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GEOMETRIC ERGODICITY FOR STOCHASTIC PDEs

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ABSTRACT

This paper examines the geometric ergodicity of a semi-linear parabolic PDE forced by a Wiener process on a separable Hilbert space. Under a dissipative assumption on the vector field and a non-degeneracy assumption on the noise, geometric ergodicity is proved with respect to the class of measurable functions bounded by $1 + \|\cdot\|^2$. The theorems apply under general conditions on the noise, both additive and multiplicative cases being considered, and apply for instance to a dissipative reaction–diffusion equation on $[0,1]$ with a globally Lipschitz nonlinearity when forced by additive space–time white noise.

1 INTRODUCTION AND MAIN RESULTS

Consider the following stochastic PDEs on a separable Hilbert space X: with additive noise,

$$du = \Big[-Au + f(u)\Big]\,dt + dW(t), \qquad u(0) = x, \tag{1.1}$$

or with multiplicative noise,

$$du = \Big[-Au + f(u)\Big]\,dt + \sigma(u)\,dW(t), \qquad u(0) = x, \tag{1.2}$$

where A is a positive definite, self adjoint, and densely defined linear operator on X with compact inverse; $W(\cdot)$ is an X valued Wiener process with non-singular covariance K; f is globally Lipschitz from X to X, in particular for some $M_1, M_2 > 0$,

$$\|f(u)\| \le M_1(1+\|u\|), \quad \|f(u_1)-f(u_2)\| \le M_2\|u_1-u_2\|, \qquad u, u_1, u_2 \in X;$$

σ is a strongly continuous mapping from X to Hilbert–Schmidt operators from X to X and, for some $M_3, M_4 > 0$,

$$\|\sigma(u)\|_{HS} \le M_3(1+\|u\|), \quad \|\sigma(u_1)-\sigma(u_2)\|_{HS} \le M_4\|u_1-u_2\|, \qquad u, u_1, u_2 \in X.$$

($\|\cdot\|_{HS}$ denotes the Hilbert–Schmidt norm of an operator). The identity operator is not Hilbert–Schmidt on infinite dimensional spaces and therefore (1.1) is not simply a subset of equations of the form (1.2). We further interpret $dW(\cdot)$ as an Itô integral. Subject to $\operatorname{Tr} KA^{-1} < \infty$, (1.1) has a mild solution in X and, because of the assumption on σ, (1.2) has a mild solution for all bounded K. Further background and definitions for many of these concepts may be found in the books of Da Prato–Zabczyk [3, 4].

Notation: Under our hypothesis, A has a complete orthonormal system of eigenfunctions $e_1, e_2, \ldots$; denote the corresponding eigenvalues by $\lambda_1, \lambda_2, \ldots$, ordered so that $0 < \lambda_1 \le \lambda_2 \ldots$. The fractional powers A^γ are well defined and the domains, $\mathcal{D}(A^\gamma)$ for $\gamma > 0$, form Hilbert spaces with inner product $\langle A^{2\gamma}\cdot,\cdot\rangle$ [8]. The underlying probability triple is $(\Omega, \mathcal{F}, \mathbf{P})$ and averages with respect to the measure $\mathbf{P}$ are denoted by $\mathbf{E}$. The solution of (1.1) for an initial condition $u(0) = x$ is denoted $u(t;x)$. $\mathcal{B}(X)$ denotes the Borel σ-algebra on X. For a measure π on $(X, \mathcal{B}(X))$ and $G \in \mathcal{L}^1(X,\pi)$, let

$$\pi(G) := \int_X G(x)\pi(dx).$$

Throughout, C will denote a generic constant.

Definition 1.1 Consider an X valued Markov process $u(t;x)$ with time $t \ge 0$ and initial condition x. Consider a measurable $\bar{G}\colon X \to [1,\infty)$ such that

$$\mathbf{E}\bar{G}(u(t;x)) < \infty.$$

Consider the [illegible]

$$\mathcal{G}_0 := \{\text{measurable } G\colon X \to \mathbb{R} \text{ such that } |G(x)| \le \bar{G}(x),\ x \in X\}.$$

The process $u(t;x)$ is *geometrically ergodic* with respect to $\bar{G}$ if there exists constants $C, p > 0$ and a measure π such that, for $G \in \mathcal{G}_0$,

$$\left|\mathbf{E}G(u(t;x)) - \pi(G)\right| \le C\bar{G}(x)e^{-pt}.$$

The precise form of this definition is motivated by [10], where a perturbation theory is developed for ergodic properties of Markov processes. When a stochastic process is geometrically ergodic and a "small" perturbation is considered, [10] establishes that averages of $G \in \mathcal{G}_1 \subset \mathcal{G}_0$ with respect to the perturbed process at a sufficiently large time approximate the ergodic average of the underlying process (where $\mathcal{G}_1$ depends on the perturbation). The class of perturbations covered by this theory include numerical approximation of (1.1) by finite difference schemes, for example. Geometric ergodicity is also used by [5] to study similar problems, though under more stringent assumptions on the perturbation than in [10].

This paper proves geometric ergodicity for (1.1) and (1.2) under a dissipative assumption and a non-degeneracy condition on the noise. For (1.1), a trace condition is also required, but one only modestly stronger than that for existence of a mild solution. For (1.2), the Hilbert–Schmidt norm of the semigroup e^{-At} must be integrable on finite time intervals: this condition is very strong but is needed to apply the theory of [9]. Geometric ergodicity is given with respect to $1+\|\cdot\|^2$ and $1+\|\cdot\|$; the result for $1+\|\cdot\|^2$ does not automatically imply that for $1+\|\cdot\|$. The results improve on [4], where a stochastic PDE with a stronger assumption on f (contractivity) is shown to be geometrically ergodic for bounded Lipschitz test functionals.

Theorem 1.2 *Suppose that the covariance operator K obeys for some $\delta > 0$*

$$\operatorname{Tr} KA^{\delta-1} < \infty,$$

and that equation (1.1) is dissipative: for some $a, b > 0$,

$$\langle -Au + f(u), u\rangle \le a - b\|u\|^2, \qquad u \in X. \tag{1.3}$$

Then, the solution $u(t;x)$ of (1.1) is geometrically ergodic with respect to $1+\|\cdot\|^2$ and with respect to $1+\|\cdot\|$.

Theorem 1.3 *Suppose that equation (1.2) is dissipative: for some $a, b > 0$,*

$$\langle -Au + f(u), u\rangle + \tfrac{1}{2}\|\sigma(u)K^{1/2}\|_{HS}^2 \le a - b\|u\|^2, \qquad u \in X; \tag{1.4}$$

that

$$\int_0^T \|e^{-At}\|_{HS}^2\, dt < \infty, \qquad T > 0; \tag{1.5}$$

and that the multiplicative noise is non-degenerate: σ is invertible and

$$\sup_{u\in X}\left(\|\sigma(u)\| + \|\sigma^{-1}(u)\|\right) < \infty. \tag{1.6}$$

Then, the solution $u(t;x)$ of (1.2) is geometrically ergodic with respect to $1+\|\cdot\|^2$ and with respect to $1+\|\cdot\|$.

Example: Consider a smooth bounded domain $\mathcal{O}\subset\mathbb{R}^n$. Let $X=L_2(\mathcal{O})$ and $A=-\nabla^2$, the Laplacian scaled to be positive definite with Dirichlet boundary conditions, so that $\mathcal{D}(A)=H^2(\mathcal{O})\cap H_0^1(\mathcal{O})$. By Rellich's Theorem [11], A^{-1} is compact for any dimension n. Consequently, Theorem 1.2 applies to the nonlinear heat equation (1.1) subject to the dissipative condition (1.3) when forced by a Wiener process with covariance K subject to $\operatorname{Tr} KA^{\delta-1}<\infty$. This applies to white noise (case $K=I$) in one dimension, as $\operatorname{Tr} KA^{\delta-1}<\infty$ for every bounded K for $0\le\delta<1/2$.

The paper is organised as follows: §2 describes the theory of Lyapunov–Foster drift conditions in relation to (1.1) and (1.2). §3 derives some estimates for the stochastic PDEs. Finally, §4 contains the proofs of Theorem 1.2 and Theorem 1.3.

2 LYAPUNOV–FOSTER DRIFT CONDITIONS

This section develops theory of Meyn–Tweedie [6] for Markov processes on X that are strong Feller and open set irreducible, properties that are understood for both the stochastic PDEs (1.1)–(1.2). This theory shows how the so-called Lyapunov–Foster drift condition on petite sets implies geometric ergodicity. The definition of petite is given in terms of the Markov process, but may be related to the topology in special circumstances. According to [6], for an aperiodic, ψ–irreducible T–process on a locally compact and separable topological space, compact sets are petite. In the present case, the topological spaces are not necessarily locally compact, and therefore we carefully review the topological parts of the Meyn–Tweedie theory. Indeed, the petite set used in our application of the drift condition is in general not compact. The presentation is intended to be self contained, except for two deep results of [6]: that an aperiodic, ψ–irreducible Markov chain has at least one ψ–positive petite set, and that when subject to a Lyapunov–Foster drift condition on a petite set such chains are geometrically ergodic.

Notation: Consider an X valued Markov chain $\{u_0,u_1,\dots\}$. Denote by $P^n(x,B)$ the probability of reaching $B\in\mathcal{B}(X)$ from x in n steps,

$$P^n(x,B):=\mathbf{P}(u_n\in B|u_0=x).$$

A key concept in [6] is the petite set; for our purposes, we need only define a "subset" of the petite sets known as small sets: a set Π is small if there exists a non-trivial measure ν and some integer $n>0$ such that

$$P^n(x,B)\ge\nu(B),\qquad x\in\Pi,\quad B\in\mathcal{B}(X).$$

A measure ψ is irreducible if $\psi(B)>0$ implies $P^n(x,B)>0$ some integer $n>0$. Sets $\{D_i\}^{i=1,\dots,d}$ form a d-cycle if (i) they are disjoint, (ii) $P^1(x,D_{i+1 \bmod d})=1$ if $x\in D_i$ for $i=1,\dots,d$, (iii) the complement of $D_1\cup\dots\cup D_d$ is ψ–trivial for every irreducible measure ψ. The Markov chain is aperiodic if the maximal d-cycle is $d=1$. The Markov chain is said to be strong Feller if $P^n(x,B)$ is continuous in x for each Borel set B and $n>0$. The Markov chain is said to be open set irreducible if $P^n(x,Z)>0$ for every open set Z and every $n>0$.

In terms of our application, the abstract Markov chain described in the previous paragraph will be the skeleton chain $\{u(0),u(T),u(2T),\dots\}$ induced by (1.1) and (1.2) for some $T>0$. This chain inherits both the strong Feller property and the open set irreducibility from the solution of the stochastic PDE. The stochastic PDE satisfies both these properties under certain hypothesis, as has been studied in [4]; this issue will be discussed further in the proofs of Theorems 1.2–1.3.

The Lyapunov–Foster drift condition has been developed for an infinitessimal T in terms of the generator [7]. This theory gives a simpler analysis of geometric ergodicity of SDEs in finite dimensions. The theory of [7] is based on locally compact spaces, and is not directly relevant to stochastic PDEs.

The following lemma is essentially taken from Meyn–Tweedie [6]: Proposition 6.1.6 and Lemma 5.2.4 (i).

Lemma 2.1 *Consider an X valued Markov chain $\{u_0,u_1,\dots\}$ that is strong Feller and open set irreducible.*

(i) There exists a non-trivial measure ψ on $\mathcal{B}(X)$ such that $P^n(x,B)>0$ for $x\in X$ and $n>1$ when $\psi(B)>0$. In particular, the chain is aperiodic, ψ–irreducible, and there exists a petite set Π with $\psi(\Pi)>0$.

(ii) Consider a Borel set Γ. If there exists a compact set Θ such that

$$P^1(x,\Theta)\ge\tfrac12,\qquad x\in\Gamma,\tag{2.1}$$

then Γ is petite.

Proof

(i) Fix $x^*\in X$. Consider B with $P^1(x^*,B)>0$. By strong Feller continuity, x^* has an open set Z such that $P^1(z,B)>0$ for $z\in Z$. For any $x\in X$ and $n>0$,

$$P^{n+1}(x,B)=\int_X P^n(x,dz)P^1(z,B)\ge\int_Z P^n(x,dz)P^1(z,B)>0$$

because $P^n(x,Z)>0$ by open set irreducibility. Consequently, the first part is proved with $\psi(\cdot) := P^1(x^*, \cdot)$. That the chain is ψ irreducible and aperiodic is clear. By Theorem 5.2.2 of [6], there exists a petite set Π for the chain such that $\psi(\Pi) > 0$.

(ii) By the petiteness of Π, for $B \in \mathcal{B}(X)$ and $x \in X$,

$$\begin{aligned} P^{n+3}(x,B) &\geq \int_{\Pi} P^3(x,dy)P^n(y,B) \\ &\geq \int_{\Pi} P^3(x,dy)\nu(B) \\ &\geq P^3(x,\Pi)\nu(B). \end{aligned}$$

Once again,

$$P^3(x,\Pi) \geq \int_{\Theta} P^1(x,\ dy)P^2(y,\Pi).$$

The set Θ is compact by hypothesis. Moreover, by part (i), $P^2(y,\Pi) > 0$ and, by the strong Feller property, is continuous in y. Therefore, $\{P^2(y,\Pi) \colon y \in \Theta\}$ has a lower bound δ strictly bigger 0. Thus,

$$P^{n+3}(x,\Pi) \geq P^1(x,\Theta)\ \delta\ \nu(B).$$

Finally, we have from (2.1) for each $x \in \Gamma$

$$P^{n+3}(x,B) \geq \frac{\delta}{2}\nu(B), \quad B \in \mathcal{B}(X),$$

so that Γ is petite.

□

Theorem 2.2 *Consider an X valued Markov process $\{u(t), t \geq 0\}$ that is strong Feller and open set irreducible. Consider $V \colon X \to [1,\infty)$ and some $T > 0$ such that for a petite set Γ of the skeleton chain $\{u(0), u(T), \dots\}$ and some $\alpha, \beta, C_1 > 0$,*

$$\mathbf{E}V(u(T;x)) - V(x) \leq -\alpha V(x) + \beta 1_{\Gamma}(x), \quad u(0) \in X \tag{2.2}$$

(1_Γ is the indicator function on the set Γ)

$$\mathbf{E}V(u(s;x)) \leq C_1 V(x), \qquad 0 \leq s \leq T. \tag{2.3}$$

Then, $u(t)$ is geometrically ergodic with respect to V.

Proof By Theorem 16.0.1 of [6], the Lyapunov–Foster drift condition (2.2) and Lemma 2.1 (i) implies geometric convergence of the skeleton chain $\{u(0), u(T), \dots\}$ with respect to V. Hence, there exists $C_2, \rho > 0$ such that for every measurable G dominated by V

$$\left|\mathbf{E}G(u(nT;x)) - \pi(G)\right| \leq C_2 V(x) e^{-\rho nT}.$$

To make a statement about continuous time, consider $nT < t < (n+1)T$ and let $t = nT + s$. Denote by μ_s^x the distribution of $u(s;x)$. Then, by (2.3) and the Markov property,

$$\begin{aligned} \left|\mathbf{E}G(u(nT+s;x)) - \pi(G)\right| &= \left|\int_X \mathbf{E}G(u(nT;y))\mu_s^x(dy) - \pi(G)\right| \\ &\leq \int_X \left|\mathbf{E}G(u(nT;y)) - \pi(G)\right|\mu_s^x(dy) \\ &\leq \int_X C_2 V(y) e^{-\rho nT}\mu_s^x(dy) \\ &\leq C_2 \mathbf{E}V(u(s;x))e^{-\rho nT} \\ &\leq C_1 C_2 V(x) e^{-\rho nT}. \end{aligned}$$

Thus, for $C := C_1 C_2 e^{\rho T}$,

$$\left|\mathbf{E}G(u(t;x)) - \pi(G)\right| \leq C V(x) e^{-\rho t}.$$

□

3 ESTIMATES

Motivated by ideas from inertial manifold theory [2, 1], we decompose the space $X = P \oplus Q$, where P is the span of the first N eigenfunctions of A. Denote the projection from X to P by $\mathbb{P}$ and from X to Q by $\mathbb{Q}$. The advantage of this decomposition is that two different techniques can be applied to the different spaces. On P, the Itô formula may be exploited by considering the functional $\|\mathbb{P}\cdot\|^2$. In general, it is not possible to apply the Itô formula to $\|\cdot\|^2$ because $\operatorname{Tr} K$ may be infinite. On Q, the linear operator e^{-At} smoothes more strongly and this will be exploited through the following elementary inequality: for every $\delta_1, \delta_2 > 0$, there exists $C > 0$ (independent of N) such that

$$\|\mathbb{Q}A^{\delta_2} e^{-At}\| \leq \frac{C}{\lambda_{N+1}^{1-\delta_1-\delta_2} t^{1-\delta_1}}. \tag{3.1}$$

We present two simple lemmas, estimating the components of the solution u in the spaces P and Q.

Lemma 3.1 *(i) Solutions of (1.1) under (1.3) obey*

$$\mathbf{E}\|\mathbb{P}u(t;x)\|^2 \leq e^{-bt}\|\mathbb{P}x\|^2 + \frac{1}{b}\left(2a + \operatorname{Tr}\mathbb{P}K\right) + \frac{M_2^2}{b}\int_0^t e^{-b(t-s)}\mathbf{E}\|\mathbb{Q}u(s;x)\|^2\, ds.$$

(ii) Solutions of (1.2) under (1.4) obey

$$\mathbf{E}\|\mathbb{P}u(t;x)\|^2 \le e^{-bt}\|\mathbb{P}x\|^2 + \frac{2a}{b} + \frac{M_2^2}{b}\int_0^t e^{-b(t-s)}\mathbf{E}\|\mathbb{Q}u(s;x)\|^2\,ds.$$

Proof Apply (1.3),

$$\begin{aligned}\langle -Au + f(u), \mathbb{P}u\rangle \le& \langle -Au + f(\mathbb{P}u), \mathbb{P}u\rangle - \langle f(\mathbb{P}u) - f(u), \mathbb{P}u\rangle\\ \le& a - b\|\mathbb{P}u\|^2 + M_2\|\mathbb{Q}u\|\cdot\|\mathbb{P}u\|\\ \le& a - b\|\mathbb{P}u\|^2 + \frac{M_2^2}{2b}\|\mathbb{Q}u\|^2 + \tfrac{1}{2}b\|\mathbb{P}u\|^2.\end{aligned}$$

Hence

$$\langle -Au + f(u), \mathbb{P}u\rangle \le a - \tfrac{1}{2}b\|\mathbb{P}u\|^2 + \frac{M_2^2}{2b}\|\mathbb{Q}u\|^2.$$

Apply Itô's formula to the functional $\|\mathbb{P}u\|^2$:

$$\begin{aligned}d\|\mathbb{P}u(t)\|^2 =& 2\langle -Au(t) + f(u(t)), \mathbb{P}u(t)\rangle\,dt + \operatorname{Tr}\mathbb{P}K\,dt\\ &+ 2\langle u(t), \mathbb{P}\,dW(t)\rangle.\end{aligned}$$

Integrate

$$\mathbf{E}\|\mathbb{P}u(t)\|^2 = \mathbf{E}\|\mathbb{P}u(0)\|^2 + \int_0^t 2\mathbf{E}\langle -Au(s) + f(u(s)), \mathbb{P}u(s)\rangle + \operatorname{Tr}\mathbb{P}K\,ds.$$

Under the dissipative condition,

$$\mathbf{E}\|\mathbb{P}u(t)\|^2 \le \mathbf{E}\|\mathbb{P}u(0)\|^2 + \int_0^t 2\Big(a - \tfrac{1}{2}b\mathbf{E}\|\mathbb{P}u(s)\|^2 + \frac{M_2^2}{2b}\mathbf{E}\|\mathbb{Q}u(s)\|^2\Big) + \operatorname{Tr}\mathbb{P}K\,ds.$$

Hence, for $C := 2a + \operatorname{Tr}\mathbb{P}K$,

$$\frac{d}{dt}\mathbf{E}\|\mathbb{P}u(t)\|^2 \le -b\mathbf{E}\|\mathbb{P}u(t)\|^2 + C + \frac{M_2^2}{b}\mathbf{E}\|\mathbb{Q}u(t)\|^2.$$

It is clear that

$$\mathbf{E}\|\mathbb{P}u(t;x)\|^2 \le e^{-bt}\|\mathbb{P}x\|^2 + \frac{C}{b} + \frac{M_2^2}{b}\int_0^t e^{-b(t-s)}\mathbf{E}\|\mathbb{Q}u(s;x)\|^2\,ds. \tag{3.2}$$

For equation (1.2), the Itô formula yields

$$\begin{aligned}d\|\mathbb{P}u(t)\|^2 =& 2\langle -Au(t) + f(u(t)), \mathbb{P}u(t)\rangle\,dt + \|\mathbb{P}\sigma(u(s))K^{1/2}\|_{HS}^2\,dt\\ &+ 2\langle \mathbb{P}u(t), \sigma(u(t))\,dW(t)\rangle.\end{aligned}$$

Therefore, the dissipative condition (1.4) can be used in a similar manner, to gain (3.2), but with $C := 2a$. □

Lemma 3.2 *Consider* $0 \le \delta < 1/4$.

(i) Solutions of (1.1), for some $C > 0$ *independent of* N, *obey*

$$\Big(\mathbf{E}\|\mathbb{Q}A^\delta u(t;x)\|^2\Big)^{1/2} \le \frac{C\|x\|}{\lambda_{N+1}^{(1-4\delta)/4}t^{1/4}} + \frac{C(1+\|x\|)}{\lambda_{N+1}^{(1-4\delta)/4}}e^{Ct} + \Big(\tfrac{1}{2}\operatorname{Tr}A^{2\delta-1}K\Big)^{1/2}.$$

Suppose that $\operatorname{Tr}A^{2\delta-1}K < \infty$. *For all* $T, \mu > 0$, *there exists* $N_*, C > 0$ *such that, for* $N > N_*$,

$$\mathbf{E}\|\mathbb{Q}A^\delta u(t;x)\|^2 \le \frac{\mu\|x\|^2 + C}{t^{1/2}}, \qquad 0 < t \le T.$$

(ii) Solutions of (1.2), for some $C > 0$ *independent of* N, *obey*

$$\Big(\mathbf{E}\|\mathbb{Q}A^\delta u(t;x)\|^2\Big)^{1/2} \le \frac{C\|x\|}{\lambda_{N+1}^{(1-4\delta)/4}t^{1/4}} + \frac{C\,(1+\|x\|)}{\lambda_{N+1}^{(1-4\delta)/4}}e^{Ct} + \frac{C(1+\|x\|)}{\lambda_{N+1}^{(1-4\delta)/4}}e^{Ct}.$$

For all $T, \mu > 0$, *there exists* $N_*, C > 0$ *such that, for* $N > N_*$,

$$\mathbf{E}\|\mathbb{Q}A^\delta u(t;x)\|^2 \le \frac{\mu\|x\|^2 + C}{t^{1/2}}, \qquad 0 < t \le T.$$

Proof One may easily prove that for some constants $C_1, C_2 > 0$

$$\Big(\mathbf{E}\|u(t;x)\|^2\Big)^{1/2} \le C_1(1 + \|x\|)e^{C_2 t}.$$

Consider $\epsilon > 0$. By exploiting (3.1) and the Variation of Constants formula for (1.1), we have

$$\begin{aligned}&\Big(\mathbf{E}\|\mathbb{Q}A^\delta u(t;x)\|^2\Big)^{1/2}\\ &\le \|\mathbb{Q}A^\delta e^{-At}\|\cdot\|x\| + \int_0^t \|\mathbb{Q}A^\delta e^{-A(t-s)}\|M_1(1 + (\mathbf{E}\|u(s;x)\|^2)^{1/2})\,ds\\ &\quad + \Big(\int_0^t \|\mathbb{Q}A^\delta e^{-A(t-s)}K^{1/2}\|_{HS}^2\,ds\Big)^{1/2}\\ &\le \frac{C}{\lambda_{N+1}^{1-\delta-\epsilon}t^{1-\epsilon}}\|x\| + \int_0^t \frac{C}{\lambda_{N+1}^{1-\delta-\epsilon}(t-s)^{1-\epsilon}}M_1(1 + C_1(1+\|x\|)e^{C_2 t})\,ds\\ &\quad + (\tfrac{1}{2}\operatorname{Tr}KA^{2\delta-1})^{1/2}\\ &\le \frac{C}{\lambda_{N+1}^{1-\delta-\epsilon}t^{1-\epsilon}}\|x\| + \frac{Ct^\epsilon}{\lambda_{N+1}^{1-\delta-\epsilon}\epsilon}M_1(1 + C_1(1+\|x\|)e^{C_2 t}) + (\tfrac{1}{2}\operatorname{Tr}KA^{2\delta-1})^{1/2}.\end{aligned}$$

We may set $\epsilon = 3/4$ and this gives the first inequality of (i). To gain the second inequality, suppose that X is infinite dimensional and use the fact that $\lambda_N \to \infty$ under the compactness of A^{-1}.

The calculation is the same for (1.2), except for the last term, which may be bounded as follows:

$$\begin{aligned}&\mathbf{E}\Big\|\int_0^t \mathbb{Q}A^\delta e^{-A(t-s)}\sigma(u(s;x))\,dW(s)\Big\|^2\\ &\quad = \int_0^t \|\mathbb{Q}A^\delta e^{-A(t-s)}\sigma(u(s;x))K^{1/2}\|_{HS}^2\,ds\end{aligned}$$

$$\le \int_0^t \|\mathbf{Q}A^\delta e^{-A(t-s)}\|^2 \cdot \mathbf{E}\|\sigma(u(s;x))\|_{HS}^2 \cdot \|K^{1/2}\|^2\, ds$$
$$\le \int_0^t \left(\frac{C}{\lambda_{N+1}^{1-\delta-(\frac{1}{2}+\epsilon)}(t-s)^{1-(\frac{1}{2}+\epsilon)}}\right)^2 2\, M_3^2(1+\mathbf{E}\|u(s;x)\|^2)\|K^{1/2}\|^2\, ds$$
$$\le \int_0^t \frac{C^2}{\lambda_{N+1}^{1-2\delta-2\epsilon}(t-s)^{1-2\epsilon}} 2\, M_3^2(1+\mathbf{E}\|u(s;x)\|^2)\|K^{1/2}\|^2\, ds$$
$$\le \frac{C^2 t^{2\epsilon}}{\lambda_{N+1}^{1-2\delta-2\epsilon}2\epsilon}\, 2\, M_3^2(1+C_1(1+\|x\|^2)e^{C_2 t})\|K^{1/2}\|^2.$$

Put $\epsilon = 1/4$, to get the result. □

4 PROOFS OF THEOREMS 1.2 AND 1.3

Lemma 4.1 *Suppose that the hypothesis of either Theorem 1.2 or Theorem 1.3 hold. For $M > 0$, let*

$$\Gamma := \left\{x \in X : \|x\|^2 \le M\right\}.$$

There exists a compact set Θ such that

$$\mathbf{P}(u(T;x) \in \Theta) \ge \tfrac{1}{2}, \qquad x \in \Gamma.$$

Proof Choose $0 < \delta < 1/4$. In case of Theorem 1.2, δ should be small enough that

$$\operatorname{Tr} K A^{\delta-1} < \infty.$$

Define a set Θ_R by

$$\Theta_R := \left\{x \in X : \|A^{\delta/2}x\| \le R\right\}.$$

The space $\mathcal{D}(A^\gamma)$ for $\gamma > 0$ is compactly embedded in X [11] and consequently Θ_R is compact. By Chebyshev's inequality,

$$P^1(x, \Theta_R) = 1 - \mathbf{P}(\|A^{\delta/2}u(T;x)\| > R) \ge 1 - \frac{\mathbf{E}\|A^{\delta/2}u(T;x)\|^2}{R^2}.$$

Now,

$$\mathbf{E}\|A^{\delta/2}u(T;x)\|^2 \le \lambda_N^\delta \mathbf{E}\|\mathbf{P}u(T;x)\|^2 + \mathbf{E}\|\mathbf{Q}A^{\delta/2}u(T;x)\|^2.$$

By Lemmas 3.1–3.2, this quantity may be bounded uniformly for $x \in \Gamma$ by a constant c_1 independent of R. Hence, for $x \in \Gamma$,

$$P^1(x, \Theta_R) \ge 1 - \frac{c_1}{R^2} \ge \frac{1}{2},$$

by taking R sufficiently large. The result is proved with $\Theta := \Theta_R$. □

Proof of Theorem 1.2 Fix $T > 0$. Choose $0 < \mu, \alpha < 1$, so that

$$e^{-bT} + \frac{2M_2^2}{b}T^{1/2}\mu + \frac{\mu}{T^{1/2}} = 1 - \alpha < 1.$$

By choosing the dimension N of P large enough, Lemma 3.2 guarantees the existence of a $C_1 > 0$ such that

$$\mathbf{E}\|\mathbf{Q}u(t;x)\|^2 \le \frac{\mu\|x\|^2 + C_1}{t^{1/2}}, \qquad 0 < t \le T.$$

From Lemma 3.1, there exists $C_2 > 0$ such that

$$\mathbf{E}\|\mathbf{P}u(T;x)\|^2 \le e^{-bT}\|\mathbf{P}x\|^2 + C_2 + \frac{M_2^2}{b}\int_0^T e^{-b(T-s)}\mathbf{E}\|\mathbf{Q}u(s;x)\|^2\, ds.$$

Hence,

$$\begin{aligned}\mathbf{E}\|\mathbf{P}u(T;x)\|^2 &\le e^{-bT}\|\mathbf{P}x\|^2 + C_2 + \frac{M_2^2}{b}\int_0^T e^{-b(T-s)}\frac{\mu\|x\|^2 + C_1}{s^{1/2}}\, ds\\ &\le e^{-bT}\|\mathbf{P}x\|^2 + C_2 + \frac{M_2^2}{b}2T^{1/2}\left(\mu\|x\|^2 + C_1\right)\\ &\le \left(e^{-bT} + \frac{2M_2^2}{b}T^{1/2}\mu\right)\|x\|^2 + C_2 + \frac{2M_2^2}{b}T^{1/2}C_1.\end{aligned}$$

Hence

$$\begin{aligned}\mathbf{E}\|u(T;x)\|^2 &= \mathbf{E}\|\mathbf{P}u(T;x)\|^2 + \mathbf{E}\|\mathbf{Q}u(T;x)\|^2\\ &\le \left(e^{-bT} + \frac{2M_2^2}{b}T^{1/2}\mu\right)\|x\|^2 + C_2 + T^{1/2}\frac{2M_2^2}{b}C_1 + \frac{\mu\|x\|^2 + C_1}{T^{1/2}}\\ &\le \left(e^{-bT} + \frac{2M_2^2}{b}T^{1/2}\mu + \frac{\mu}{T^{1/2}}\right)\|x\|^2 + C_2 + T^{1/2}\frac{2M_2^2}{b}C_1 + \frac{C_1}{T^{1/2}}\\ &\le (1-\alpha)\|x\|^2 + C,\end{aligned}$$

for $C := C_2 + 2C_1T^{1/2}M_2^2/b + C_1/T^{1/2}$. For $V(u) = 1 + \|u\|^2$,

$$\begin{aligned}\mathbf{E}V(u(T;x)) &\le 1 + (1-\alpha)(V(x) - 1) + C\\ &\le (1 - \tfrac{1}{2}\alpha)V(x) + (C + \alpha - \tfrac{1}{2}\alpha V(x))\\ &\le (1 - \tfrac{1}{2}\alpha)V(x) + (C + \alpha)\mathbf{1}_\Gamma(x),\end{aligned}$$

where

$$\Gamma := \left\{x \in X : \|x\|^2 \le \frac{2C + \alpha}{\alpha}\right\}.$$

This is the drift condition (2.2) with respect to Γ.

The process u generated by (1.1) is strong Feller continuous (Theorem 7.2.4 of [4]) and open set irreducible (Theorem 7.4.2 of [4]). (These properties depend on the covariance operator being non-singular). Consequently, the theory of §2 applies:

by Lemma 2.1 (ii) and Lemma 4.1, the set Γ is petite with respect to the skeleton chain $\{u(0), u(T), \dots\}$. Finite time regularity with respect to V is easy to prove and so Theorem 2.2 completes the proof of geometric ergodicity with respect to $1+\|\cdot\|^2$.

Geometric ergodicity holds with respect to $1+\|\cdot\|$, giving a stronger bound on the convergence rate for the set of test functionals dominated by $1+\|\cdot\|$. This is because a Lyapunov–Foster drift condition for $1+\|\cdot\|$ follows from the drift condition for $1+\|\cdot\|^2$, as given by Lemma 15.2.9 of [6]. Again the theory of §2 applies. □

Proof of Theorem 1.3 The same as above, by using the corresponding estimates given in Lemmas 3.1–3.2, because the solution of (1.2) under (1.5)–(1.6) is strong Feller continuous and open set irreducible by Theorems 1.2–1.3 [9]. □

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