NUMERICAL METHODS FOR STOCHASTIC PARABOLIC PDEs

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Abstract

We develop a convergence theory for finite difference approximations of reaction diffusion equations forced by an additive space–time white noise. Special care is taken to develop the estimates in terms of non-smooth initial data and over a long time interval, motivated by an abstract approximation theory of ergodic properties developed by Shardlow & Stuart.

1 INTRODUCTION

This paper considers a nonlinear reaction diffusion equation driven by an additive space–time noise on the domain $(0,1)$ with Dirichlet conditions. The reaction term is assumed to be globally Lipschitz from $L_2(0,1)$ to itself. We examine a finite difference approximation, applying the θ method in time, the standard three point discrete Laplacian in space, and a spectral approximation to the noise. We develop a strong convergence theory, to gain estimates of the following form: let $u(t)$ for $0 \leq t \leq T$ denote the solution to the stochastic PDE, and u_n denote the solution to the numerical method at step n with time step Δt and grid spacing Δx. The numerical solution u_n is a function on the continuous space, defined by lifting the discrete numerical solution up to a function on $L_2(0,1)$. For each $\epsilon > 0$, we find that

$$(\mathbf{E}\|u(n\Delta t) - u_n\|^2)^{1/2} = \mathcal{O}(\Delta x^{1/2-\epsilon})\,.$$

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($\|\cdot\|$ is the $L_2(0,1)$ norm.) This estimate is obtained by taking limits with $\Delta t/\Delta x^2$ fixed subject to the stability condition $\Delta t/\Delta x^2(1-\theta) < 1/4$. (Recall that without the noise, these methods would converge like $\mathcal{O}(\Delta x^2)$ and with respect to a stability condition $\Delta t/\Delta x^2(1-2\theta) < 1/2$.)

Further details of the above estimate are given in the Theorem 3.7. In particular, we account for initial data in $L_2(0,1)$ by introducing a time $t = 0$ singularity into the estimate, and describe the exponential growth of the estimate in large time. These features are familiar from deterministic theories. They were specifically included here, to allow the present convergence theory to fit into the framework of Shardlow–Stuart [8]. In [8], two abstract properties of a stochastic process and its approximation are shown to imply a weak convergence of the approximation to the invariant measure of the process. The first assumption is that of finite time convergence, in the sense developed in this paper; the second is geometric ergodicity—see [7, 2] for a discussion of this topic for stochastic PDEs.

The paper is arranged as follows: Theorem 2.9 examines the linear deterministic problem: this situation is well understood, but the proof is given to bring out an estimate for initial data in $L_2(0,1)$ (normally initial data is taken with four derivatives, to gain a Δx^2 convergence uniformly up to the origin of a finite time interval). Theorem 2.10 examines the strong numerical approximation of a linear stochastic PDE. Theorem 3.7 extends this result to account for Lipschitz nonlinearities.

Previous works on strong convergence of finite difference schemes for stochastic PDEs include Gyöngy [5] and Davie & Gaines [3]. Both papers discuss finite differences for parabolic stochastic PDEs on a bounded domain in one dimension. Gyöngy proves detailed estimates related to L^p convergence, and gains convergence (without rates) for non-Lipschitz nonlinearities. Davie & Gaines prove similar result, but also prove the rates attained are the best available.

1.1 Notation

Denote by $\|\cdot\|$ the $L_2(0,1)$ norm; and by $\|\cdot\|_{HS}$ the Hilbert–Schmidt norm of an operator from $L_2(0,1)$ to $L_2(0,1)$. Recall that $\sqrt{2}\sin(k\pi x)$ for $k = 1, 2, \ldots$ is a complete orthonormal system for $L_2(0,1)$. Thus, for $\Psi : L_2(0,1) \to L_2(0,1)$,

$$\|\Psi\|_{HS}^2 = 2\sum_{k=1}^{\infty} \|\Psi \sin(k\pi\cdot)\|^2, \tag{1.1}$$

or, if Ψ is written as integral with respect to a kernel $K(x,\xi)$,

$$(\Psi f)(x) = \int_0^1 K(x,\xi) f(\xi)\, d\xi, \qquad 0 < x < 1;$$

then

$$\|\Psi\|_{HS}^2 = \int_0^1 \int_0^1 |K(x,\xi)|^2\, dx\, d\xi.$$

We recall a few generalities concerning Wiener processes in infinite dimensions. Let $B(t)$ be a Wiener process with covariance Q. This process may be considered in terms of its Fourier series. Suppose that Q has eigenvalues $\alpha_k > 0$ and corresponding eigenfunctions $\sin(k\pi\cdot)$. Let β_k for $k = 1, 2, \ldots$ be a sequence of independent Brownian motions. Then,

$$B(t) = \sum_{i=1}^{\infty} \sqrt{2}\sqrt{\alpha_k} \sin(k\pi\cdot)\beta_k(t), \tag{1.2}$$

is a Wiener process with covariance Q.

Space–time white noise is simply the Wiener process with $Q = I$. The space–time white noise process does not exist in $L_2(0,1)$, but stochastic integrals can be defined with respect to B, when the integrand smoothes the noise process sufficiently. We derive the numerical method estimates for the case $Q = I$, for simplicity of notation. The case of a general bounded linear operator Q on $L_2(0,1)$ can be approximated in an analogous way.

The Ito isometry for a Wiener process of covariance Q states that, for an integrand $\Psi\colon [0,T] \to \mathcal{L}(L_2(0,1), L_2(0,1))$ (i.e., a function on $[0,T]$ taking values in the linear operators from $L_2(0,1)$ to $L_2(0,1)$)

$$\mathbf{E}\|\int_0^T \Psi(s)\, dB(s)\|^2 = \int_0^T \|\Psi(s)Q\|_{HS}^2\, ds. \tag{1.3}$$

In particular, the stochastic integral $\int_0^T \Psi(s)\, dB(s)$ makes sense only when $\|\Psi(s)Q\|_{HS}$ is square integrable. This result will be applied frequently to the integrand $e^{-A(t-s)}$.

2 LINEAR PROBLEM

Consider the equation, for initial data $U \in L_2(0,1)$,

$$du + Au\, dt = dB(t), \qquad u(0) = U \tag{2.1}$$

in the case that $A = -\partial^2/\partial x^2$ with domain $H^2(0,1) \cap H_0^1(0,1)$ and $B(t)$ is a Wiener process. For simplicity, we work with a space–time white noise so that the covariance $Q = I$.

Consider a time step Δt and a grid size $\Delta x = 1/J$, some $J \in \mathbf{N}$. We construct a numerical approximation to (2.1) as follows: The Wiener process is approximated by truncating its Fourier expansion to $J-1$ terms. Let $\mathbb{P}$ denote the operator taking f to its first $J-1$ Fourier modes,

$$\mathbb{P}f := 2\sum_{j=1}^{J-1} \langle f, \sin(\pi j\cdot)\rangle \sin(j\pi\cdot).$$

Then, we approximate the Wiener process over the time step $(n\Delta t, (n+1)\Delta t)$ by

$$dB_{\Delta t}(n) := \int_{n\Delta t}^{(n+1)\Delta t} \mathbb{P}\, dB(s). \tag{2.2}$$

This gives an $L_2(0,1)$ function; the numerical method evaluates this function at the grid point $i\Delta x$ for $i = 1, \dots, J-1$. In practise, this may be done efficiently using the Fast Fourier Transform (FFT). The initial condition is also approximated by using $u_0 := \mathbb{P}U$.

The numerical method is

$$\begin{aligned} \mathbf{u}_{n+1} - \mathbf{u}_n + \frac{\Delta t}{\Delta x^2} A((1-\theta)\mathbf{u}_n + \theta \mathbf{u}_{n+1}) &= \begin{bmatrix} dB_{\Delta t}(n)(\Delta x) \\ \vdots \\ dB_{\Delta t}(n)((J-1)\Delta x) \end{bmatrix}, \\ \mathbf{u}_0 = [u_0(\Delta x), \dots, u_0((J-1)\Delta x)]^T, \quad A &:= \begin{bmatrix} 2 & -1 & & \\ -1 & 2 & -1 & \\ & -1 & 2 & -1 \\ & & -1 & 2 \end{bmatrix} \end{aligned} \tag{2.3}$$

The quantity $dB_{\Delta t}$ is well defined in $L_2(0,1)$, as follows.

Lemma 2.1

$$\mathbf{E}\|dB_{\Delta t}(n)\|^2 \le \frac{\Delta t}{\Delta x}.$$

Proof By the Ito isometry (1.3), by (1.1), and as $\Delta x = 1/J$,

$$\mathbf{E}\|\, dB_{\Delta t}(n)\|^2 = \int_{n\Delta t}^{(n+1)\Delta t} \|\mathbb{P}\|_{HS}^2\, ds \le \Delta t(J-1) \le \frac{\Delta t}{\Delta x}.$$

□

We'd like to consider (2.3) on the continuous space $L_2(0,1)$. Let $\mathbf{u}_n = [u_n^1, \dots, u_n^{J-1}]^T$. To do this, we replace $\mathbf{u}_n$ by $u_n \in L_2(0,1)$ such that

$$u_n(i\Delta x) = u_n^i, \quad i = 1, \dots, J-1. \tag{2.4}$$

Every vector in (2.3) must be replaced a continuous function that equals the vector at the grid points $i\Delta x$ for $i = 1, \ldots, J-1$. Replace the vector

$$[\ dB_{\Delta x}(n)(\Delta x), \ldots]^T$$

by the $L_2(0,1)$ function $dB_{\Delta x}(n)$. The matrix operator A must be replaced by a continuous operator $\tilde{A}$ on $L_2(0,1)$ such that, if (2.4) holds, then

$$(\tilde{A}u_n)(i\Delta x) = (A\mathbf{u}_n)^i, \quad i = 1, \ldots, J-1. \tag{2.5}$$

(the superscript i denotes the ith component of the vector.) Denote by $\mathbf{e}_k$ the grid function with components $e_k^j = \sin(kj\pi\Delta x)$. The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_{J-1}$ are an orthogonal basis for $\mathbb{R}^{J-1}$. Clearly, $\mathbf{e}_{j+J} = \mathbf{e}_j$.

In Fourier space, the matrix A is represented by

$$A\mathbf{e}_k = 4\sin^2(k\pi\Delta x/2)\mathbf{e}_k.$$

Define the operator $\tilde{A}$ as

$$\tilde{A}\sin(k\pi\cdot) = \lambda_k \sin(k\pi\cdot), \quad k = 1, \ldots,$$

where λ_k are to be determined. Consider $m = k + nJ$, where $k = 1, \ldots, J-1$ and $n = 0, 1 \ldots$. Now to satisfy (2.5), we require, for $l = 1, \ldots, J-1$,

$$(\tilde{A}\sin(m\pi\cdot))(l\Delta x) = (A\mathbf{e}_m)^l = (A\mathbf{e}_k)^l = 4\sin^2(k\pi\Delta x/2)\sin(kl\pi\Delta x).$$

Hence, for $k = 1, \ldots, J-1$ and $n = 0, \ldots,$

$$\lambda_m = \lambda_{k+nJ} = 4\sin^2(k\pi\Delta x/2). \tag{2.6}$$

This specifies the operator $\tilde{A}$.

The continuous version of the method (2.3) is

$$\begin{aligned} u_{n+1} - u_n + \frac{\Delta t}{\Delta x^2}\tilde{A}\Big[(1-\theta)u_n + \theta u_{n+1}\Big] &= dB_{\Delta t}(n), \\ u_0 &= \mathbb{P}U. \end{aligned} \tag{2.7}$$

Rearrange this equation, to get

$$\Big[I + \frac{\Delta t}{\Delta x^2}\theta\tilde{A}\Big]u_{n+1} = \Big[I - \frac{\Delta t}{\Delta x^2}(1-\theta)\tilde{A}\Big]u_n + \ dB_{\Delta t}(n).$$

Hence,

$$u_{n+1} = (I - \mathcal{C}\Delta t)u_n + Q_{\Delta x}\ dB_{\Delta t}(n), \tag{2.8}$$

where

$$\mathcal{C} =\frac{1}{\Delta t}\Big(I-\Big[I+\frac{\Delta t}{\Delta x^2}\theta\tilde{A}\Big]^{-1}\Big[I-\frac{\Delta t}{\Delta x^2}(1-\theta)\tilde{A}\Big]\Big) \tag{2.9}$$

$$Q_{\Delta x} =\Big[I+\frac{\Delta t}{\Delta x^2}\theta\tilde{A}\Big]^{-1}. \tag{2.10}$$

From (2.6) and (2.9), the eigenvalue of $\mathcal{C}$ corresponding to the eigenfunction $\sin(k\pi\cdot)$ is

$$\mu_k := \frac{1}{\Delta t}\Big(1-\frac{(1-\nu(1-\theta)\lambda_k)}{1+\nu\theta\lambda_k}\Big) = \frac{1}{\Delta t}\frac{\nu\lambda_k}{1+\nu\theta\lambda_k}, \quad \nu := \frac{\Delta t}{\Delta x^2}. \tag{2.11}$$

Clearly, $\mu_1 \le \mu_2 \le \dots \mu_{J-1}$. The corresponding eigenvalue of $Q_{\Delta x}$ is

$$\frac{1}{1+\theta\nu\lambda_k}, \quad k=1,\dots.$$

Note that $\|Q_{\Delta x}\| \le 1$.

We prove a sequence of lemmas concerning the properties of $Q_{\Delta x}$, the eigenvalues μ_k, the smoothing properties of e^{-At}, and the properties of $\mathcal{C}$.

Lemma 2.2 *(i) For each $\gamma > 5/4$, there exists a constant c_γ such that*

$$\|A^{-\gamma}(I-Q_{\Delta x}\mathbb{P})\|_{HS}^2 \le c_\gamma \Delta x^4.$$

(ii) For each $\epsilon > 0$, there exists a constant C_ϵ with

$$\|e^{-At}(I-Q_{\Delta x})\| \le \frac{C_\epsilon}{t^{1+\epsilon}}\Delta x^2.$$

Proof For (i), consider $j = 1,\dots,J-1$ and set $\gamma = (5+\epsilon)/4$,

$$\begin{aligned}\|A^{-\gamma}(I-Q_{\Delta x}\mathbb{P})\sqrt{2}\sin(j\pi\Delta x)\|^2 \le&\frac{1}{(j^2\pi^2)^{2\gamma}}\Big(1-\frac{1}{1+4\theta\nu\sin^2(j\pi\Delta x/2)}\Big)^2\\ \le&\frac{1}{(j^2\pi^2)^{2\gamma}}16\theta^2\nu^2\sin^4(j\pi\Delta x/2)\\ \le&16\theta^2\nu^2\frac{1}{(j^2\pi^2)^{2\gamma}}\Big(\frac{j\pi\Delta x}{2}\Big)^4\\ \le&\theta^2\nu^2\pi^{4(1-\gamma)}j^{4(1-\gamma)}\Delta x^4\\ =&\theta^2\nu^2\pi^{4(1-\gamma)}\frac{\Delta x^4}{j^{1+\epsilon}}.\end{aligned}$$

For $j = J,\dots,$

$$\begin{aligned}\|A^{-\gamma}(I-Q_{\Delta x}\mathbb{P})\sqrt{2}\sin(j\pi\Delta x)\|^2 \le&\frac{1}{(j^2\pi^2)^{2\gamma}}\\ =&\frac{1}{\pi^{4\gamma}j^4}\frac{1}{j^{1+\epsilon}}\\ \le&\frac{\Delta x^4}{\pi^{4\gamma}}\frac{1}{j^{1+\epsilon}}.\end{aligned}$$

Consequently, for all $\epsilon > 0$, there exists a C_ϵ with

$$\|A^{-\gamma}(I - Q_{\Delta x})\|_{HS}^2 \le \theta^2\nu^2\pi^{4(1-\gamma)}\Delta x^4 \sum_{j=1}^{J-1} \frac{1}{j^{1+\epsilon}} + \frac{\Delta x^4}{\pi^{4\gamma}} \sum_{j=J}^{\infty} \frac{1}{j^{1+\epsilon}} \le C_\epsilon \Delta x^4.$$

For (ii), consider $j = 1, \dots, J-1$ and $n = 1, \dots$ and let $k = j + nJ$. Consider $\gamma > 1$. Then, as above, for a constant C,

$$\begin{aligned}\|e^{-At}(I - Q_{\Delta x})\sqrt{2}\sin(k\pi\cdot)\| &\le e^{-k^2\pi^2 t}\Big(1 - \frac{1}{1 + 4\theta\nu\sin^2(j\pi\Delta x/2)}\Big)\\ &\le e^{-k^2\pi^2 t} 4\theta\nu \sin^2(j\pi\Delta x/2)\\ &\le C e^{-j^2\pi^2 t} j^2 \Delta x^2 \le \frac{C}{(j^2 t)^\gamma} j^2\Delta x^2 \le \frac{C}{t^\gamma}\Delta x^2.\end{aligned}$$

□

Lemma 2.3 *For $0 \le \theta \le 1$ and $\Delta t/\Delta x^2$ fixed, there exists γ such that*

$$\pi^2 - \gamma^2\Delta x^2 j^2 \le \frac{\mu_j}{j^2} \le \pi^2, \quad j = 1, \dots, J-1.$$

Thus $\pi^2 - \gamma^2\Delta x^2 \le \mu_1 \le \mu_j$.

Proof Let

$$X := \frac{\sin^2(j\pi\Delta x/2)}{(\Delta x/2)^2},$$

By Taylor expanding, we can show that

$$j^2\pi^2 - \frac{1}{12} j^4\pi^4\Delta x^2 \le X \le j^2\pi^2.$$

Now, from (2.11),

$$\mu_j = \frac{X}{1 + \theta\Delta t X} = X - \frac{\theta X^2 \Delta t}{1 + \theta X \Delta t}.$$

Hence,

$$\mu_j \le j^2\pi^2 - \frac{\theta X^2\Delta t}{1 + \theta X\Delta t} \le j^2\pi^2.$$

Now, because $\theta X^2\Delta t/(1 + \theta X\Delta t) \le \theta j^4\pi^4\Delta t$,

$$\begin{aligned}\mu_j &\ge j^2\pi^2 - \frac{1}{12}j^4\pi^4\Delta x^2 - \theta j^4\pi^4\Delta t\\ &= j^2\pi^2 - j^4\pi^4\Delta x^2\Big(\frac{1}{12} + \theta\frac{\Delta t}{\Delta x^2}\Big)\\ &= j^2\pi^2 - j^4\Delta x^2\gamma^2,\end{aligned}$$

for $\gamma^2 := \pi^4(1/12 + \theta\Delta t/\Delta x^2)$. □

Proposition 2.4 *Consider $t > 0$; for a constant $C_\gamma > 0$ depending on γ,*

$$\|A^{-\gamma}(I - e^{-At})\| \le C_\gamma\, t^\gamma, \qquad 0 \le \gamma \le 1;$$

$$\|A^\gamma e^{-At}\| \le C_\gamma\, t^{-\gamma}, \qquad \gamma > 0.$$

Proof This is a standard result on powers of sectorial operators [6]. □

Lemma 2.5 *For constants $C, \gamma > 0$,*

(i) $\|(e^{-At} - e^{-\mathcal{C}t})\mathbb{P}\| \le Ct^{-\gamma}\Delta x^{2\gamma}$.

(ii) $\|\mathcal{C}^{-2}(e^{-\mathcal{C}\Delta t} - (I - \mathcal{C}\Delta t))\mathbb{P}\| \le C\Delta t^2$.

(iii) $\|\mathcal{C}^\gamma e^{-\mathcal{C}t}\mathbb{P}\| \le C\, t^{-\gamma}$.

Proof The eigenvalues of A are $j^2\pi^2$ and those of $\mathcal{C}$ are μ_j. (i) For $j = 1, \dots, J-1$ and a constant C, (using $|e^{-x} - e^{-y}| \le e^{-(x\wedge y)}|x - y|$ for $x, y \ge 0$)

$$|e^{-\pi^2 j^2 t} - e^{-\mu_j t}| \le e^{-(\mu_j \wedge \pi^2 j^2)t} \frac{|\pi^2 j^2 - \mu_j|}{(j^2\pi^2)^{1+\gamma}}\, t\, (j^2\pi^2)^{1+\gamma}$$

(using $(j^2\pi^2 - \mu_j)/j^2\pi^2 \le C\Delta x^2 j^2$ for $j = 1, \dots, J-1$ by Lemma 2.3)

$$\le Ce^{-(\mu_j \wedge \pi^2 j^2)t} \frac{\Delta x^2 j^2}{(j^2\pi^2)^\gamma} t\, (j^2\pi^2)^{1+\gamma}$$

(as $j^2\Delta x^2 \le 1$ and $j = 1, \dots, J-1$)

$$\le Ce^{-(\mu_j \wedge \pi^2 j^2)t} \frac{\Delta x^{2\gamma}}{\pi^{2\gamma}}\, t\, (j^2\pi^2)^{1+\gamma}$$

(as $\mu_j \wedge \pi^2 j^2 = \pi^2 j^2$ by Lemma 2.3)

$$\le C\frac{\Delta x^{2\gamma}}{\pi^{2\gamma}}\, t\, (j^2\pi^2)^{1+\gamma} e^{-j^2\pi^2 t}.$$

Hence,

$$\|(e^{-At} - e^{-\mathcal{C}t})\sqrt{2}\sin(k\pi\cdot)\| \le C\frac{\Delta x^{2\gamma}}{\pi^{2\gamma}} t\|A^{(1+\gamma)}e^{-At}\sqrt{2}\sin(k\pi\cdot)\|.$$

Finally apply Proposition 2.4, to gain (i).

(ii) For $j = 1, \dots, J-1$ and a constants C,

$$|e^{-\mu_j\Delta t} - (1 - \mu_j\Delta t)| \le C\mu_j^2\Delta t^2.$$

Since $\mathcal{C}$ has eigenvalues μ_j, the result follows. (iii) For $j = 1, \dots,$

$$\|\mathcal{C}^\gamma e^{-\mathcal{C}t} \sqrt{2}\sin(j\pi\cdot)\| = \mu_j^\gamma e^{-\mu_j t} = \frac{1}{t^\gamma}(t\mu_j)^\gamma e^{-\mu_j t}.$$

There exists C with $x^\gamma e^{-x} \le C$ for $x \ge 0$. Hence,

$$\|\mathcal{C}^\gamma e^{-\mathcal{C}t} \sqrt{2}\sin(j\pi\cdot)\| \le C t^{-\gamma}.$$

□

Lemma 2.6 *Let Z_1, Z_2 be bounded linear operators $L_2(0,1) \to L_2(0,1)$.*

$$\mathbf{E}\langle Z_1\, dB_{\Delta t}(i), Z_2\, dB_{\Delta t}(j)\rangle = 0, \qquad i \ne j.$$

Proof This is simply because $dB_{\Delta t}(i)$ and $dB_{\Delta t}(j)$ are independent $L_2(0,1)$ random variables for $i \ne j$. □

The stability properties of the method are established in the following Lemma.

Lemma 2.7 *Let $\nu = \Delta t/\Delta x^2$. Subject to $\nu(1-\theta) < 1/4$, the following hold*

(i) $0 \le \mu_k \Delta t \le 1$.

(ii) $\|(I - \mathcal{C}\Delta t)e^{\mathcal{C}\Delta t}\| \le 1$.

(iii) there exists $\mu, \Delta x^* > 0$ such that the following holds uniformly for $\Delta x < \Delta x^*$,

$$\|(I - \mathcal{C}\Delta t)^n\| \le e^{-\mu n \Delta t}.$$

(iv) for $0 < \gamma < 1/2$, there exists a constant $C_\gamma > 0$, such that, for $n \ge 1$,

$$\mathbf{E}\|A^\gamma \sum_{i=0}^{n-1} (I - \mathcal{C}\Delta t)^{n-i-1} Q_{\Delta x}\, dB_{\Delta t}(i)\|^2 \le C_\gamma G(\Delta x, \gamma)$$

where $G(\Delta x, \gamma)$ is defined by

$$G(\Delta x, \gamma) := \begin{cases} 1, & \gamma < 1/4; \\ -\log \Delta x, & \gamma = 1/4; \\ \Delta x^{1-4\gamma}, & \gamma > 1/4. \end{cases}$$

Proof For (i),

$$\begin{aligned}(1-\mu_j\Delta t) =& \frac{1-4(1-\theta)\nu\sin^2(j\pi\Delta x/2)}{1+4\theta\nu\sin^2(j\pi\Delta x/2)}\\ =& 1-\frac{4\nu\sin^2(j\pi\Delta x/2)}{1+4\theta\nu\sin^2(j\pi\Delta x/2)}\end{aligned}$$

Hence, $0\le 1-\mu_j\Delta t\le 1$ when

$$\frac{4\nu\sin^2(j\pi\Delta x/2)}{1+4\theta\nu\sin^2(j\pi\Delta x/2)}<1,$$

which holds for $\nu(1-\theta)<1/4$.

(ii) follows because μ_j are the eigenvalues of $\mathcal{C}$ and by using the inequality $|1-x|\le e^{-x}$ for $0<x<1$.

(iii) The eigenvalues μ_k may be bounded below by $\mu>0$ uniformly for $\Delta x\le\Delta x^*$, some $\Delta x^*>0$ (Lemma 2.3). Thus,

$$\|I-\mathcal{C}\Delta t\|\le e^{-\mu\Delta t},$$

by using again $|1-x|\le e^{-x}$ for $0\le x\le 1$.

(iv) By using $\|Q_{\Delta x}\|\le 1$ and (2.2) and Lemma 2.6

$$\begin{aligned}&\mathbf{E}\Big\|\sum_{i=0}^{n-1}A^{\gamma}(I-\mathcal{C}\Delta t)^{n-1-i}Q_{\Delta x}\,dB_{\Delta t}(i)\Big\|^2\\ =&\sum_{i=0}^{n-1}\sum_{k=1}^{J-1}(k^2\pi^2)^{2\gamma}(1-\mu_k\Delta t)^{2(n-i-1)}\Delta t\\ \le&\sum_{i=0}^{n-1}\sum_{k=1}^{J-1}(k^2\pi^2)^{2\gamma}e^{-2\mu_k\Delta t(n-i-1)}\Delta t\\ \le&\sum_{k=1}^{J-1}(k^2\pi^2)^{2\gamma}\frac{1-e^{-2\mu_k n\Delta t}}{1-e^{-2\mu_k\Delta t}}\Delta t\\ \le&\sum_{k=1}^{J-1}C\frac{(k^2\pi^2)^{2\gamma}}{2\mu_k}.\end{aligned}$$

This is finite for $\gamma<1/4$ by Lemma 2.3. For $1/4<\gamma<1/2$, the term grows like

$$\begin{aligned}\sum_{k=1}^{J-1}\frac{k^{4\gamma}}{\mu_k}\le& C\sum_{k=1}^{J-1}k^{4\gamma-2}\\ \le& CJ^{4\gamma-1}\\ \le& C\Delta x^{1-4\gamma}.\end{aligned}$$

For $\gamma = 1/4$, the term grows like

$$\sum_{k=1}^{J-1} \frac{k^{4\gamma}}{\mu_k} \le C \sum_{k=1}^{J-1} \frac{1}{k} \le C \log J \le -C \log \Delta x.$$

The three cases are accounted for by the definition of $G(\cdot,\cdot)$. □

Lemma 2.8 *For linear operators Ψ_1, Ψ_2 from $L_2(0,1)$ to itself,*

$$\|\Psi_1\Psi_2\|_{HS} \le \|\Psi_1\| \cdot \|\Psi_2\|_{HS},$$

where $\|\Psi_1\|$ denotes the L_2 operator norm.

Proof Elementary. □

Theorem 2.9 *Suppose that initial data $U \in L_2(0,1)$ and that $(\Delta x, \Delta t) \to 0$ under the stability condition*

$$\frac{\Delta t}{\Delta x^2}(1-\theta) \le \frac{1}{4}. \tag{2.12}$$

Consider (2.3) applied to (2.1), with the noise term $B = 0$. For each $0 < s < 1$, there exists a constant C with

$$\|u(n\Delta t) - u_n\| \le C\Delta x^{2s} \frac{1}{(n\Delta t)^s} \|U\|.$$

Proof Expand $u(t)$ as

$$u(t,x) := \sqrt{2} \sum_{j=1}^{\infty} a_j(t) \sin(j\pi x).$$

Introduce $v(t,x)$,

$$v(t,x) := \sqrt{2} \sum_{j=1}^{J-1} b_j(t) \sin(j\pi x),$$

where b_j solves the equation

$$db_j(t) = -\mu_j b_j(t)\, dt, \qquad b_j(0) = a_j(0).$$

We estimate $\|u(n\Delta t) - u_n\|^2$, by applying the triangle inequality as follows

$$\begin{aligned}\|u(n\Delta t) - u_n\|^2 \le& 3\|u(n\Delta t) - \mathbb{P}u(n\Delta t)\|^2 + 3\|\mathbb{P}u(n\Delta t) - v(n\Delta t)\|^2 \\ &+ 3\|v(n\Delta t) - u_n\|^2.\end{aligned}$$

The estimates below use a generic constant C, which is independent of Δx, Δt. The constant C will absorb other constants, including ν and θ without comment.

Term 1.

$$u(n\Delta t, x) - \mathbb{P}u(n\Delta t, x) = \sqrt{2}\sum_{k=J}^{\infty} a_k(n\Delta t)\sin(k\pi x),$$

where $a_k(t) = e^{-k^2\pi^2 t}a_k(0)$. Therefore,

$$\begin{aligned}\|u(n\Delta t,\cdot) - \mathbb{P}u(n\Delta t,\cdot)\|^2 \leq& \sum_{k=J}^{\infty} a_k^2(0)e^{-2\pi^2k^2n\Delta t}\\ \leq& e^{-2J^2\pi^2n\Delta t}\sum_{k=J}^{\infty} e^{2(J^2-k^2)\pi^2n\Delta t}a_k^2(0)\end{aligned}$$

(because $x^{2s}e^{-x}$ may be bounded uniformly in $x \geq 0$ for $s > 0$)

$$\leq C\frac{1}{\pi^{4s}J^{4s}(n\Delta t)^{2s}}\|U\|^2 \leq C\frac{\Delta x^{4s}}{\pi^{4s}}\frac{1}{(n\Delta t)^{2s}}\|U\|^2.$$

Term 2.

$$\mathbb{P}u(n\Delta t, x) - v(n\Delta t, x) = \sqrt{2}\sum_{k=1}^{J-1}\Big(a_k(n\Delta t) - b_k(n\Delta t)\Big)\sin(k\pi x),$$

where

$$a_k(n\Delta t) = e^{-k^2\pi^2n\Delta t}a_k(0), \qquad b_k(n\Delta t) = e^{-\mu_k n\Delta t}a_k(0).$$

Now, $|e^{-x} - e^{-y}| \leq e^{-(x\wedge y)}|x-y|$ for $x, y > 0$ and $\mu_k \leq k^2\pi^2$ by Lemma 2.3, so that

$$\begin{aligned}&\sum_{k=1}^{J-1}(a_k(n\Delta t) - b_k(n\Delta t))^2\\ \leq&\sum_{k=1}^{J-1} e^{-2\mu_k n\Delta t}|k^2\pi^2 - \mu_k|^2\ (n\Delta t)^2a_k^2(0)\\ =&\sum_{k=1}^{J-1}(\mu_k n\Delta t)^{2+2s}e^{-2\mu_k n\Delta t}\frac{|k^2\pi^2-\mu_k|^2(n\Delta t)^2}{(\mu_k n\Delta t)^{2+2s}}a_k^2(0)\\ \leq&\sum_{k=1}^{J-1} C\frac{|k^2\pi^2-\mu_k|^2}{\mu_k^2}\frac{1}{(n\Delta t)^{2s}\mu_k^{2s}}a_k^2(0)\\ \leq&\sum_{k=1}^{J-1} C\Delta x^4\mu_k^2\frac{1}{(n\Delta t)^{2s}\mu_k^{2s}}a_k^2(0)\\ \leq&\sum_{k=1}^{J-1} C\Delta x^4\mu_k^{2(1-s)}\frac{1}{(n\Delta t)^{2s}}a_k^2(0)\\ \leq& C\Delta x^{4s}\frac{1}{(n\Delta t)^{2s}}\|U\|^2.\end{aligned}$$

Term 3.

$$v(n\Delta t,x)-u_n(x)=\sqrt{2}\sum_{k=1}^{J-1}\Big(b_k(n\Delta t)-A_k(n\Delta t)\Big)\sin(k\pi x)$$

where

$$b_k(n\Delta t)=e^{-\mu_k n\Delta t}a_k(0)$$

and $A_k(n\Delta t)=A_{n,k}$ where (from (2.8))

$$A_{n,k}=(1-\mu_k\Delta t)^n a_k(0).$$

Here we exploit the fact that, for the linear problem, the numerical solution has only $J-1$ terms in its Fourier expansion; this happens because we have taken initial condition $u_0=\mathbb{P}U$. When $n\geq 1$,

$$|b_k(n\Delta t)-A_k(n\Delta t)|^2=\Big\{e^{-\mu_k n\Delta t}-(1-\mu_k\Delta t)^n\Big\}^2 a_k^2(0)$$

(because $a^n-b^n=(a-b)\sum_{j=0}^{n-1}a^{n-j-1}b^j$)

$$\leq\Big(e^{-\mu_k\Delta t}-(1-\mu_k\Delta t)\Big)^2\Big(\sum_{j=0}^{n-1}e^{-\mu_k(n\Delta t-(j+1)\Delta t)}(1-\mu_k\Delta t)^j\Big)^2 a_k^2(0)$$

(because $|1-x|\leq e^{-x}$ for $0\leq x\leq 1$ and because the stability condition ensures $0\leq\mu_k\Delta t\leq 1$)

$$\leq\Big(e^{-\mu_k\Delta t}-(1-\mu_k\Delta t)\Big)^2\Big(\sum_{j=0}^{n-1}e^{-\mu_k(n-1)\Delta t)}\Big)^2 a_k^2(0).$$

Now,

$$\begin{aligned}&|b_k(n\Delta t)-A_k(n\Delta t)|^2\\ \leq&C\ \mu_k^4\Delta t^4\ \Big(\sum_{j=0}^{n-1}e^{-\mu_k(n\Delta t-\Delta t)}\Big)^2 a_k^2(0)\\ \leq&C\ \Delta t^4\mu_k^{2-2s}\Big(\sum_{j=0}^{n-1}\frac{\mu_k^{1+s}(n\Delta t-\Delta t)^{1+s}}{(n\Delta t-\Delta t)^{1+s}}e^{-\mu_k(n-1)\Delta t}\Big)^2 a_k^2(0)\end{aligned}$$

(by bounding $x^{1+s}e^{-x}$ uniformly in $x>0$)

$$\begin{aligned}\leq&C\ \Delta t^2\mu_k^{2-2s}\Big(\sum_{j=0}^{n-1}\frac{1}{(n\Delta t-\Delta t)^{1+s}}\Delta t\Big)^2 a_k^2(0)\\ \leq&C\ \Delta t^2\mu_k^{2-2s}\Big(\frac{1}{(n\Delta t)^s}\Big)^2 a_k^2(0).\end{aligned}$$

Thus,

$$\sum_{k=1}^{J-1} |b_k(n\Delta t) - A_k(n\Delta t)|^2 \leq C \sum_{k=1}^{J-1} \Delta t^2 \mu_k^{2-2s} \frac{1}{(n\Delta t)^{2s}} a_k^2(0)$$

(because by Lemma 2.3 and for $s < 1$, $\mu_k^{2-2s} \leq \pi^{2(2-s)} \Delta x^{4s-4}$ for $k = 1, \dots, J-1$)

$$\leq C \Delta x^{4s-4} \Delta t^2 \frac{1}{(n\Delta t)^{2s}} \|U\|^2$$

And by taking $\Delta t / \Delta x^2 = \nu$ fixed and $s \leq 1$, for a constant C we have that

$$\|v(n\Delta t, x) - u_n(x)\|^2 \leq C \Delta x^{4s} \frac{\nu^2}{(n\Delta t)^{2s}} \|U\|^2.$$

□

Theorem 2.10 *For initial data U in $L_2(0,1)$, the numerical method (2.3) converges strongly to a weak solution of the stochastic PDE (2.1), subject to the stability condition (2.12). Indeed, for all $\epsilon > 0$, there exists C_ϵ such that, for all $n\Delta t > 0$,*

$$\left(\mathbf{E}\|u(n\Delta t) - u_n\|^2\right)^{1/2} \leq C_\epsilon \Big(1 + \|U\|\Big) \left(1 + \frac{1}{(n\Delta t)^{1-\epsilon}}\right) \Delta x^{(1-\epsilon)/2}.$$

Proof Consider the variation of constants formula for the stochastic PDE (2.1) and its numerical approximation (2.3):

$$u(n\Delta t) = \exp(-An\Delta t)U + \int_0^{n\Delta t} \exp(-A(n\Delta t - s))\, dB(s),$$

and

$$u_n = (I - \mathcal{C}\Delta t)^n \mathbb{P} U + \sum_{i=0}^{n-1} (I - \mathcal{C}\Delta t)^{n-i-1} Q_{\Delta x}\, dB_{\Delta t}(i).$$

In the light of Theorem 2.9, we may take $U = 0$; hence we examine

$$\begin{aligned}
u(n\Delta t) - u_n = &\int_{(n-1)\Delta t}^{n\Delta t} e^{-A(n\Delta t - s)}\, dB(s) \\
&+ \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} \left(e^{-A(n\Delta t - s)} - e^{-A(n-i-1)\Delta t}\right) dB(s) \\
&+ \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t} \left(I - Q_{\Delta x}\mathbb{P}\right) dB(s) \\
&+ \sum_{i=0}^{n-2} \left(e^{-A(n-i-1)\Delta t} - e^{-\mathcal{C}(n-i-1)\Delta t}\right) Q_{\Delta x}\, dB_{\Delta t}(i) \\
&+ \sum_{i=0}^{n-2} \left(e^{-\mathcal{C}(n-i-1)\Delta t} - (I - \mathcal{C}\Delta t)^{n-i-1}\right) Q_{\Delta x}\, dB_{\Delta t}(i) \\
&- Q_{\Delta x}\, dB_{\Delta t}(n-1).
\end{aligned}$$

The final part of the convolutions are treated separately, because there is no smoothing in the numerical term; thus we only have the smallness associated with the time step. The terms $i = 0, \dots, n-2$ make use of the smoothing of e^{-At}, to regularise the white noise.

The first integral. By the Ito isometry (1.3),

$$\begin{aligned}\mathbf{E}\|\int_{(n-1)\Delta t}^{n\Delta t} e^{-A(n\Delta t - s)}\, dB(s)\|^2 &= \int_{(n-1)\Delta t}^{n\Delta t} \|e^{-A(n\Delta t-s)}\|_{HS}^2\, ds\\ &= \int_{(n-1)\Delta t}^{n\Delta t} \sum_{k=1}^{\infty} e^{-2k^2\pi^2(n\Delta t - s)}\, ds\\ &= \sum_{k=1}^{\infty} \frac{1}{2k^2\pi^2}(1 - e^{-2k^2\pi^2\Delta t})\end{aligned}$$

(as for a constant C, $(1 - e^{-2\lambda\pi^2}) \le C\lambda$ for $0 \le \lambda \le 1$)

$$\begin{aligned}&\le \sum_{k=1}^{\lfloor 1/\Delta t^{1/2}\rfloor} \frac{Ck^2\Delta t}{2k^2\pi^2} + \sum_{k=1+\lfloor 1/\Delta t^{1/2}\rfloor}^{\infty} \frac{1}{2k^2\pi^2}\\ &\le C\Delta t^{1/2} + \frac{\Delta t^{1/2}}{2\pi^2}.\end{aligned}$$

The second integral. By the Ito isometry (1.3),

$$\begin{aligned}&\mathbf{E}\left\| \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} (e^{-A(n\Delta t - s)} - e^{-A(n-i-1)\Delta t})\, dB(s)\right\|^2\\ &\quad= \mathbf{E}\left\| \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t}(e^{-A((i+1)\Delta t - s)} - I)\, dB(s)\right\|^2\\ &\quad= \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} \|e^{-A(n-i-1)\Delta t}(e^{-A((i+1)\Delta t-s)} - I)\|_{HS}^2\, ds.\end{aligned}$$

Take $0 < \gamma < 1$; then

$$\mathbf{E}\left\| \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} (e^{-A(n\Delta t - s)} - e^{-A(n-i-1)\Delta t})\, dB(s)\right\|^2$$

(by using Lemma 2.8)

$$\le \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} \|A^{-\gamma}\|_{HS}^2 \cdot \|A^{2\gamma} e^{-A(n-i-1)\Delta t}\|^2 \cdot \|A^{-\gamma}(e^{-A((i+1)\Delta t - s)} - I)\|^2\, ds$$

(using Proposition 2.4)

$$\leq C\Big(\sum_{k=1}^{\infty}(k^2\pi^2)^{-2\gamma}\Big)\sum_{i=0}^{n-2}\frac{\Delta t}{((n-i-1)\Delta t)^{4\gamma}}\ \Delta t^{2\gamma}.$$

Consequently, by taking $\gamma = (\epsilon+1)/4$, there exists C_ϵ for each $0 < \epsilon < 1$ with

$$\begin{aligned}\mathbf{E}\Big\|\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t}(e^{-A(t-s)}-e^{-A(n-i-1)\Delta t})\ dB_s\Big\|^2 &\leq C_\epsilon \Delta t^{-\epsilon}\Delta t^{(\epsilon+1)/2}\\ &\leq C_\epsilon \Delta t^{(1-\epsilon)/2}.\end{aligned}$$

The third integral. By the Ito isometry (1.3) and Lemma 2.8 ,

$$\begin{aligned}&\mathbf{E}\Big\|\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t}(I-Q_{\Delta x}\mathbb{P})\ dB(s)\Big\|^2\\ &\qquad\leq \sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t}\|A^{\gamma}e^{-A(n-i-1)\Delta t}\|^2\times\Big(\|A^{-\gamma}(I-Q_{\Delta x}\mathbb{P})\|_{HS}^2\Big)\ ds\end{aligned}$$

(take $\gamma > 5/4$ and apply Proposition 2.4)

$$\begin{aligned}&\leq\Big(\|A^{-\gamma}(I-Q_{\Delta x}\mathbb{P})\|_{HS}^2\Big)\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t}\frac{1}{((n-i-1)\Delta t)^{2\gamma}}\ ds\\ &\leq\Big(\|A^{-\gamma}(I-Q_{\Delta x}\mathbb{P})\|_{HS}^2\Big)\int_{\Delta t}^{n\Delta t}\frac{1}{s^{2\gamma}}\ ds\\ &\leq\Big(\|A^{-\gamma}(I-Q_{\Delta x}\mathbb{P})\|_{HS}^2\Big)\frac{\Delta t^{1-2\gamma}}{2\gamma-1}.\end{aligned}$$

Then, with Lemma 2.2 and $\gamma = (5+\epsilon)/4$ and $\Delta t = \nu\Delta x^2$, the third integral satisfies

$$\begin{aligned}&\mathbf{E}\Big\|\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t}(I-Q_{\Delta x}\mathbb{P})\ dB(s)\Big\|^2\\ &\quad\leq C_\epsilon\Delta x^4\Delta x^{(-\epsilon-3)}\leq C_\epsilon\Delta x^{1-\epsilon}.\end{aligned}$$

The fourth integral. Because $\mathbb{P}\ dB_{\Delta t}(i) =\ dB_{\Delta t}(i)$ by (2.2) and Lemma 2.6 and $\|Q_{\Delta x}\| \leq 1$,

$$\begin{aligned}&\mathbf{E}\Big\|\sum_{i=0}^{n-2}(e^{-A(n-i-1)\Delta t}-e^{-\mathcal{C}(n-i-1)\Delta t})\mathbb{P}Q_{\Delta x}\ dB_{\Delta t}(i)\Big\|^2\\ &\leq \mathbf{E}\sum_{i=0}^{n-2}\Big\|(e^{-A(n-i-1)\Delta t}-e^{-\mathcal{C}(n-i-1)\Delta t})\mathbb{P}\Big\|^2\Big\|\ dB_{\Delta t}(i)\Big\|^2\end{aligned}$$

(by Lemma 2.5 and Lemma 2.1)

$$
\begin{aligned}
&\leq 2\sum_{i=0}^{n-2} C\left((n-i-1)\Delta t\right)^{-2\gamma}\Delta x^{4\gamma}\frac{\Delta t}{\Delta x}\\
&\leq C\Delta x^{4\gamma-1}\frac{\Delta t^{1-2\gamma}}{2\gamma-1}\leq C\Delta x.
\end{aligned}
$$

by taking $\gamma > 1/2$ and using $\Delta t = \nu\Delta x^2$.

The fifth integral. By Lemma 2.6 and the identity

$$
a^n - b^n = (a-b)\sum_{j=0}^{n-1} a^{n-j-1}b^j,
$$

we may estimate the fifth integral as follows:

$$
\begin{aligned}
&\mathbf{E}\left\|\sum_{i=0}^{n-2}(e^{-\mathcal{C}(n-i-1)\Delta t} - (I-\mathcal{C}\Delta t)^{n-i-1})\mathbb{P}Q_{\Delta x}\, dB_{\Delta t}(i)\right\|^2\\
&\quad\leq \|(e^{-\mathcal{C}\Delta t} - (I-\mathcal{C}\Delta t))\mathcal{C}^{-2-\epsilon}\|^2\\
&\qquad\qquad \times\sum_{i=0}^{n-3}\left\|\sum_{j=1}^{n-i-2}\mathcal{C}^{2+\epsilon}e^{-\mathcal{C}(n-i-j-1)\Delta t}(I-\mathcal{C}\Delta t)^j\mathbb{P}\right\|^2\\
&\qquad\qquad \times\|Q_{\Delta x}\|\cdot\mathbf{E}\|\, dB_{\Delta t}(i)\|^2\\
&\qquad + \mathbf{E}\|(e^{-\mathcal{C}\Delta t} - (I-\mathcal{C}\Delta t))\mathbb{P}Q_{\Delta x}\, dB_{\Delta t}(i)\|^2.
\end{aligned}
$$

By the stability condition and $\|Q_{\Delta x}\| \leq 1$, the second term here is bounded by a constant times $\mathbf{E}\|\, dB_{\Delta t}(i)\|^2$, which, by Lemma 2.1, is bounded by $C\Delta t/\Delta x$. The first is bounded by applying the triangle inequality, Lemma 2.7 (ii), and Lemma 2.1 again. Hence, we get,

$$
\begin{aligned}
&\mathbf{E}\left\|\sum_{i=0}^{n-2}(e^{-\mathcal{C}(n-i-1)\Delta t} - (I-\mathcal{C}\Delta t)^{n-i-1})\mathbb{P}Q_{\Delta x}\, dB_{\Delta t}(i)\right\|^2\\
&\quad\leq \|(e^{-\mathcal{C}\Delta t} - (I-\mathcal{C}\Delta t))\mathcal{C}^{-2-\epsilon}\|^2\\
&\qquad\qquad \times\sum_{i=0}^{n-3}\left(\sum_{j=1}^{n-i-2}\|\mathcal{C}^{2+\epsilon}e^{-\mathcal{C}(n-i-1)\Delta t}\mathbb{P}\|\right)^2\frac{\Delta t}{\Delta x} + \frac{C\Delta t}{\Delta x}
\end{aligned}
$$

(by Lemma 2.5)

$$
\begin{aligned}
&\le C\Delta t^4\Big(\sum_{i=0}^{n-3}\frac{(n-i-1)}{[(n-i-1)\Delta t]^{2+\epsilon}}\Big)^2\frac{\Delta t}{\Delta x}+\frac{C\Delta t}{\Delta x}\\
&\le \frac{C\Delta t}{\Delta x}\Big(\sum_{i=0}^{n-3}\frac{(n-i-1)\Delta t^2}{[(n-i-1)\Delta t]^{2+\epsilon}}\Big)^2+\frac{C\Delta t}{\Delta x}\\
&\le \frac{C\Delta t}{\Delta x}\Big(\sum_{i=0}^{n-3}\frac{\Delta t}{[(n-i-1)\Delta t]^{1+\epsilon}}\Big)^2+\frac{C\Delta t}{\Delta x}\\
&\le \frac{C\Delta t}{\Delta x}+\frac{C\Delta t}{\Delta x}\frac{\Delta t^{-\epsilon}}{\epsilon}.
\end{aligned}
$$

Final term. The square of the expectation of the final term is bounded by $C\Delta t/\Delta x \le C\nu^2\Delta x$ by using Lemma 2.1.

The small time singularity of the error estimate results from non zero initial data in $L_2(0,1)$. The dependence on the $L_2(0,1)$ norm of the initial data is accounted for by Theorem 2.9. □

3 NONLINEAR PROBLEM

The ideas already presented suggest an approach to the nonlinear equation

$$
\begin{aligned}
du(t) =&\Big[-Au(t)+f(u(t))\Big]\,dt+\,dB(t);\\
u(0) =&U.
\end{aligned}
\tag{3.1}
$$

To make sense of this equation, we must work with mild solutions:

Definition 3.1 A predictable $L_2(0,1)$-valued process $u(\cdot)$ on $[0,T]$ is a mild solution of (3.1) if

(i)

$$\mathbf{P}\Big(\int_0^T \|u(s)\|^2\,ds<\infty\Big)=1.$$

(ii) for $0<t<T$,

$$u(t)=e^{-At}U+\int_0^t e^{-A(t-s)}f(u(s))\,ds+\int_0^t e^{-A(t-s)}\,dB(s). \tag{3.2}$$

Theorem 3.2 *Consider* $f\colon L_2(0,1) \to L_2(0,1)$ *subject to, for a constant* C,

$$\|f(x) - f(y)\| \le C\|x - y\|, \qquad x, y \in L_2(0,1);$$

$$\|f(x)\|^2 \le C(1 + \|x\|^2), \qquad x \in L_2(0,1).$$

If the consider initial data U *belongs to* $L_2(0,1)$, *then there exists a unique mild solution with a continuous modification in* $\mathcal{C}([0,T]; L_2(0,1)))$.

The solution satisfies, for a constant C,

$$\sup_{0\le t\le T} \mathbf{E}\|u(t)\|^2 \le C(1 + \mathbf{E}\|U\|^2).$$

Proof This is proved in Da Prato–Zabczyk [1]. □

Corresponding to (2.3) and in particular to the formulation of the numerical method (2.7) in the continuous space $L_2(0,1)$, we have the following discretization of the nonlinear equation (3.1):

$$\begin{aligned} u_{n+1} - (I - \mathcal{C}\Delta t)u_n =& Q_{\Delta x}\, dB_{\Delta t}(n) + Q_{\Delta x} f_\theta(n)\Delta t \\ u_0 =& \mathbb{P}U, \end{aligned} \tag{3.3}$$

where $f_\theta(n)$ is the $L_2(0,1)$ function defined by

$$f_\theta(n) := (1-\theta)f(u_n) + \theta f(u_{n+1}).$$

Hypothesis 3.3 The function f is Lipschitz as follows: there exist constants K_1, K_2 such that

$$\begin{aligned} \|(f(x) - f(y))\| \le& K_1\|A^\gamma(x-y)\|, \quad x, y \in L_2(0,1); \\ \|f(x)\| \le& K_2 + K_1\|x\|, \quad x \in L_2(0,1); \\ \|A^{1/4}f(x)\| \le& K_2 + K_1\|A^{1/4}x\|, \quad x \in L_2(0,1). \end{aligned}$$

This hypothesis is satisfied for functions f defined by $f(u)(x) := p(x)$, where the function $p\colon [0,1] \to \mathbb{R}$ has two uniformly bounded derivatives.

We require two lemmas: one gives boundedness of the numerical solutions in terms of the initial data; the second describes the divergence of trajectories of the stochastic PDE over an interval of length Δt.

Lemma 3.4 *Let Hypothesis 3.3 hold. For $0 \le \gamma < 1/2$, there exists $C_\gamma, \alpha > 0$ such that the following bounds hold: the solution of the numerical method (3.3) satisfies, for $n\Delta t \ge 0$,*

$$\Big(\mathbf{E}\|A^\gamma u_n\|^2\Big)^{1/2} \le C_\gamma e^{\alpha n\Delta t} G(\Delta x, \gamma)\Big(1 + \frac{1}{(n\Delta t)^\gamma}\Big)(1 + \|U\|). \tag{3.4}$$

($G(\Delta x, \gamma)$ is defined in Lemma 2.7.) The stochastic PDE (2.1) satisfies, for $t \ge 0$ and $0 \le \gamma < 1/4$,

$$\Big(\mathbf{E}\|A^\gamma u(t)\|^2\Big)^{1/2} \le C_\gamma e^{\alpha t}\Big(1 + \frac{1}{t^\gamma}\Big)(1 + \|U\|). \tag{3.5}$$

Remark The estimate on the numerical method reflects the fact that the stochastic integral $\int_0^t e^{-A(t-s)}\, dW(s)$ takes values in $H^\gamma(0,1)$, only when $\gamma < 1/2$. The $H^\gamma(0,1)$ norm, for $\gamma \ge 1/2$, of a solution of the numerical method grows as $\Delta x \to 0$.

Proof The Variation of Constants formula for the numerical method (3.3) states that

$$\begin{aligned} u_n =& (I - \mathcal{C}\Delta t)^n \mathbb{P} U \\ &+ \sum_{i=0}^{n-1} (I - \mathcal{C}\Delta t)^{n-i-1} Q_{\Delta x} f_\theta(i) \Delta t \\ &+ \sum_{i=0}^{n-1} (I - \mathcal{C}\Delta t)^{n-i-1} Q_{\Delta x}\, dB_{\Delta t}(i). \end{aligned}$$

Consequently, the triangle inequality gives

$$\begin{aligned} (\mathbf{E}\|A^\gamma u_n\|^2)^{1/2} \le& \|A^\gamma (I - \mathcal{C}\Delta t)^n \mathbb{P}\| \cdot \|U\| \\ &+ \sum_{i=0}^{n-1} \|(I - \mathcal{C}\Delta t)^{n-i-1}\| \cdot \|Q_{\Delta x}\| \cdot (\mathbf{E}\|A^\gamma f_\theta(i)\|^2)^{1/2} \Delta t \\ &+ \Big(\mathbf{E}\Big\|A^\gamma \sum_{i=0}^{n-1} (I - \mathcal{C}\Delta t)^{n-i-1} Q_{\Delta x}\, dB_{\Delta t}(i)\Big\|^2\Big)^{1/2}. \end{aligned}$$

Now, the first term: for $j = 1, \ldots, J-1$,

$$\begin{aligned} \|A^\gamma (I - \mathcal{C}\Delta t)^n \sqrt{2}\sin(j\pi\cdot)\| =& (k^2\pi^2)^\gamma (1 - \mu_k \Delta t)^n \\ \le& (k^2\pi^2)^\gamma e^{-\mu_k n\Delta t} \\ \le& C\mu_k^\gamma e^{-\mu_k n\Delta t} \\ \le& \frac{C_\gamma}{(n\Delta t)^\gamma}. \end{aligned}$$

The second term: by the Lipschitz property of Hypothesis 3.3, Lemma 2.7 and $\|Q_{\Delta x}\| \leq 1$,

$$\begin{aligned}&\|(I-\mathcal{C}\Delta t)^{n-i-1}\|\|Q_{\Delta x}\|(\mathbf{E}\|A^\gamma f_\theta(i)\|^2)^{1/2}\\ \leq& e^{-\mu(n-i-1)\Delta t}(K_1\theta(\mathbf{E}\|A^\gamma u_i\|^2)^{1/2}+K_1(1-\theta)(\mathbf{E}\|A^\gamma u_{i+1}\|^2)^{1/2}+K_2).\end{aligned}$$

The third term is bounded by (Lemma 2.7) $C_\gamma G(\Delta x,\gamma)$, for a constant C_γ.

Hence, putting the three bounds together, we have

$$\begin{aligned}(\mathbf{E}\|A^\gamma u_n\|^2)^{1/2} \leq& C_\gamma G(\Delta x,\gamma)\Big(1+\frac{1}{(n\Delta t)^\gamma}\Big)\Big(1+\|U\|\Big)\\ &+\sum_{i=0}^{n-1} e^{-\mu\Delta t(n-i-1)}\Big(K_2+K_1\theta(\mathbf{E}\|A^\gamma u_i\|^2)^{1/2}\\ &+K_1(1-\theta)(\mathbf{E}\|A^\gamma u_{i+1}\|^2)^{1/2}\Big)\Delta t.\end{aligned}$$

The generalised discrete Gronwall's inequality [4] gives the desired result.

The same steps lead to a bound for solutions of the stochastic PDE. The variation of constants formula for the stochastic PDE employs convolutions with respect to e^{-At} instead of $(I-\mathcal{C}\Delta t)^n$; but this term may be estimated just as easily by using Proposition 2.4. The stochastic convolution term (corresponding to the third term here) is bounded uniformly by the Ito isometry in $H^\gamma(0,1)$ norm for $\gamma<1/4$. It is not possible to gain an estimate for $\mathbf{E}\|A^\gamma u(s)\|^2$ for $\gamma\geq 1/4$. □

Lemma 3.5 *Denote the solution to (3.1) with initial condition $u(0)=x$ by $u(t;x)$. For $0<\gamma\leq 1/4$, there exists $C,\alpha>0$ with*

$$\Big(\mathbf{E}\|u(t;x)-x\|^2\Big)^{1/2} \leq Ct^\gamma(1+(\mathbf{E}\|A^\gamma x\|^2)^{1/2})e^{\alpha t}.$$

Proof By Lemma 3.4 and Proposition 2.4,

$$\begin{aligned}\Big(\mathbf{E}\|u(t;x)-x\|^2\Big)^{1/2} \leq& \|(e^{-At}-I)x\|+\int_0^t\Big(\mathbf{E}\|e^{-A(t-s)}f(u(s))\|^2\Big)^{1/2}\,ds+\Big(\int_0^t\|e^{-A(t-s)}\|_{HS}^2\,ds\Big)^{1/2}\\ \leq& C_\gamma t^\gamma\|A^\gamma x\|+tC(1+\|x\|)e^{\alpha t}+\Big(\int_0^t\|e^{-A(t-s)}\|_{HS}^2\,ds\Big)^{1/2}.\end{aligned}$$

But, of course,

$$\begin{aligned}\int_0^t \|e^{-A(t-s)}\|_{HS}^2 \, ds =& \sum_{k=1}^{\infty} \int_0^t e^{-2\pi^2 k^2 (t-s)} \, ds \\ =& \sum_{k=1}^{\infty} \frac{1}{2\pi^2 k^2}(1 - e^{-2\pi^2 k^2 t}) \\ \leq & \sum_{k=1}^{\lfloor 1/t^{1/2} \rfloor} \frac{(1 - e^{-2\pi^2 k^2 t})}{2\pi^2 k^2} + \sum_{k=\lfloor 1/t^{1/2} \rfloor + 1}^{\infty} \frac{(1 - e^{-2\pi^2 k^2 t})}{2\pi^2 k^2} \\ \leq & C \frac{t}{t^{1/2}} + C \frac{1}{1/t^{1/2}}.\end{aligned}$$

Together, we have

$$\left(\mathbf{E}\|u(t;x) - x\|^2\right)^{1/2} \leq C\, t^{\gamma} \|A^{\gamma} x\| + C\, t\, (1 + \|x\|) e^{Ct} + C t^{1/4}.$$

□

Lemma 3.6 *Let Hypothesis 3.3 hold. For* $0 < \gamma < 1/4$*, there exists* C *such that*

$$\begin{aligned}\left(\mathbf{E}\left\| \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t} (f(u(s)) - (1-\theta) f(u(i\Delta t)) - \theta f(u((i+1)\Delta t))) \, ds \right\|^2 \right)^{1/2} \\ \leq C \Delta t^{\gamma} e^{\alpha n \Delta t} (1 + \|U\|).\end{aligned}$$

Proof Consider only the case $\theta = 0$.

$$\begin{aligned}&\left(\mathbf{E}\left\| \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t} (f(u(s)) - f(u(i\Delta t))) \, ds \right\|^2 \right)^{1/2} \\ &\leq \sum_{i=1}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} \|e^{-A(n-i-1)\Delta t}\| \cdot K_1 (\mathbf{E}\|u(s) - u(i\Delta t)\|^2)^{1/2} \, ds + C\Delta t(1 + \|U\|)\end{aligned}$$

(by Lemma 3.5 and 3.4)

$$\begin{aligned}&\leq \sum_{i=1}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} C(s - i\Delta t)^{\gamma} (1 + \frac{1}{(i\Delta t)^{\gamma}}) e^{\alpha i \Delta t} (1 + \|U\|) \, ds + C\Delta t(1 + \|U\|) \\ &\leq \sum_{i=1}^{n-1} C\Delta t^{\gamma} \left(1 + \frac{1}{(i\Delta t)^{\gamma}}\right) e^{\alpha i \Delta t} \Delta t (1 + \|U\|) + C\Delta t(1 + \|U\|) \\ &\leq C\Delta t^{\gamma} \int_{\Delta t}^{n\Delta t} \left(1 + \frac{1}{s^{\gamma}}\right) e^{\alpha s} \, ds (1 + \|U\|) + C\Delta t(1 + \|U\|) \\ &\leq C\Delta t^{\gamma} e^{\alpha n \Delta t} (1 + \|U\|),\end{aligned}$$

increasing α if necessary □

Theorem 3.7 *For f satisfying Hypothesis 3.3, initial data U in $L_2(0,1)$, and $(\Delta t, \Delta x) \to 0$ subject to the stability condition (2.12), the following convergence holds: for $0 < \epsilon < 1$, there exist constants $C_\epsilon, \alpha > 0$ so that, for $n\Delta t > 0$,*

$$\Big(\mathbf{E}\|u(i\Delta t) - u_i\|^2\Big)^{1/2} \leq C_\epsilon \Delta x^{(1-2\epsilon)/2} e^{\alpha i \Delta t}(1 + \|U\|)(1 + (i\Delta t)^{(\epsilon-1)/4}).$$

Proof Write an expression for the error using the Variation of Constants formula: Denote the errors resulting from the linear terms and the noise term by Υ; we know from the previous theorem how to bound $\mathbf{E}\|\Upsilon\|^2$.

$$\begin{aligned}
u(n\Delta t) - u_n =& \Upsilon \\
&+ \int_{(n-1)\Delta t}^{n\Delta t} e^{-A(n\Delta t - s)} f(u(s))\, ds \\
&+ \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} (e^{-A(n\Delta t - s)} - e^{-A(n-i-1)\Delta t}) f(u(s))\, ds \\
&+ \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t} \Big(f(u(s)) - Q_{\Delta x} f_\theta(i)\Big)\, ds \\
&+ \sum_{i=0}^{n-2} (e^{-A(n-i-1)\Delta t} - e^{-\mathcal{C}(n-i-1)\Delta t}\mathbb{P}) Q_{\Delta x} f_\theta(i) \Delta t \\
&+ \sum_{i=0}^{n-2} (e^{-\mathcal{C}(n-i-1)\Delta t}\mathbb{P} - (I - \mathcal{C}\Delta t)^{n-i-1}) Q_{\Delta x} f_\theta(i) \Delta t \\
&- Q_{\Delta x} f_\theta(n-1)\Delta t.
\end{aligned}$$

The first two terms correspond to a term in the proof of Theorem 2.10, and may be bounded in an analogous way. For example,

$$\begin{aligned}
&\Big(\mathbf{E}\Big\| \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} (e^{-A(n\Delta t - s)} - e^{-A(n-i-1)\Delta t}) f(u(s))\, ds \Big\|^2\Big)^{1/2} \\
&\qquad \leq \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} \|e^{-A(n\Delta t - s)} - e^{-A(n-i-1)\Delta t}\| (\mathbf{E}\|f(u(s))\|^2)^{1/2}\, ds.
\end{aligned}$$

Lemma 3.4 provides a bound on $\mathbf{E}\|u(t)\|^2$ and hence, by the Lipschitz Hypothesis 3.3, a bound on $\mathbf{E}\|f(u(t))\|^2$. Therefore, following the steps in Theorem 2.10, this term may be bounded by the same power of Δx as in the previous argument multiplied by the bound on $\mathbf{E}\|f(u(t))\|^2$. The conclusion is that

$$\begin{aligned}
&\Big(\mathbf{E}\Big\| \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} (e^{-A(n\Delta t - s)} - e^{-A(n-i-1)\Delta t}) f(u(s))\, ds \Big\|^2\Big)^{1/2} \\
&\qquad \leq \Delta x^{1/2} e^{\alpha n \Delta t}(1 + \mathbf{E}\|U\|).
\end{aligned}$$

The third term. The third term needs estimating in terms of $\mathbf{E}\|u(i\Delta t) - u_i\|^2$, for Gronwall's sake. Consider $i\Delta t \le s \le (i+1)\Delta t$. We split the third term into three parts according to

$$\begin{aligned}
&e^{-A(n-i-1)\Delta t}(f(u(s)) - Q_{\Delta x} f_\theta(i)) \\
\le & e^{-A(n-i-1)\Delta t}\Big(f(u(s)) - (1-\theta)f(u(i\Delta t)) - \theta f(u((i+1)\Delta t))\Big) \\
&+ e^{-A(n-i-1)\Delta t}(I - Q_{\Delta x}) \cdot \Big((1-\theta)f(u(i\Delta t)) + \theta f(u((1+i)\Delta t))\Big) \\
&+ Q_{\Delta x} \cdot e^{-A(n-i-1)\Delta t}\Big((1-\theta)f(u(i\Delta t)) + \theta f(u((i+1)\Delta t)) - f_\theta(i)\Big).
\end{aligned}$$

Consider the first term: for $i\Delta t \le s \le (i+1)\Delta t$ and $\gamma = 1/4$,

$$\begin{aligned}
&\Big(\mathbf{E}\Big\|\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t}\Big(f(u(s)) \\
&\qquad - (1-\theta)f(u(i\Delta t)) - \theta f(u((i+1)\Delta t))\Big)\, ds\Big\|^2\Big)^{1/2}
\end{aligned}$$

(by Lemma 3.6 and Lemma 3.4)

$$\le C\Delta t^\gamma e^{\alpha n\Delta t}(1 + \|U\|).$$

Consider the second term:

$$\begin{aligned}
&\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t}(\mathbf{E}\|e^{-A(n-i-1)\Delta t}(I - Q_{\Delta x}) \\
&\qquad ((1-\theta)f(u(i\Delta t)) + \theta f(u((i+1)\Delta t)))\|^2)^{1/2}\, ds \\
&\le \sum_{i=0}^{n-2}\|e^{-A(n-i-1)\Delta t}(I - Q_{\Delta x})\| \cdot \\
&\qquad \Big(\mathbf{E}\|(1-\theta)f((u(i\Delta t))) + \theta f(u((i+1)\Delta t))\|^2\Big)^{1/2}\Delta t
\end{aligned}$$

(by Lemma 2.2)

$$\begin{aligned}
&\le \sum_{i=0}^{n-2} C_\epsilon \frac{\Delta x^2}{[(n-i-1)\Delta t]^{1+\epsilon}} e^{\alpha n\Delta t}(1 + \|U\|)\Delta t \\
&\le C_\epsilon e^{\alpha n\Delta t}(1 + \|U\|)\Delta x^2 \Delta t^{-\epsilon}.
\end{aligned}$$

Consider the third term:

$$\begin{aligned}
&\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t}\Big(\mathbf{E}\|Q_{\Delta x} e^{-A(n-i-1)\Delta t}\Big((1-\theta)f(u(i\Delta t)) \\
&\qquad + \theta f(u((i+1)\Delta t)) - f_\theta(i)\Big)\|^2\Big)^{1/2}\, ds
\end{aligned}$$

(because $\|Q_{\Delta x}\| \leq 1$ and because of the Lipschitz Hypothesis 3.3)

$$\leq \sum_{i=0}^{n-2} Ce^{-\pi^2(n-i-1)\Delta t}\Big((1-\theta)(K_1\mathbf{E}\|u(i\Delta t)-u_i\|^2)^{1/2} + K_1\theta(\mathbf{E}\|u((i+1)\Delta t)-u_{i+1}\|^2)^{1/2}\Big)\Delta t.$$

Hence, these three terms sum to give

$$\Big(\mathbf{E}\Big\|\sum_{i=0}^{n-2}\int_{i\Delta t}^{(i+1)\Delta t} e^{-A(n-i-1)\Delta t}\Big(f(u(s)) - Q_{\Delta x}f_\theta(i)\Big)\,ds\Big\|^2\Big)^{1/2}$$
$$\leq C_\epsilon e^{\alpha n\Delta t}(1+\|U\|)\Delta x^{1-2\epsilon} + K_1\sum_{i=0}^{n-1}\Big(\theta(\mathbf{E}\|u(i\Delta t)-u_i\|^2)^{1/2} + (1-\theta)(\mathbf{E}\|u((i+1)\Delta t)-u_{i+1}\|^2)^{1/2}\Big)e^{-\pi^2(n-i-1)\Delta t}\Delta t.$$

Fourth term. By Lemma 3.4 and the Lipschitz Hypothesis 3.3,

$$\Big(\mathbf{E}\|\sum_{i=0}^{n-2}(e^{-A(n-i-1)\Delta t} - e^{-\mathcal{C}(n-i-1)\Delta t}\mathbb{P})Q_{\Delta x}f_\theta(i)\Delta t\|^2\Big)^{1/2}$$
$$\leq \sum_{i=0}^{n-1}\|(e^{-A(n-i-1)\Delta t} - e^{-\mathcal{C}(n-i-1)\Delta t})\mathbb{P}\|\cdot\|Q_{\Delta x}\|\;C\;e^{\alpha n\Delta t}(1+\|U\|)\Delta t$$
$$+\sum_{i=0}^{n-1}\|e^{-A(n-i-1)\Delta t}(I-\mathbb{P})\|\cdot\|Q_{\Delta x}\|\;C\;e^{\alpha n\Delta t}(1+\|U\|)\Delta t.$$

By following the steps used for the fourth integral in Theorem 2.10, the first part of this term may be bounded by $C\Delta x^{1/2}e^{\alpha n\Delta t}(1+\|U\|)$. The second is bounded as follows:

$$\sum_{i=0}^{n-1}\|e^{-A(n-i-1)\Delta t}(I-\mathbb{P})\|\Delta t \leq \sum_{i=0}^{n-2} e^{-J^2(n-i-1)\Delta t}\Delta t \leq C\frac{1}{J^2} \leq C\Delta x^2,$$

and hence

$$\sum_{i=0}^{n-1}\|e^{-A(n-i-1)\Delta t}(I-\mathbb{P})\|\cdot\|Q_{\Delta x}\|Ce^{\alpha n\Delta t}(1+\|U\|)\Delta t$$
$$\leq Ce^{\alpha n\Delta t}\Delta x^2(1+\|U\|).$$

Fifth term.

$$\mathbf{E}\|\sum_{i=0}^{n-2}\Big((I-\mathcal{C}\Delta t)^{n-i-1} - e^{-\mathcal{C}(n-i-1)\Delta t}\mathbb{P}\Big)Q_{\Delta x}f_\theta(i)\Delta t\|^2$$
$$\leq 2\mathbf{E}\Big\|\sum_{i=0}^{n-2}\Big((I-\mathcal{C}\Delta t)^{n-i-1} - e^{-\mathcal{C}(n-i-1)\Delta t}\Big)\mathbb{P}Q_{\Delta x}f_\theta(i)\Delta t\Big\|^2$$
$$+ 2\mathbf{E}\Big\|\sum_{i=0}^{n-2}(I-\mathcal{C}\Delta t)^{n-i-1}(I-\mathbb{P})Q_{\Delta x}f_\theta(i)\Delta t\Big\|^2$$

By exploiting the boundedness of $f_\theta(i)$ in terms of the initial data given by Lemma 3.4, the first term can be estimated using the techniques applied to integral five in Theorem 2.10. The bound is $Ce^{2\alpha n\Delta t}\Delta x(1+\|U\|)^2$.

For the second term, argue as follows:

$$\Big(\mathbf{E}\|\sum_{i=0}^{n-2}(I-\mathcal{C}\Delta t)^{n-i-1}(I-\mathbb{P})Q_{\Delta x}f_\theta(i)\Delta t\|^2\Big)^{1/2}$$
$$\leq\sum_{i=0}^{n-2}\|(I-\mathcal{C}\Delta t)\|^{n-i-1}\|Q_{\Delta x}\|(\mathbf{E}\|(I-\mathbb{P})f_\theta(i)\|^2)^{1/2}\Delta t$$

(because of Lemma 2.7)

$$\leq\sum_{i=0}^{n-2}e^{-\mu\Delta t(n-i-1)}(\mathbf{E}\|(I-\mathbb{P})f_\theta(i)\|^2)^{1/2}\Delta t.$$

But, for $\gamma=1/4$,

$$\begin{aligned}\mathbf{E}\|(I-\mathbb{P})f_\theta(i)\|^2 \leq&\|A^{-\gamma}(I-\mathbb{P})\|^2\mathbf{E}\|A^\gamma f_\theta(i)\|^2\\ \leq&\frac{C}{J^{4\gamma}}\Big(K_1\theta(\mathbf{E}\|A^\gamma u_i\|^2)^{1/2}+K_2(1-\theta)(\mathbf{E}\|A^\gamma u_{i+1}\|^2)^{1/2}+K_2\Big)\end{aligned}$$

(by Lemma 3.4)

$$\leq C_\gamma\Delta x^{4\gamma}\Big(1+\frac{1}{((i+1)\Delta t)^\gamma}\Big)G(\Delta x,\gamma)e^{2\alpha n\Delta t}(1+\|U\|)^2.$$

Hence,

$$\Big(\mathbf{E}\|\sum_{i=0}^{n-2}(I-\mathcal{C}\Delta t)^{n-i-1}(I-\mathbb{P})Q_{\Delta x}f_\theta(i)\Delta t\|^2\Big)^{1/2}$$
$$\leq C_\gamma\ \Delta x^{2\gamma}\ G(\Delta x,\gamma)(1+\|U\|)e^{\alpha n\Delta t}\sum_{i=0}^{n-2}e^{-\mu(n-i-1)\Delta t}\Big(1+\frac{1}{((i+1)\Delta t)^\gamma}\Big)\Delta t.$$

The sum may be bounded uniformly in n. Thus, for $\gamma=(2-\epsilon)/4$,

$$\Big(\mathbf{E}\|\sum_{i=0}^{n-2}(I-\mathcal{C}\Delta t)^{n-i-1}(I-\mathbb{P})Q_{\Delta x}f_\theta(i)\Delta t\|^2\Big)^{1/2}\leq C\Delta x^{1/2-\epsilon}e^{\alpha n\Delta t}(1+\|U\|).$$

Summing up. Thus, for each $\epsilon > 0$, we have shown that there exists C_ϵ with

$$\begin{aligned}&\Big(\mathbf{E}\|u(n\Delta t) - u_n\|^2\Big)^{1/2}\\ \le & C\Delta x^{1/2-\epsilon}\Big(1+\|U\|\Big)e^{\alpha n\Delta t}\Big(1+\frac{1}{(n\Delta t)^{(1-2\epsilon)/4}}\Big)\\ &+\sum_{i=0}^{n-1}\Big(\theta(\mathbf{E}\|u(i\Delta t)-u_i\|^2)^{1/2}\\ &\qquad + (1-\theta)(\mathbf{E}\|u((i+1)\Delta t)-u_{i+1}\|^2)^{1/2}\Big)e^{-\mu(n-i-1)\Delta t}\Delta t.\end{aligned}$$

The generalized Gronwall Lemma [4] applies to give the following estimate (the constant α may be larger)

$$(\mathbf{E}\|u(i\Delta t)-u_i\|^2)^{1/2} \le C(1+\|U\|)e^{\alpha i\Delta t}\Delta x^{(1-\epsilon)/2}(1+(i\Delta t)^{(2\epsilon-1)/4}).$$

□

4 NUMERICAL EXPERIMENTS

Plots are shown in Figures 1–2 of the convergence of method (2.3) approximating (3.1) with $f = \frac{1}{2}(u-u^3)$ and $f = 0$. The parameters used in the experiment are $\theta = 1$ (the backward Euler method) and $\nu = \Delta t/\Delta x^2$ fixed equal to 1. Solutions $\mathbf{u}_1, \dots, \mathbf{u}_n$ are computed for grid sizes $\Delta x_1, \dots, \Delta x_n$, where $\Delta x_i = 1.0/(2^i - 1)$. The approximations $dB_{\Delta t}(n)$ are computed using the FFT (the choice of Δx was made, to make this convenient). The true solution is take to be u_n and the error $e_i := (\mathbf{E}\|\mathbf{u}_n - \mathbf{u}_i\|^2)^{1/2}$, where the mean is simply the average over the number of trials taken. Two experiments are plotted, both indicate sublinear convergence but stronger than the rate of $\Delta x^{1/2}$.

References

[1] G. Da Prato and J. Zabczyk, *Stochastic equations in infinite dimensions*, vol. 44 of Encyclopedia of Mathematics and its Applications, Cambridge University Press, Cambridge, 1992.

[2] G. Da Prato and J. Zabczyk, *Ergodicity for infinite-dimensional systems*, vol. 229 of London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1996.

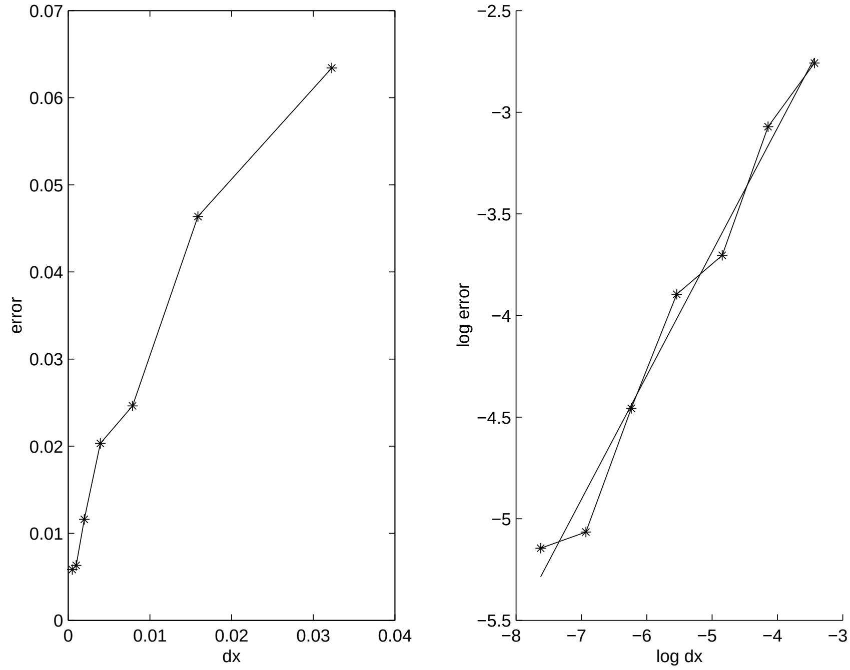

Figure 1: $f(u) = 0$, average over 10 trials on time interval $[0, 0.01]$. The slope of the best linear fit to the log-log plot is 0.61.

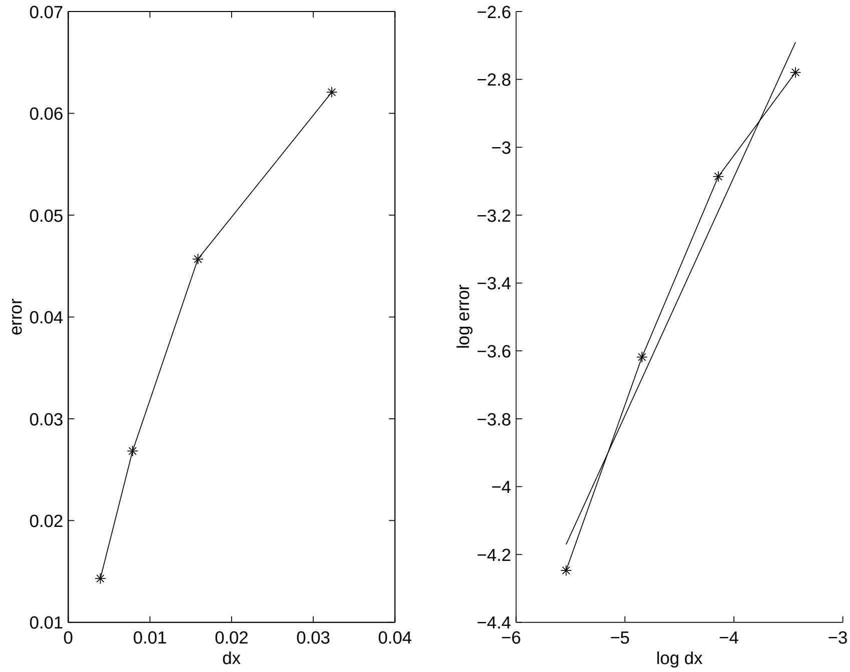

Figure 2: $f(u) = \frac{1}{2}(u - u^3)$, average over 10 trials on time interval $[0, 0.01]$. The slope of the best linear fit to the log-log plot is 0.71

[3] A. M. Davie and J. G. Gaines, *Convergence of implicit schemes for numerical solution of parabolic stochastic partial differential equations.* Work In Progress, 1996.

[4] C. M. Elliott and S. Larsson, *Error estimates with smooth and nonsmooth data for a finite element method for the Cahn–Hilliard equation*, Mathematics of Computation, 58 (1991), pp. 603–630.

[5] I. Gyöngy, *Lattice approximations for stochastic quasi–linear parabolic partial differential equations driven by space–time white noise I and II.* Submitted.

[6] A. Pazy, *Semi groups of Linear Operators and Applications to Partial Differential Equations*, vol. 44 of Applied Mathematical Sciences, Springer–Verlag, 1983.

[7] T. Shardlow, *Geometric ergodicity for stochastic PDEs*, Stoch. Anal. App., 17 (1999).

[8] T. Shardlow and A. M. Stuart, *A perturbation theory for ergodic properties of Markov chains.* To appear, SIAM J. Num. Anal.