsilon_n, \sigma_n \to 0$ subject to $\sigma_n/\epsilon_n^{1/2}$ being bounded. Hence, there exists a constant C_1 with

$$\begin{aligned}\Big(\mathbf{E}\| \int_0^t (e^{-\epsilon_n^2 A(t-s)} - e^{-\epsilon_m^2 A(t-s)}) f(u_{\epsilon_n,\sigma_n}(s))\, ds\|_{-r}^2\Big)^{1/2} \\ \leq C_1(\epsilon_m^2 - \epsilon_n^2)^r \int_0^t (t-s)^r\, ds.\end{aligned}$$

Consider the second term:

$$\begin{aligned}\Big(\mathbf{E}\| \int_0^t e^{-\epsilon_m^2 A(t-s)} \Big[f(u_{\epsilon_n,\sigma_n}(s)) - f(u_{\epsilon_m,\sigma_m}(s))\Big]\, ds\|_{-r}^2\Big)^{1/2} \\ \leq K \int_0^t \Big(\mathbf{E}\|u_{\epsilon_n,\sigma_n}(s) - u_{\epsilon_m,\sigma_m}(s)\|_{-r}^2\Big)^{1/2}\, ds.\end{aligned}$$

Consider the third term: By the Itô isometry,

$$\begin{aligned}\mathbf{E}\| \int_0^t e^{-\epsilon_n^2 A(t-s)}\, dW(s)\|_{-r}^2 &= \int_0^t \|A^{-r} e^{-\epsilon_n^2 A(t-s)}\|_{HS}^2\, ds \\ &= \int_0^t \sum_{k=1}^{\infty} \frac{1}{(k^2\pi^2)^{2r}} e^{-2\epsilon_n^2 k^2 \pi^2 (t-s)}\, ds,\end{aligned}$$

which is finite for $r > 1/4$.

Consider the fourth term: by Lemma 2.2 and the Itô isometry, we have for $0 \leq t \leq T$,

$$\begin{aligned}&\mathbf{E}\| \int_0^t (e^{-\epsilon_n^2 A(t-s)} - e^{-\epsilon_m^2 A(t-s)})\, dW(s)\|_{-r}^2 \\ &\leq \int_0^t \|A^{-r}(I - e^{(\epsilon_n^2 - \epsilon_m^2)A(t-s)})\|^2 \|e^{-\epsilon_n^2 A(t-s)}\|_{HS}^2\, ds \\ &\leq \|A^{-r}(I - e^{(\epsilon_n^2-\epsilon_m^2)At})\|^2 \int_0^t \|e^{-\epsilon_n^2 A(t-s)}\|_{HS}^2\, ds \\ &\leq C^2(\epsilon_n^2 - \epsilon_m^2)^{2r} t^{2r} \frac{C_T}{\epsilon_n} \\ &\leq \frac{C^2 C_T t^{2r}}{\epsilon_n} (\epsilon_n^2 - \epsilon_m^2)^{2r}\end{aligned}$$

Thus, taking a limit $(\sigma, \epsilon) \to 0$ with σ^2/ϵ bounded above, there exists a constant C_2 such that for $0 \leq t \leq T$,

$$\begin{aligned}\Big(\mathbf{E}\|u_{\epsilon_n,\sigma_n}(t) - u_{\epsilon_m,\sigma_m}(t)\|_{-r}^2\Big)^{1/2} &\leq C_2((\epsilon_n^2 - \epsilon_m^2)^r + (\sigma_n - \sigma_m)) \\ &+ \int_0^t K\Big(\mathbf{E}\|u_{\epsilon_n,\sigma_n}(s) - u_{\epsilon_m,\sigma_m}(s)\|_{-r}^2\Big)^{1/2}\, ds.\end{aligned}$$

Gronwall's inequality now gives, for a constant C_3

$$\Big(\mathbf{E}\sup_{0\le t\le T}\|u_{\epsilon_n,\sigma_n}(t)-u_{\epsilon_m,\sigma_m(t)}\|_{-r}^2\Big)^{1/2}\le C((e_m^2-e_n^2)^r+(\sigma_n-\sigma_m))e^{KT}.$$

If ϵ_n,σ_n are Cauchy, the sequences u_{ϵ_n,σ_n} are Cauchy with respect to

$$\Big(\sup_{0\le t\le T}\mathbf{E}\|\cdot\|_{-r}^2\Big)^{1/2}$$

and thus a limiting process exists. The above formula also gives uniqueness for if $u_{\epsilon_n,\sigma_n}\to u_1$ and $u_{\epsilon_m,\sigma_m}\to u_2$ where (ϵ_n,σ_n) and (ϵ_m,σ_m) are both Cauchy, then, by the above,

$$\begin{aligned}\Big(\mathbf{E}\sup_{0\le t\le T}\|u_1(t)-u_2(t)\|_{-r}^2\Big)^{1/2}\le&\Big(\mathbf{E}\sup_{0\le t\le T}\|u_{\epsilon_n,\sigma_n}(t)-u_{\epsilon_m,\sigma_m}(t)\|_{-r}^2\Big)^{1/2}\\&+\Big(\mathbf{E}\sup_{0\le t\le T}\|u_{\epsilon_n,\sigma_n}(t)-u_1(t)\|_{-r}^2\Big)^{1/2}\\&+\Big(\mathbf{E}\sup_{0\le t\le T}\|u_2(t)-u_{\epsilon_m,\sigma_m}(t)\|_{-r}^2\Big)^{1/2}\\\to&0.\end{aligned}$$

(iii) Suppose that $\mathbf{E}\|u_{\epsilon,\sigma}(t)\|^2<\infty$ uniformly as $\sigma/\epsilon^{1/2}\to\infty$ for $0\le t\le T$. For simplicity take $u_0=0$. Then, argue for a contradiction as follows: from the Variation of Constants formula and the Itô isometry,

$$\begin{aligned}\mathbf{E}\|u_{\epsilon,\sigma}(t)\|^2=&\mathbf{E}\Big[\|\int_0^t e^{-\epsilon^2A(t-s)}f(u_{\epsilon,\sigma}(s))\,ds\|^2\Big]\\&+2\sigma\mathbf{E}\Big[\Big\langle\int_0^t e^{-\epsilon^2A(t-s)}f(u_{\epsilon,\sigma}(s))\,ds,\int_0^t e^{-\epsilon^2A(t-s)}\,dW(s)\Big\rangle\Big]\\&+\sigma^2\mathbf{E}\Big[\int_0^t\|e^{-\epsilon^2A(t-s)}\|_{HS}^2\,ds\Big].\end{aligned}$$

The third term is positive and order σ^2/ϵ by Lemma 2.1; the first term is positive; thus, to gain a contradiction, we show the second has lower order than σ^2/ϵ. Indeed, for $0\le t\le T$

$$\begin{aligned}&\sigma\mathbf{E}\Big[\Big\langle\int_0^t e^{-\epsilon^2A(t-s)}f(u_{\epsilon,\sigma}(s))\,ds,\int_0^t e^{-\epsilon^2A(t-s)}\,dW(s)\Big\rangle\Big]\\\le&\sigma\mathbf{E}\Big[\int_0^t e^{-\epsilon^2A(t-s)}f(u(s))^2\,ds\Big]^{1/2}\mathbf{E}\Big[\int_0^t\|e^{-2\epsilon^2A(t-s)}\|_{HS}^2\,ds\Big]^{1/2},\\\le&\sigma\frac{C_T^{1/2}}{\epsilon^{1/2}}K\sup_{0\le t\le T}\Big(\mathbf{E}\|u(t)\|^2\Big)^{1/2},\end{aligned}$$

which is clearly order $\sigma/\epsilon^{1/2}$. o

3 Formal derivation of an SDE

The positions of the interfaces h_i are well defined in the deterministic case $\sigma = 0$ as the contours of $u = (s_+ + s_-)/2$. In the case $\sigma > 0$, the interface may be wrinkled, making the contour ill defined. We choose $\mathbf{h}$ by solving the following minimisation problem: let $\mathbf{h}$ minimise

$$\|u - u^h\| \tag{3.1}$$

over $\mathbf{h} \in \mathbf{R}^N$ with $|h_{i+1} - h_i| \geq \epsilon$ and over $N = 1, 2, \ldots$. In this case, letting

$$V := u - u^h, \qquad \phi_i := \frac{\alpha_i}{\epsilon} U'\Big(\frac{\alpha_i(x - h_i)}{\epsilon}\Big),$$

we have by differentiating (3.1) with respect to h_i

$$\langle \phi_i, V \rangle = 0.$$

We'll need the following asymptotic properties as we go along [12]:

(i) The solution U of (1.3) satisfies

$$U(x) = s_+ - a_+ e^{-\sigma_+ x}, \qquad x \to \infty;$$

$$U(x) = s_- + a_- e^{\sigma_- x}, \qquad x \to -\infty.$$

where $s\pm$ are the zeros of f; $\sigma_\pm = (-f'(s_\pm))^{1/2}$;

$$\log a_\pm = \log(\pm s_\pm) + \int_0^{s_\pm} \Big(\frac{s_\pm}{(2F(\eta))^{1/2}} + \frac{1}{\eta - s_\pm} \Big)\, d\eta.$$

(ii)

$$\|U'\|^2 \approx \int_{-\infty}^{\infty} U'(x)^2\, dx = \int_{s_-}^{s_+} (2F(x))^{1/2}\, dx.$$

(iii) For case $f(u) = \frac{1}{2}(u - u^3)$; these quantities evaluate to $s_\pm = \pm 1$; $\|U'\|^2 = 2/3$; $a_\pm = 2$; $\sigma_\pm = 1$.

Assume that $\mathbf{h}$ obeys the Itô equation

$$d\mathbf{h} = \psi(\mathbf{h}, t, \omega)\, dt + \Theta(\mathbf{h}, t, \omega)\, d\beta(t), \tag{3.2}$$

where $\Theta = \mathrm{diag}(\theta_1, \ldots, \theta_N)$ and $\psi = (\psi_1, \ldots, \psi_N)^T$ and $\beta(t)$ is a vector of N Brownian motions, to be specified later in terms of $W(t)$.

Apply the Itô formula to $u = u^h + V$ using (1.4) and (3.2),

$$\begin{aligned} du &= -\sum_i \phi_i\, dh_i - \tfrac{1}{2} \sum_i \phi_{ix} \theta_i^2\, dt + dV \\ &= -\Big(\sum_i \phi_i \psi_i + \tfrac{1}{2} \phi_{ix} \theta_i^2 \Big)\, dt + \sum_i \phi_i \theta_i\, d\beta_i(t) + dV, \end{aligned}$$

where $\phi_{ix} = (\phi_i)_x$. Take the inner product with ϕ_i:

$$\langle \phi_i,\ du\rangle = \Big\{ -\psi_i \|\phi_i\|^2 + \sum_i \tfrac{1}{2}\langle \phi_{ix}, \phi_i\rangle \Big\}\ dt - \|\phi_i\|\theta_i\ d\beta_i(t) + \sum_{i\neq j} \theta_j \langle \phi_i, \phi_j\rangle\ d\beta_j(t).$$

Note that

$$\langle \phi_{ix}, \phi_i\rangle = \big[\phi_i^2\big]_0^1.$$

This quantity is very small and is neglected as the asymptotics of U show that ϕ_i is exponentially small away from the layers. Similarly, $\langle \phi_i, \phi_j\rangle$ is negligible for $i \neq j$. Hence, we'll work with

$$\langle \phi_i,\ du\rangle = -\psi_i \|\phi_i\|^2\ dt - \|\phi_i\|\theta_i\ d\beta_i(t) \tag{3.3}$$

To compare, multiply (1.1) by ϕ_i:

$$\langle \phi_i,\ du\rangle = \langle \phi_i, \epsilon^2 u_{xx} + f(u)\rangle\ dt + \sigma \langle \phi_i,\ dW(t)\rangle. \tag{3.4}$$

Let

$$\beta_i(t) := \frac{1}{\|\phi_i\|} \int_0^t \langle \phi_i,\ dW(s)\rangle.$$

The $\beta_i(t)$ are continuous martingales with variance

$$\mathbf{E}\beta_i(t)^2 = \frac{1}{\|\phi_i\|^2} \int_0^t \|\langle \phi_i, \cdot\rangle\|_{HS}^2\ ds = \frac{1}{\|\phi_i\|^2} \int_0^t \|\phi_i\|^2\ ds = t.$$

Therefore $\beta_i(t)$ are standard Brownian motions. Moreover, the processes $\beta_i(t)$ are independent (upto exponentially small terms), because

$$\mathbf{E}\langle \beta_i(t), \beta_j(t)\rangle = t \frac{\langle \phi_i, \phi_j\rangle}{\|\phi_i\| \cdot \|\phi_j\|}.$$

Thus (3.4) becomes

$$\langle \phi_i,\ du\rangle = \langle \phi_i, \epsilon^2 u_{xx} + f(u)\rangle\ dt + \sigma\ d\beta_i(t). \tag{3.5}$$

Equate coefficients in (3.3) and (3.5):

$$-\psi_i \|\phi_i\|^2 = \langle \phi_i, \epsilon^2 u_{xx} + f(u)\rangle, \tag{3.6}$$

$$-\theta_i \|\phi_i\|^2 = \sigma \|\phi_i\|. \tag{3.7}$$

Expand the RHS of (3.6):

$$\langle \phi_i, \epsilon^2 u_{xx} + f(u)\rangle = \langle \phi_i, \epsilon^2 u_{xx}^h + f(u^h)\rangle + \langle \phi_i, L^h V\rangle + \mathcal{O}(\|V\|^2)\,, \tag{3.8}$$

where $L^h u = \epsilon^2 u_{xx} +\ df(u^h)u$. Write the first term

$$\langle \phi_i, \epsilon^2 u_{xx}^h + f(u^h)\rangle = \Big\langle \phi_i, \epsilon^2 u_{xx}^h + \sum_{i=1}^N f\Big(U\Big(\frac{\alpha_i(x-h_i)}{\epsilon}\Big)\Big)\Big\rangle + \Big\langle \phi_i, E\Big\rangle, \tag{3.9}$$

where

$$E := f(u^h) - \sum_i f(U(\alpha_i(x-h_i)/\epsilon)).$$

Because U solves (1.3), the first term is zero and, by the asymptotic analysis in [12], the quantity

$$\langle \phi_i, E\rangle \approx 2\epsilon\Big(\tilde{\mu}_{i+1}e^{-\sigma_{i+1}\epsilon^{-1}\ell_{i+1}} - \tilde{\mu}_i e^{-\sigma_i\epsilon^{-1}\ell_i}\Big), \quad i=1,\dots,N \tag{3.10}$$

where $\ell_i := h_i - h_{i-1}$ (recall $h_0 := 0$ and $h_{N+1} := 1$) and $\tilde{\mu}_i := (a_i\sigma_i)^2$ and $\tilde{\mu}_1 = \tilde{\mu}_{N+1} = 0$ and

$$a_i = \begin{cases} a_+, & \text{if } \alpha_i = 1; \\ a_-, & \text{if } \alpha_i = -1; \end{cases}, \qquad \sigma_i = \begin{cases} \sigma_-, & \text{if } \alpha_i = 1; \\ \sigma_+, & \text{if } \alpha_i = -1; \end{cases}.$$

Consider the second term in (3.8): let $\mathcal{L}(u) = \epsilon^2 u_{xx} + f(u)$ so that

$$\langle \phi_i, L^h V\rangle = \langle L^h\phi_i, V\rangle + B_i = \langle \mathcal{L}(u^h)_{h_i}, V\rangle + B_i \tag{3.11}$$

where B_i are boundary terms, which may be neglected for internal layers [12]. We wish to compute $\mathcal{L}(u^h)_{h_i}$. First note that

$$\mathcal{L}(u^h) = \sum_i \langle \phi_i, \mathcal{L}(u^h)\rangle \frac{\phi_i}{\|\phi_i\|^2} + \text{lower order terms}.$$

Consequently, from (3.9)

$$\langle \mathcal{L}(u^h)_{h_i}, V\rangle = \langle \phi_i, \mathcal{L}(u^h)\rangle \frac{\langle \phi_{ix}, V\rangle}{\|\phi_i\|^2} = \langle \phi_i, \mathcal{L}(u^h)\rangle A_i, \tag{3.12}$$

where $A_i := \langle \phi_{ix}, V\rangle / \|\phi_i\|^2$.

Collecting (3.6), (3.8), and (3.12), we have

$$-\psi_i \|\phi_i\|^2 = \langle \phi_i, E\rangle + \langle \phi_i, E\rangle A_i,$$

and so

$$\psi_i = \frac{1}{1-A_i}\frac{\langle \phi_i, E\rangle}{\|\phi_i\|^2} + \mathcal{O}(\|V\|^2)\,. \tag{3.13}$$

Finally, from (3.10), (3.13), and (3.7), the SDE is

$$dh_i = \frac{1}{1-A_i}\frac{2}{\|\phi_i\|^2}\Big(\tilde{\mu}_{i+1}e^{-\sigma_{i+1}\epsilon^{-1}\ell_{i+1}} - \tilde{\mu}_i e^{-\sigma_i\epsilon^{-1}\ell_i}\Big)\,dt + \frac{\sigma}{\|\phi_i\|}\,d\beta_i(t).$$

The term $\|\phi_i\|$ is independent of i (upto exponentially small terms) and hence we write $\|\phi_i\| = \epsilon^{-1/2}\|U'\|$ giving

$$dh_i = \frac{1}{1-A_i}\frac{2\epsilon}{\|U'\|^2}\Big(\tilde{\mu}_{i+1}e^{-\sigma_{i+1}\epsilon^{-1}\ell_{i+1}} - \tilde{\mu}_i e^{-\sigma_i\epsilon^{-1}\ell_i}\Big)\,dt + \frac{\sigma\epsilon^{1/2}}{\|U'\|}\,d\beta_i(t).$$

Similarly, A_i may be better written

$$A_i = \frac{C}{\epsilon^{1/2}} \langle \mathbf{e}_i, V \rangle,$$

where

$$C := \frac{\|U''\|}{\|U'\|^2}, \qquad \mathbf{e}_i(x) = \frac{\phi_{ix}}{\|\phi_{ix}\|} \approx \frac{U''((x-h_i)/\epsilon)}{\|U''((x-h_i)/\epsilon)\|}.$$

In the case where h_i is a neighbour of the boundary ($i = 1$ or $i = N$), one term drops out (viz., $\tilde{\mu}_1 = 0$ or $\tilde{\mu}_{N+1} = 0$) and the boundary terms B_i in (3.11) should be evaluated to give the lowest contribution. [12] computes the contribution from B_i and this contribution is not effected by the stochastic perturbation: let $\mu_i = (a_i \sigma_i)^2$ and $\delta_{i,j}$ denote the Kronecker delta, then for $i = 1, \dots, N$

$$dh_i = \frac{1}{1-A_i} \frac{2\epsilon}{\|U'\|^2} \left(\mu_{i+1} e^{-\sigma_{i+1}(1+\delta_{i,N})\epsilon^{-1}\ell_{i+1}} - \mu_i e^{-\sigma_i(1+\delta_{i,1})\epsilon^{-1}\ell_i} \right) dt + \frac{\sigma \epsilon^{1/2}}{\|U'\|} \, d\beta_i(t).$$

4 Numerical Experiments

We would like to show that the trajectories of the interfaces described by (1.2) and (1.7) converges weakly on a finite time interval in the small $\sigma/\epsilon^{1/2}$ limit. To this end, consider an initial condition $u_0 = u^{\mathbf{h}}$ where $\mathbf{h} = (0.4)$. We compute the mean and variance of the deviation of the interface position from $x = 0.4$ for both (1.2) (with initial condition u_0) and (1.7) (with initial condition h_0). Clearly, the average at time t is taken over realisations where the single interface persists at time t. The diagrams show the mean and variance on a time interval $[0, 200]$ for parameter values $(\epsilon, \sigma) = (0.08, 0.01)$ and $(0.08, 0.03)$. The trajectory of the interface for the noise free problem ($\epsilon = 0.08$, $\sigma = 0$) is shown for reference.

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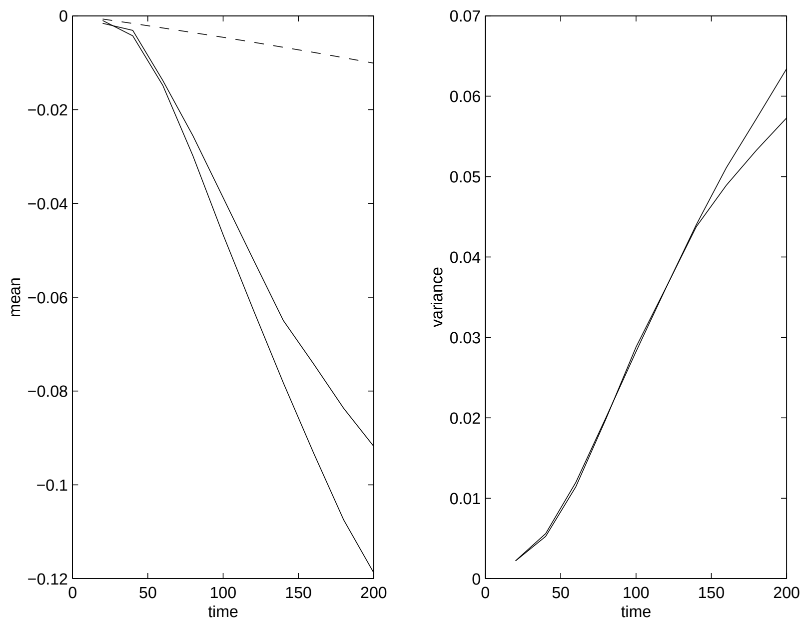

Figure 7: For $\sigma = 0.03$, plots of mean and variance of the position of the interface relative to its initial position given by (1.2) and (1.7); 18500 trials are taken to compute for (1.2) and 20,000 for (1.7). The dashed line represents the motion of the interface with $\sigma = 0$.

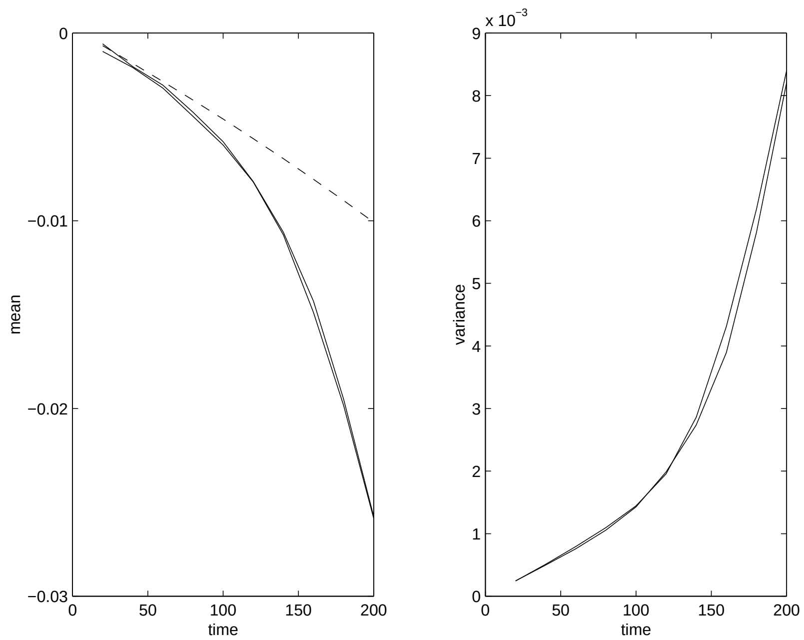

Figure 8: For $\sigma = 0.01$, plots of mean and variance of the position of the interface relative from its initial position given by (1.2) and (1.7); 500 trials are taken in both case. The dashed line represents the motion of the solution with $\sigma = 0$.

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