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Well-posedness for a regularised inertial Dean–Kawasaki model for slender particles in several space dimensions

Federico Cornalba [a,*], Tony Shardlow [b], Johannes Zimmer [c]

[a] Institute of Science and Technology Austria (IST Austria), Austria
[b] University of Bath, United Kingdom
[c] Technische Universität München, Germany

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Abstract

A stochastic PDE, describing mesoscopic fluctuations in systems of weakly interacting inertial particles of finite volume, is proposed and analysed in any finite dimension $d \in \mathbb{N}$. It is a regularised and inertial version of the Dean–Kawasaki model. A high-probability well-posedness theory for this model is developed. This theory improves significantly on the spatial scaling restrictions imposed in an earlier work of the same authors, which applied only to significantly larger particles in one dimension. The well-posedness theory now applies in d-dimensions when the particle-width ϵ is proportional to $N^{-1/\theta}$ for $\theta > 2d$ and N is the number of particles. This scaling is optimal in a certain Sobolev norm. Key tools of the analysis are fractional Sobolev spaces, sharp bounds on Bessel functions, separability of the regularisation in the d-spatial dimensions, and use of the Faà di Bruno's formula.

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Keywords: Well-posedness of stochastic PDEs; Multi-dimensional Dean–Kawasaki model; von Mises kernel; Spatial regularisation; Fractional Sobolev spaces; Bessel functions of first kind

* Corresponding author.
E-mail addresses: federico.cornalba@ist.ac.at (F. Cornalba), T.Shardlow@bath.ac.uk (T. Shardlow), jz@ma.tum.de (J. Zimmer).

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1. Introduction

Fluctuating hydrodynamics is a class of models describing fluctuations around the hydrodynamic limit of a many-particle system; a particular example is the Dean–Kawasaki model [10,17], which describes the evolution of finitely many particles governed by over-damped Langevin dynamics. At its core, this model is a stochastic PDE for the empirical density, comprising a diffusion equation that is stochastically perturbed by a mass-preserving multiplicative space-time white noise; see (6) below. Equations of fluctuating hydrodynamics are widely used in physics and other sciences (e.g., in the description of active matter [24,5], thermal advection [19], neural networks [22], and agent based models [11]), and are currently being investigated numerically [16]. Still, the mathematical analysis of these equations is in its infancy. A truly remarkable recent result [18] shows that a solution for the original Dean–Kawasaki model (as derived in [10] and given in (6) below) only exists when the initial datum is a superposition of a finite-number N of Dirac delta functions and the diffusion coefficient is $\frac{1}{2}N$; if such an initial datum is ever so slightly mollified, then no solution exists. Given the numerous applications of equations of fluctuating hydrodynamics, this apparent mathematical instability is particularly puzzling.

In light of this, several *regularised* Dean–Kawasaki models (featuring smooth noise coefficient and coloured driving noise) have been proposed and studied [13,11,14,7,8]. In recent work [7,8], the authors have derived and analysed stochastic PDE models for the empirical density of N-particles following second-order Langevin dynamics and interacting weakly. The models are derived from particles as entities of finite size rather than Dirac delta functions and this regularisation is crucial for the mathematical theory. We refer to this PDE as the Regularised Inertial Dean–Kawasaki (RIDK) model. In particular, we have established that RIDK has a well-defined mild solution in one-dimension with probability converging to one in the limit as $N \to \infty$ and the particle width $\epsilon \to 0$, subject to particles being wide enough (as given by the scaling condition $N\epsilon^{\theta} = 1$ for a given θ). In this paper, we establish well-posedness for RIDK in any finite spatial dimension and significantly improve the scaling condition (relax conditions on θ) in the one-dimensional case. To the best of our knowledge, this is the first proof of well-posedness for RIDK or any Dean–Kawasaki model in several space dimensions.

1.1. Setting and main result

We consider N-weakly interacting particles on the d-dimensional torus $\mathbb{T}^d := [0, 2\pi)^d$. The particles are identified by position and momentum $(\boldsymbol{q}_i, \boldsymbol{p}_i)_{i=1}^{N} \in \mathbb{T}^d \times \mathbb{R}^d$, and satisfy the stochastic differential equation

$$\dot{\boldsymbol{q}}_i = \boldsymbol{p}_i, \qquad \dot{\boldsymbol{p}}_i = -\gamma\, \boldsymbol{p}_i - N^{-1} \sum_{j=1}^{N} \nabla U(\boldsymbol{q}_i - \boldsymbol{q}_j) + \sigma\, \dot{\boldsymbol{b}}_i, \qquad i = 1, \dots, N, \tag{1}$$

where γ, σ are positive constants, $U\colon \mathbb{T}^d \to \mathbb{R}$ is a smooth pairwise interaction potential, and $\{\boldsymbol{b}_i\}_{i=1}^{N}$ is a family of independent standard d-dimensional Brownian motions. We work under the key modelling assumption that the particles have a finite size. Specifically, we describe their spatial occupancy by means of a kernel $w_\epsilon\colon \mathbb{T}^d \to [0, \infty)$ indexed by $\epsilon > 0$, which may be thought of as the particle width. We propose RIDK as a model for the particle density and momentum density

$$(\rho_\epsilon(\boldsymbol{x},t),\boldsymbol{j}_\epsilon(\boldsymbol{x},t)) := \left(N^{-1}\sum_{i=1}^{N} w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t)),\, N^{-1}\sum_{i=1}^{N}\boldsymbol{p}_i(t)\, w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t))\right),$$

where $(\boldsymbol{x},t)\in\mathbb{T}^d\times[0,T]$.

In particular, RIDK defines an approximate particle and momentum density $(\tilde{\rho}_\epsilon,\tilde{\boldsymbol{j}}_\epsilon)\colon \mathbb{T}^d\times[0,T]\to\mathbb{R}\times\mathbb{R}^d$ by the stochastic PDE

$$\begin{cases}\partial_t\tilde{\rho}_\epsilon = -\nabla\cdot\tilde{\boldsymbol{j}}_\epsilon,\\ \partial_t\tilde{\boldsymbol{j}}_\epsilon = -\gamma\,\tilde{\boldsymbol{j}}_\epsilon - \dfrac{\sigma^2}{2\gamma}\nabla\tilde{\rho}_\epsilon - \tilde{\rho}_\epsilon\,(\nabla U * \tilde{\rho}_\epsilon) + \sigma\, N^{-1/2}\left(\sqrt{\tilde{\rho}_\epsilon}\,P^{1/2}_{\sqrt{2}\epsilon}\,\xi_1,\ldots,\sqrt{\tilde{\rho}_\epsilon}\,P^{1/2}_{\sqrt{2}\epsilon}\,\xi_d\right),\end{cases} \tag{2}$$

subject to $(\tilde{\rho}_\epsilon(\cdot,0),\tilde{\boldsymbol{j}}_\epsilon(\cdot,0)) = (\tilde{\rho}_0,\tilde{\boldsymbol{j}}_0)$ for initial densities $\tilde{\rho}_0$ and $\tilde{\boldsymbol{j}}_0$, where $\{\xi_\ell\}_{\ell=1}^d$ are independent space-time white noises, and P_ϵ is the convolution operator $P_\epsilon\colon L^2(\mathbb{T}^d)\to L^2(\mathbb{T}^d)\colon f\mapsto P_\epsilon f(\cdot) = \int_{\mathbb{T}^d} w_\epsilon(\cdot-\boldsymbol{y})\,f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$. The operator P_ϵ describes the spatial correlation of the stochastic noise and is intrinsically linked to the spatial occupancy of the particles through the regularising kernel w_ϵ. This model is of inertial type (meaning that it keeps track of both density and momentum density), and is a generalisation of the models studied in [7,8] to higher dimensions. For w_ϵ, we choose the von Mises kernel

$$w_\epsilon(\boldsymbol{x}) := Z_\epsilon^{-d}\exp\left\{-\frac{\sum_{\ell=1}^d \sin^2(x_\ell/2)}{\epsilon^2/2}\right\}, \qquad Z_\epsilon := \int_{\mathbb{T}}\exp\left(-\frac{\sin^2(y/2)}{\epsilon^2/2}\right)\mathrm{d}y. \tag{3}$$

Any non-negative function $w\colon[-\pi,\pi)\to\mathbb{R}$ can be written $w(x)=\exp(-V(\sin(x/2)))$ for a function $V\colon[-1,1]\to\mathbb{R}$. For $x\approx 0$, $V(\sin(x/2))\approx V(0)+V'(0)\sin(x/2)+\frac{1}{2}V''(0)\sin^2(x/2)$. Assuming V is symmetric for extending periodically, we find that

$$w(x)\approx\exp(-V(0))\exp\left(-\frac{1}{2}V''(0)\sin^2(x/2)\right).$$

The values of $\exp(-V(0)) = 1/Z_\epsilon$ and $V''(0)=4/\epsilon^2$ are chosen so that the moments agree with $N(0,\epsilon^2)$ and there is convergence to the Dirac delta function. The periodic extension to $\mathbb{T}^d$ defines the von Mises kernel w_ϵ.

For regularity purposes which will become clear later, it is convenient to replace the square-root in (2) with a smooth function $h_\delta\colon\mathbb{R}\to\mathbb{R}$ such that $h_\delta(z)=\sqrt{|z|}$ for $|z|\geq\delta/2$, for some small and fixed $\delta>0$. Following this change, the RIDK equation (2) is rewritten in the abstract stochastic PDE notation

$$\begin{cases}\mathrm{d}X_{\epsilon,\delta}(t) = A\,X_{\epsilon,\delta}(t)\,\mathrm{d}t + \alpha_U(X_{\epsilon,\delta}(t))\,\mathrm{d}t + B_{N,\delta}(X_{\epsilon,\delta}(t))\,\mathrm{d}W_{\mathrm{per},\epsilon},\\ X_{\epsilon,\delta}(0) = X_0,\end{cases} \tag{4}$$

where $X_{\epsilon,\delta}=(\tilde{\rho}_{\epsilon,\delta},\tilde{\boldsymbol{j}}_{\epsilon,\delta})$, $X_0=(\tilde{\rho}_0,\tilde{\boldsymbol{j}}_0)$, A is a linear operator describing the deterministic drift excluding the interaction-potential term $\alpha_U(X_{\epsilon,\delta})$, $B_{N,\delta}$ is the stochastic integrand associated with the introduction of h_δ, and $W_{\mathrm{per},\epsilon}$ is a Q-Wiener representation of the noise $\left(P^{1/2}_{\sqrt{2}\epsilon}\,\xi_1,\ldots,P^{1/2}_{\sqrt{2}\epsilon}\,\xi_d\right)$. More details concerning (4), as well as a sketch of its derivation from the Langevin particle dynamics (1), are given in Section 2.

Throughout the paper, we work under the general scaling

$$N\epsilon^{\theta} = 1, \qquad \theta > \theta_0 := 2d. \tag{5}$$

From a modelling point of view, (5) imposes the particle size (comparatively to N), where increasing θ implies increasing particle size. If θ is close to the limiting case $\theta_0 = 2d$, the scaling (5) is approximately only dependent on the volume υ of each particle: specifically, since each particle is roughly of size ϵ in each direction, then (5) corresponds to $N\upsilon^2 \approx 1$. The purpose of the scaling is regularisation and smoothing of the densities, and it is necessary that particles overlap in the limit so that conservation of volume is not possible. The condition (5) provides sufficient regularity for the Sobolev space analysis in Theorem 1.1 and is optimal in that sense as we show in Remark 1.2.

From an analytical perspective, (5) affects the spectral properties of the noise $W_{\mathrm{per},\epsilon}$ in (4) through the operator P_ϵ.

We state the main result of this paper.

Theorem 1.1 *(Well-posedness of RIDK on $\mathbb{T}^d$). Let $\delta > 0$, $h_\delta \in C^{\lceil d/2\rceil+2}(\mathbb{R})$, $\nu \in (0,1)$, and $U \in C^1$. Fix $\theta > \theta_0 = 2d$ such that $(\theta-\theta_0)/2 < \lceil (d+1)/2\rceil - d/2$. Pick $\eta \in (0, \min\{(\theta-\theta_0)/2, C(d)\})$ for some small enough $C(d) \in (0, 1/2)$ (see Lemma B.4). Set $s := d/2 + \eta$. Let $(\tilde{\rho}_0, \tilde{\boldsymbol{j}}_0)$ be a deterministic initial condition belonging to the fractional Sobolev space $\mathcal{W}^s := H^s(\mathbb{T}^d) \times [H^s(\mathbb{T}^d)]^d$ such that $\min_{\boldsymbol{x}\in\mathbb{T}^d} \tilde{\rho}_0(\boldsymbol{x}) > \delta$.*

There exists $T = T(\tilde{\rho}_0)$, a large enough N, a unique $\mathcal{W}^s$-valued process $X_{\epsilon,\delta} = (\tilde{\rho}_{\epsilon,\delta}, \tilde{\boldsymbol{j}}_{\epsilon,\delta})$, and a set F_ν of probability at least $1-\nu$ such that $\min_{\boldsymbol{x}\in\mathbb{T}^d, s\in[0,T]} \tilde{\rho}_{\epsilon,\delta}(\boldsymbol{x}, s) \geq \delta$ on F_ν, and $X_{\epsilon,\delta}$ solves (4) pathwise on F_ν in the sense of mild solutions [9, Chapter 7]. As a consequence, $X_{\epsilon,\delta}$ also solves the RIDK equations (2) pathwise on F_ν in the sense of mild solutions.

The proof exploits a small-noise analysis, by obtaining the solution to (4) as a small perturbation of the strictly positive solution of the noise-free dynamics (i.e., the damped wave equation). When the perturbations are small and the initial data is everywhere larger than δ, the solution to (4) remains outside the regularisation regime $(-\infty, \delta/2)$ for h_δ and the regularisation is bypassed, resulting in a well-defined solution of (2). The C^0-norm is used to measure the perturbations and keep track of whether the solution falls into the regularisation region. To do this, the parameter s is chosen so the mild solutions take values in the Sobolev space $\mathcal{W}^s$, which is embedded continuously in $C^0 \times [C^0]^d$.

With this in mind, the proof of Theorem 1.1 (see Section 4) is built upon three conceptual blocks, developed in Section 3. Firstly, A is proven to generate a C_0-semigroup with respect to the $\mathcal{W}^s$-norm (see Subsection 3.1, Lemma 3.1). Secondly, the stochastic integrand $B_{N,\delta}$ is shown to be locally Lipschitz and sublinear ($d = 1$) or locally Lipschitz and locally bounded ($d > 1$) with respect to the Hilbert–Schmidt $L_2^0(\mathcal{W}^s)$-norm (Subsection 3.3, Lemma 3.4). These two blocks give rigour to the application of the mild solution theory. Thirdly, sharp bounds for the trace of $W_{\mathrm{per},\epsilon}$ with respect to the $\mathcal{W}^s$-norm are provided via spectral analysis of $P_{\sqrt{2}\epsilon}$ (see Subsection 3.2, Lemma 3.2). In combination with Lemma 3.4, this guarantees the vanishing-noise regime for (4) in the $\mathcal{W}^s$-norm as $N \to \infty$.

Theorem 1.1 carries two significant contributions. Firstly, it provides a well-posedness theory for the multi-dimensional RIDK model; to the best of our knowledge, this is the first paper to give an existence and uniqueness theory for such a model. Secondly, it improves an analogous one-dimensional result [7,8] by significantly relaxing the scaling threshold in (5) from $\theta_0 = 7$ to

$\theta_0 = 2$. The more restrictive threshold for θ resulted from a suboptimal analysis with respect to the $\mathcal{W}^1$ norm. The $\theta_0 = 7$ scaling is inconveniently restrictive, as it only allows for rather large particles (comparatively to N). Specifically, θ is significantly away from the null value, which *formally* corresponds to representing particles by Dirac delta functions.

The main technical novelties that we introduce in the proof of Theorem 1.1 are the following. First, we deploy improved estimates for the spectral properties of $P_{\sqrt{2}\epsilon}$, which rely on refined bounds for modified Bessel functions of the first kind. Secondly, we set the analysis in the 'least restrictive' Sobolev space $\mathcal{W}^s$ that embeds continuously in the space of continuous functions, and this corresponds to considering $s = d/2+\eta$ for arbitrarily small positive η. Thirdly, we extend the analysis to higher dimensions by relying on the separability of the kernel w_ϵ in its d variables, the boundedness and periodicity of the spatial domain, and the considerations from the one-dimensional case. The boundedness of the spatial domain is crucially used also in the derivation of technical tools related to fractional Sobolev spaces and Faà di Bruno's formula, which are deferred to Appendix B. Relevant elementary algebraic tools are summarised in Appendix A. Additionally, the proof of Theorem 1.1 in Section 4 is finalised with a localisation procedure argument. Crucially, the same techniques adopted to deal with the superlinear interaction α_U (analogous to those developed in [8, Section 4]) also allow to deal with the locally bounded noise in $d > 1$.

Remark 1.2. The justification of the scaling assumptions of Theorem 1.1 is found in Lemma 3.2. There, each index s is associated to a relevant value $\theta_c(s) := 2s + d$, and the trace of $W_{\mathrm{per},\epsilon}$ with respect to the $\mathcal{W}^s$-norm is bounded by $\epsilon^{-\theta_c(s)}$. In combination with Lemma 3.4, this implies that the $\mathcal{W}^s$-norm of the stochastic noise of (4) vanishes as $N \to \infty$ for any $\theta > \theta_c(s)$. As our well-posedness theory relies on the embedding $\mathcal{W}^s \subset C^0 \times [C^0]^d$, we require the equality $s = d/2 + \eta = (\theta_c(s) - d)/2$ to hold, giving $\theta > 2d + 2\eta$. As η may be chosen arbitrarily small, we obtain the threshold $\theta_0 = 2d$.

Furthermore, for each s, the value $\theta_c(s)$ is optimal, in the sense that $\theta_c(s)$ is also the minimum value for which $\mathbb{E}[\|\rho_\epsilon(\cdot,t)\|^2_{H^s}]$ (where, we recall, ρ_ϵ denotes the true particle density) is uniformly bounded in N and ϵ, at least in the case of independent particles given by $U \equiv 0$. Namely, it is easy to proceed as in [7] and argue that, under reasonable assumptions on the law of the particle dynamics,

$$0 < C_1 < \lim_{N\to\infty,\,\epsilon\to 0} \left\{ N\,\epsilon^{2s+d}\,\mathbb{E}\left[\|\rho_\epsilon(\cdot,t)\|^2_{H^s}\right]\right\} < C_2,$$

for some constants C_1, C_2 independent of N and ϵ. Crucially, we obtain scaling agreement for fluctuations on microscopic and mesoscopic scale; here *microscopic* means particle-level dynamics, see (1) above, while *mesoscopic* means the Dean–Kawasaki dynamics (2). As a result, the value θ_0 is also optimal, as $\lim_{s\to d/2}\theta_c(s) = \theta_0$. We stress that this notion of scaling optimality is only understood with respect to the evaluation of Sobolev norms for the densities ρ_ϵ and $\tilde{\rho}_\epsilon$.

Remark 1.3. The RIDK model (2) may be regarded as the regularised inertial analogue of the original over-damped Dean–Kawasaki model [10,17]

$$\partial_t \rho_N = \frac{N}{2}\,\Delta\rho_N + \nabla\cdot\left(\sqrt{\rho_N}\,\xi\right) \quad \text{in } \mathbb{R}^d \times (0,\infty), \tag{6}$$

where ξ is a space-time white noise. As stated before, (6) admits nothing but an atomic solution, and only in the integer regime $N \in \mathbb{N}$. In this case, the solution is $\rho_N(\boldsymbol{x},t) = N^{-1}\sum_{i=1}^{N}\delta(\boldsymbol{x}-\boldsymbol{B}_i(t))$, where $(\boldsymbol{B}_i)_{i=1}^{N}$ are independent Brownian walkers.

We consider the RIDK model (2) instead of (6) for a number of reasons. Firstly, while both models describe mesoscopic fluctuations in particle systems of physical relevance (such as those treated, e.g., in the description of active matter [24,5] and thermal advection [19]), the RIDK model does so while also capturing core inertial effects (i.e., Newton's law of motion). Secondly, it bypasses any problematic interpretations arising from taking a formal divergence of the stochastic noise. Finally, it allows to work with smooth rather than atomic solutions.

1.2. Comparison with classical over-damped Dean–Kawasaki model and open problems

A rigorous connection between (2) and (6) is, to the present day, still lacking. Firstly, it is not at all clear if (6) can be recovered via an over-damped limit (i.e., by taking $\gamma \to \infty$) in (2): to the best of our knowledge, there are no known mathematical results in this context, and, additionally, (2) admits smooth solutions while (6) only admits atomic solutions. Secondly, we have no clear indication as to how to recover the square root singularity in (2) (i.e., how to perform the limit $\delta \to 0$), even in the context of strictly positive solution to the noise-free dynamics of (2) considered in the paper. The closest result on the subject is given in [13]. The macroscopic limit $N \to \infty$ (which, in the case of (2), also forces the removal of the regularisation $\epsilon \to 0$ due to the scaling (5)) is, on the contrary, better understood. The solution to (2) converges to the solution of a noiseless wave equation as a consequence of Theorem 1.1; on the other hand, the solution to (6) converges to the solution of a deterministic parabolic equation, at least for reasonable initial configurations (this follows from the definition of ρ_N and the law of large numbers).

In the case $\theta \le 2d$ (currently out of the scope of our well-posedness theory), or indeed for any other scaling of N and ϵ, the over-damped limit in (2) is just as open a question as for the case $\theta > 2d$, while the macroscopic limit $N \to \infty$ is unknown.

We now briefly turn to possible future improvements of our RIDK model. The derivation of the RIDK model (2) heavily relies on boundedness and periodicity of the spatial domain $\mathbb{T}^d$. In this case, the spectrum of the convolution operator P_ϵ is known: as explained in Section 2, this is a consequence of the one-dimensional analysis treated in [8], of suitable multiplication rules for the kernel w_ϵ, and of its separability in the d variables on $\mathbb{T}^d$.

The analysis of the RIDK model takes the spectral properties of P_ϵ merely as starting points, and, therefore, is a relatively independent and self-contained argument. It is reasonable to expect that it could be extended to general bounded domains (and, if applicable, to different boundary conditions) by adapting the analysis of the spectrum of P_ϵ. Improvements on the scaling requirement (5) will likely come from using less restrictive notions of solutions. Questions such as the long time behaviour of solutions and the invariant measures for (2) are hard to answer, and will likely require a radically different approach.

1.3. Basic notation

We work with periodic functions on the d-dimensional torus $\mathbb{T}^d = [0,2\pi)^d$ for $d \in \mathbb{N}$. We never specify the dependence of any function space on d, as this is always clear from the context. Bold face characters always denote vectors. For $m \in \mathbb{N}_0$ and $p \in [1,\infty]$, we denote by $W^{m,p}$ the standard Sobolev space of periodic functions on $\mathbb{T}^d$ with derivatives up to order m belonging to L^p. For $0 < s \notin \mathbb{N}$ and $p \in [1,\infty)$, we define the fractional spaces $W^{s,p}$ via the norm

$$\|u\|_{W^{s,p}} := \|u\|_{W^{\lfloor s\rfloor,p}} + \max_{|z|=\lfloor s\rfloor} \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{|\partial^z u(\boldsymbol{x}) - \partial^z u(\boldsymbol{y})|^p}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)p}}\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}, \tag{7}$$

where $\lfloor s\rfloor := \max\{n\in\mathbb{N}_0 \colon n\le s\}$. We also set $\lceil y\rceil := \min\{n\in\mathbb{N} : y\le n\}$. We consider the fractional Hilbert spaces H^s and $\boldsymbol{H}^s := \left[H^s\right]^d$ identified by the Fourier-type inner products

$$\langle u, v\rangle_{H^s} := \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \hat{u}_{\boldsymbol{j}}\,\overline{\hat{v}_{\boldsymbol{j}}}\,(1+|\boldsymbol{j}|^2)^s, \qquad \hat{u}_{\boldsymbol{j}} := (2\pi)^{-d}\int_{\mathbb{T}^d} e^{-\mathrm{i}\boldsymbol{j}\cdot\boldsymbol{x}}\,u(\boldsymbol{x})\,\mathrm{d}\boldsymbol{x}, \qquad u,v\in H^s,$$

$$\langle \boldsymbol{u}, \boldsymbol{v}\rangle_{\boldsymbol{H}^s} := \sum_{\ell=1}^{d} \langle u_\ell, v_\ell\rangle_{H^s}, \qquad \boldsymbol{u},\boldsymbol{v}\in\boldsymbol{H}^s, \tag{8}$$

and we define the norm on $\mathcal{W}^s := H^s\times\boldsymbol{H}^s$ as $\|(u,\boldsymbol{v})\|_{\mathcal{W}^s} := (\|u\|^2_{H^s} + \|\boldsymbol{v}\|^2_{\boldsymbol{H}^s})^{1/2}$, for $(u,\boldsymbol{v})\in\mathcal{W}^s$. The norms $\|\cdot\|_{H^s}$ and $\|\cdot\|_{W^{s,2}}$ are equivalent; see [3, Proposition 1.3]. We define the space $\boldsymbol{V}^{s+1} := \{\boldsymbol{v}\in\boldsymbol{H}^s \colon \nabla\cdot\boldsymbol{v}\in H^s\} \supset \boldsymbol{H}^{s+1}$, and recall the integration-by-parts formula

$$\langle -\nabla\cdot\boldsymbol{v}, u\rangle_{H^s} = \langle \boldsymbol{v}, \nabla u\rangle_{\boldsymbol{H}^s}, \qquad \forall u\in H^{s+1}, \forall\boldsymbol{v}\in\boldsymbol{V}^{s+1}. \tag{9}$$

In dimension $d=1$, we trivially have $V^{s+1}\equiv H^{s+1}$. We denote by $L(\mathcal{W}^s)$ (respectively, $L_2^0(\mathcal{W}^s)$) the set of continuous linear functionals mapping $\mathcal{W}^s$ into itself (respectively, the set of Hilbert–Schmidt operators from $P^{1/2}_{\sqrt{2}\epsilon}\mathcal{W}^s\subset\mathcal{W}^s$ into $\mathcal{W}^s$), with the convolution operator $P_{\sqrt{2}\epsilon}$ as defined after (2).

For each $\alpha\in\mathbb{N}$, we define

$$\begin{aligned}
\Pi_\alpha &:= \{\text{set of partitions of } \{1,\dots,\alpha\}\}, \\
B(\pi) &:= \{\text{set of blocks forming partition } \pi\}, \qquad \pi\in\Pi_\alpha, \\
|\pi| &:= \#B(\pi) = \text{number of blocks forming partition } \pi,
\end{aligned}$$

where # denotes the number of elements in a set. Furthermore, for every partition $\pi\in\Pi_\alpha$, we set

$$\begin{aligned}
\beta_j(\pi) &:= \#\{b\in B(\pi) \colon |b| = j\}, \qquad j\in\{1,\dots,\alpha\}, \\
J(\pi) &:= \{j\in\{1,\dots,\alpha\} \colon \beta_j(\pi) > 0\}.
\end{aligned}$$

As an immediate consequence of the definitions, we have $\sum_{j\in J(\pi)} j\,\beta_j(\pi) = \alpha$.

We use C as a generic constant whose value may change from line to line (with dependence on relevant parameters highlighted whenever necessary, for example $C(s)$). In addition, we denote the embedding constant of $H^s\subset C^0$ by $K_{H^s\to C^0}$. Finally, we use the subscript notation to link specific constants with the lemmas where they are defined; for example, $K_{B.1}$ is the constant introduced in Lemma B.1.

2. Derivation of RIDK

We now derive the RIDK model (4) by following the methodology outlined in [8]. Consider the second-order Langevin system (1), as well as the quantities $(\rho_\epsilon(\boldsymbol{x},t), \boldsymbol{j}_\epsilon(\boldsymbol{x},t)) = (N^{-1}\sum_{i=1}^N w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t)), N^{-1}\sum_{i=1}^N \boldsymbol{p}_i(t) w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t)))$ defined via the kernel (3). Simple Itô computations imply that ρ_ϵ and $\boldsymbol{j}_\epsilon$ satisfy the system

$$\begin{cases} \partial_t \rho_\epsilon(\boldsymbol{x},t) = -\nabla\cdot \boldsymbol{j}_\epsilon(\boldsymbol{x},t), \\ \partial_t \boldsymbol{j}_\epsilon(\boldsymbol{x},t) = -\gamma\, \boldsymbol{j}_\epsilon(\boldsymbol{x},t) - \boldsymbol{j}_{2,\epsilon}(\boldsymbol{x},t) + \boldsymbol{I}_U(\boldsymbol{x},t) + \dot{\boldsymbol{Z}}_N(\boldsymbol{x},t), \end{cases} \tag{10}$$

where the ℓth component of terms on the right are defined by

$$\begin{aligned} [\boldsymbol{j}_{2,\epsilon}(\boldsymbol{x},t)]_\ell &:= N^{-1}\sum_{i=1}^N p_{\ell,i}^2(t)\, \partial_{x_\ell} w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t)) \\ &\qquad + N^{-1}\sum_{i=1}^N \sum_{k\neq\ell} p_{\ell,i}(t)\, p_{k,i}(t)\, \partial_{x_\ell} w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t)), \\ [\boldsymbol{I}_U(\boldsymbol{x},t)]_\ell &:= -N^{-1}\sum_{i=1}^N N^{-1}\sum_{j=1}^N \partial_{x_\ell} U(\boldsymbol{q}_i(t)-\boldsymbol{q}_j(t))\, w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t)), \\ \left[\dot{\boldsymbol{Z}}_N(\boldsymbol{x},t)\right]_\ell &:= \sigma N^{-1}\sum_{i=1}^N w_\epsilon(\boldsymbol{x}-\boldsymbol{q}_i(t))\, \dot{b}_{\ell,i}. \end{aligned}$$

The terms $\boldsymbol{j}_{2,\epsilon}$, $\boldsymbol{I}_U$, and $\dot{\boldsymbol{Z}}_N$ are not closed in the leading quantities $(\rho_\epsilon, \boldsymbol{j}_\epsilon)$, and approximations are used to close the system of equations. We now sketch how the approximations in [7,8] extend to the multi-dimensional case.

The term $\boldsymbol{j}_{2,\epsilon}$ is dealt with under a local-equilibrium assumption [12, Corollary 3.2]. In this situation, the probability density function of $(\boldsymbol{q}_i(t), \boldsymbol{p}_i(t))$ is approximately separable in the position variable $\boldsymbol{q}_i(t)$ and momentum variable $\boldsymbol{p}_i(t)$ due to the structure of the Gibbs invariant measure. In addition, the momentum variable is distributed according to a Gaussian of mean zero and diagonal covariance matrix $(\sigma^2/2\gamma)\, I_d$. Furthermore, under the additional assumption $\sigma^2 \ll 2\gamma$, the approximation $\sigma^2/(2\gamma) = \mathbb{E}[p_{\ell,i}^2(t)] \approx p_{\ell,i}^2(t)$ is legitimate. All these considerations imply that $\mathbb{E}[\boldsymbol{j}_{2,\epsilon}] \approx \sigma^2/(2\gamma)\,\mathbb{E}[\nabla\rho_\epsilon]$ and this motivates the replacement $\boldsymbol{j}_{2,\epsilon} \approx (\sigma^2/2\gamma)\,\nabla\rho_\epsilon$.

The interaction term $\boldsymbol{I}_U$ may be approximated as $\boldsymbol{I}_U \approx -\rho_\epsilon\,(\nabla U * \rho_\epsilon)$, following the lines of [8, Proposition 3.5].

Finally, one may substitute the noise $\dot{\boldsymbol{Z}}_N(\boldsymbol{x},t)$ with $\dot{\boldsymbol{Y}}_N(\boldsymbol{x},t)$, where

$$\left[\dot{\boldsymbol{Y}}_N(\boldsymbol{x},t)\right]_\ell := \sigma\, N^{-1/2}\sqrt{\rho_\epsilon(\boldsymbol{x},t)}\, P_{\sqrt{2}\epsilon}^{1/2}\, \xi_\ell(\boldsymbol{x},t), \tag{11}$$

where P_ϵ is the convolution operator $P_\epsilon\colon L^2(\mathbb{T}^d)\to L^2(\mathbb{T}^d)\colon f \mapsto P_\epsilon f(\cdot) = \int_{\mathbb{T}^d} w_\epsilon(\cdot-\boldsymbol{y}) \times f(\boldsymbol{y})\,\mathrm{d}\boldsymbol{y}$ introduced above, and where $\{\xi_\ell\}_{\ell=1}^d$ are independent space-time white noises. The substitution of $\dot{\boldsymbol{Z}}_N(\boldsymbol{x},t)$ with $\dot{\boldsymbol{Y}}_N(\boldsymbol{x},t)$ relies on the two noises being approximately equivalent in distribution. This is a consequence of the following approximate multiplication rule for von Mises kernels

$$w_\epsilon(\boldsymbol{x}_1 - \boldsymbol{q}_i(t))w_\epsilon(\boldsymbol{x}_2 - \boldsymbol{q}_i(t)) \approx w_{\sqrt{2}\epsilon}(\boldsymbol{x}_1 - \boldsymbol{x}_2)w_{\epsilon/\sqrt{2}}\left(\frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{2} - \boldsymbol{q}_i(t)\right), \tag{12}$$

where $\boldsymbol{x}_1, \boldsymbol{x}_2 \in \mathbb{T}^d$, which can be deduced from its one-dimensional analogue [8] thanks to the crucial fact that the kernel w_ϵ is separable in its d variables on $\mathbb{T}^d$.

In addition, the stochastic independence of the d components of each member of the family $\{\boldsymbol{b}_i\}_{i=1}^N$ is reflected in the stochastic independence of the $\{\xi_\ell\}_{\ell=1}^d$. Taking all into account, we obtain our multi-dimensional RIDK system (2).

The noise $\dot{\boldsymbol{Y}}_N(\boldsymbol{x}, t)$ can be explicitly expanded using the spectral properties of the operator P_ϵ, which, due to the separability of the kernel w_ϵ, are readily available from the one-dimensional case [7, Section 4.2]. More specifically, with $\{e_j\}_{j\in\mathbb{Z}}$ being the trigonometric system

$$e_j(x) := \begin{cases} \sqrt{\frac{1}{\pi}}\cos(jx), & \text{if } j > 0, \\ \sqrt{\frac{1}{\pi}}\sin(jx), & \text{if } j < 0, \\ \sqrt{\frac{1}{2\pi}}, & \text{if } j = 0, \end{cases}$$

it is not difficult to see that the family $\{f_{\boldsymbol{j},s}\}_{\boldsymbol{j}\in\mathbb{Z}^d}$ defined as

$$f_{\boldsymbol{j},s}(\boldsymbol{x}) := C(d)\left\{\prod_{\ell=1}^d e_{j_\ell}(x_\ell)\right\}\left(1 + |\boldsymbol{j}|^2\right)^{-s/2}, \quad \boldsymbol{j} \in \mathbb{Z}^d, \tag{13}$$

is, for some choice of normalisation constant $C(d)$, an H^s-orthonormal basis of eigenfunctions for $P_{\sqrt{2}\epsilon}$ for any $\epsilon > 0$. Furthermore, the eigenvalue of $P_{\sqrt{2}\epsilon}$ corresponding to the eigenfunction $f_{\boldsymbol{j},s}$ is

$$\lambda_{\boldsymbol{j},\epsilon} = \prod_{\ell=1}^d \lambda_{j_\ell,\epsilon}, \tag{14}$$

where the eigenvalues from the one-dimensional case are given by

$$\lambda_{j,\epsilon} = \begin{cases} Z^{-1}_{\sqrt{2}\epsilon}\int_{\mathbb{T}} e^{-\frac{\sin^2(x/2)}{\epsilon^2}}\cos(jx)\mathrm{d}x = I_j\left(\{2\epsilon^2\}^{-1}\right)/I_0\left(\{2\epsilon^2\}^{-1}\right), & \text{if } j \neq 0, \\ 1, & \text{if } j = 0, \end{cases} \tag{15}$$

with I_j denoting the modified Bessel function of first kind and order j [1, Eq. (9.6.19)]. As a result, the stochastic process

$$W_{\mathrm{per},\epsilon} := \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \sqrt{\alpha_{\boldsymbol{j},s,\epsilon}}\,(0, f_{\boldsymbol{j},s}, 0, \ldots, 0)\,\beta_{1,\boldsymbol{j}} + \cdots$$

$$+ \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \sqrt{\alpha_{\boldsymbol{j},s,\epsilon}}\,(0, \ldots, 0, f_{\boldsymbol{j},s})\,\beta_{d,\boldsymbol{j}}, \qquad \alpha_{\boldsymbol{j},s,\epsilon} := (1+|\boldsymbol{j}|^2)^s \lambda_{\boldsymbol{j},\epsilon}, \tag{16}$$

with *iid* families $\{\beta_{\ell,\boldsymbol{j}}\}_{\ell=1}^{d}$ of independent Brownian motions, is a $\mathcal{W}^s$-valued Q-Wiener process representation of the $\mathbb{R}\times\mathbb{R}^d$-valued stochastic noise $(0, \dot{\boldsymbol{Y}}_N(\boldsymbol{x},t))$. It follows that, upon swapping $\dot{\boldsymbol{Z}}_N(\boldsymbol{x},t)$ with $\dot{\boldsymbol{Y}}_N(\boldsymbol{x},t)$, we can write (10) in the abstract stochastic PDE form

$$\begin{cases} \mathrm{d}X_\epsilon(t) = A\,X_\epsilon(t)\,\mathrm{d}t + \alpha_U(X_{\epsilon,\delta}(t))\,\mathrm{d}t + B_N(X_\epsilon(t))\,\mathrm{d}W_{\mathrm{per},\epsilon}, \\ X_\epsilon(0) = X_0, \end{cases} \tag{17}$$

where $X_\epsilon = (\tilde{\rho}_\epsilon, \tilde{\boldsymbol{j}}_\epsilon)$, A is the wave-type differential operator given by

$$A\,X := \Big(-\nabla\cdot\boldsymbol{j},\, -\gamma\,\boldsymbol{j} - (\sigma^2/2\gamma)\,\nabla\rho\Big), \qquad X = (\rho, \boldsymbol{j}),$$

the interaction potential is $\alpha_U(X_{\epsilon,\delta}) := -\tilde{\rho}_\epsilon\,(\nabla U * \tilde{\rho}_\epsilon)$, and the stochastic integrand B_N is given by

$$B_N(\rho, \boldsymbol{j})(a, \boldsymbol{b}) := \sigma\,N^{-1/2}\left(0, \sqrt{\rho}\,b_1, \ldots, \sqrt{\rho}\,b_d\right).$$

For some $h_\delta \in C^{\lceil d/2\rceil+2}(\mathbb{R})$ regularising the square function in $[0,\delta]$, we substitute B_N with the smoothed stochastic integrand

$$B_{N,\delta}((\rho, \boldsymbol{j}))(a, \boldsymbol{b}) := \sigma\,N^{-1/2}\,(0, h_\delta(\rho)\,b_1, \ldots, h_\delta(\rho)\,b_d) \tag{18}$$

in (17), and we finally obtain the following equation in $X_{\epsilon,\delta} = (\tilde{\rho}_{\epsilon,\delta}, \tilde{\boldsymbol{j}}_{\epsilon,\delta})$

$$\begin{cases} \mathrm{d}X_{\epsilon,\delta}(t) = A\,X_{\epsilon,\delta}(t)\,\mathrm{d}t + \alpha_U(X_{\epsilon,\delta}(t))\,\mathrm{d}t + B_{N,\delta}(X_{\epsilon,\delta}(t))\,\mathrm{d}W_{\mathrm{per},\epsilon}, \\ X_{\epsilon,\delta}(0) = X_0, \end{cases}$$

which is exactly (4).

3. Main technical results for the proof of Theorem 1.1

We develop the three main technical tools upon which we base the main proof in Section 4. We investigate the cases $d = 1$ and $d > 1$ separately.

3.1. Semigroup analysis of operator A in $\mathcal{W}^s$

Lemma 3.1. *Let $\mathcal{D}(A) := H^{s+1} \times \boldsymbol{V}^{s+1}$. The operator $A\colon \mathcal{D}(A) \subset \mathcal{W}^s \to \mathcal{W}^s$ defines a C_0-semigroup of contractions.*

Proof of Lemma 3.1 in dimension $d = 1$**.** The proof is identical to the one provided in [7, Lemma 4.2], simply with all relevant spaces H^{α} being replaced by $H^{\alpha-1+s}$. We assume $\sigma^2/(2\gamma) := 1$ for simplicity, even though the proof is analogous for the general case $\sigma^2/(2\gamma) > 0$.

We verify the assumptions of the Hille–Yosida Theorem, as stated in [21, Theorem 3.1].

Step 1: A is a closed operator, and $\mathcal{D}(A)$ is dense in $\mathcal{W}^s$. This is easily checked.

Step 2: The resolvent set of A contains the positive half line. Let $\lambda > 0$. We show that the operator $A_\lambda := A - \lambda I$ is injective. Assume that $A_\lambda(\rho, j) = (0, 0)$. We take the H^s-inner product of the first component of $A_\lambda(\rho, j)$ with ρ and of the second component of $A_\lambda(\rho, j)$ with j, and we obtain

$$0 = \langle -j' - \lambda\,\rho, \rho\rangle_{H^s} + \langle -(\lambda+\gamma)\,j - \rho', j\rangle_{H^s} = -\lambda\,\|\rho\|^2_{H^s} - (\lambda+\gamma)\,\|j\|^2_{H^s},$$

where we have used (9). Since $\lambda, \gamma > 0$, we deduce that $(\rho, j) = (0, 0)$. We now show that A_λ^{-1} is a bounded operator. Consider $A_\lambda^{-1}(a, b) = (\rho, j)$. This implies

$$\lambda\rho = -a - j', \tag{19}$$

$$(\lambda+\gamma)\,j = -b - \rho'. \tag{20}$$

Taking the H^s-inner product of (19) (respectively, of (20)) with ρ (respectively, with j), we get

$$\lambda\,\|(\rho, j)\|^2_{\mathcal{W}^s} \le \lambda\,\|\rho\|^2_{H^s} + (\lambda+\gamma)\,\|j\|^2_{H^s} = \langle -a, \rho\rangle_{H^s} + \langle -b, j\rangle_{H^s}. \tag{21}$$

We use the Cauchy–Schwartz and Young inequalities to deduce $\|A_\lambda^{-1}\|_{\mathcal{L}(\mathcal{W}^s,\mathcal{W}^s)} \le \lambda^{-1}$, hence the boundedness of A_λ^{-1}. We now show that $\mathrm{Dom}(A_\lambda^{-1})$ is dense in $\mathcal{W}^s$. Let us fix $(a, b) \in H^s \times H^{s+1}$. The system of equations $A_\lambda(\rho, j) = (a, b)$ reads

$$-j' - \lambda\,\rho = a, \qquad -(\lambda+\gamma)\,j - \rho' = b,$$

which promptly gives

$$\frac{\rho''}{\lambda+\gamma} - \lambda\,\rho = a - \frac{b'}{\lambda+\gamma} \in H^s. \tag{22}$$

A Fourier series expansion argument provides existence of a unique solution $\rho \in H^{s+2}$ for (22). From $-(\lambda+\gamma)\,j = \rho' + b$, we immediately deduce that $j \in H^{s+1}$. We have shown that, for every (a, b) in the dense subset $H^s \times H^{s+1} \subset \mathcal{W}^s$, the operator A_λ^{-1} is well-defined.

Step 3: Inequality [21, (3.1)] is satisfied: This is precisely $\|A_\lambda^{-1}\|_{\mathcal{L}(\mathcal{W}^s,\mathcal{W}^s)} \le \lambda^{-1}$, which we already proved. □

Proof of Lemma 3.1 in dimension $d > 1$**.** Steps 2 and 3 are readily adapted, as the Fourier analysis is unchanged. We only need to justify the validity of Step 1. As for the density of $\mathcal{D}(A)$ in $\mathcal{W}^s$, this is implied by the density of H^{s+1} into H^s and $\boldsymbol{H}^{s+1}$ into $\boldsymbol{H}^s$, as well as by the inclusion $\boldsymbol{H}^{s+1} \subset \boldsymbol{V}^{s+1}$. As for the closedness of the operator A, this follows from the consistency of the first component of A and of the definition of $\boldsymbol{V}^{s+1}$. More specifically, consider a sequence

$\mathcal{D}(A) \ni (\rho_n, \boldsymbol{j}_n) \to (\rho, \boldsymbol{j})$ in $\mathcal{W}^s$, such that $A(\rho_n, \boldsymbol{j}_n) \to (x, \boldsymbol{y})$ in $\mathcal{W}^s$. This immediately implies that $\nabla \cdot \boldsymbol{j}_n$ converges in H^{s-1} to both $-x$ and $\nabla \cdot \boldsymbol{j}$, forcing them to agree. In particular, $\boldsymbol{j} \in \boldsymbol{V}^{s+1}$. Similarly, $\nabla \rho_n$ converges in $\boldsymbol{H}^{s-1}$ to both $\nabla \rho$ and $-\gamma \boldsymbol{j} - \boldsymbol{y}$, forcing them to agree. In particular, $\rho \in H^{s+1}$. Therefore, $(\rho, \boldsymbol{j}) \in \mathcal{D}(A)$ and $A(\rho, \boldsymbol{j}) = (x, \boldsymbol{y})$. □

3.2. Improved bounds on trace of $W_{\mathrm{per},\epsilon}$ in $\mathcal{W}^s$-norm

Lemma 3.2. *Let $\{\lambda_{\epsilon,\boldsymbol{j}}\}_{\boldsymbol{j}\in\mathbb{Z}^d}$ be the eigenvalues of $P_{\sqrt{2}\epsilon}$, see (14) and (15). Let $\alpha \in (0,1)$ and $\beta \in (0,1)$ such that $\alpha + \beta \geq 1$, and let $s \geq 0$.*

(i) The following bound holds

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \{\lambda_{\boldsymbol{j},\epsilon}\}(1+|\boldsymbol{j}|^2)^s \leq C(s,d)\left\{\epsilon^{-2\beta(2s+1)} + \epsilon^{-2\alpha(2s+1)} + \epsilon^{-2\alpha-4\beta s}\right\}\epsilon^{-(d-1)}. \tag{23}$$

(ii) The right-hand side of (23) is minimised, among all admissible pairs (α, β), by choosing $(\alpha, \beta) = (1/2, 1/2)$. In this case, the right-hand side of (23) is proportional to $\epsilon^{-\theta_c(s)}$, where $\theta_c(s) = 2s + d$ was introduced in Remark 1.2.

Proof of Lemma 3.2 in dimension $d = 1$. We denote by $I_j(x)$ the j-th modified Bessel function of the first kind evaluated at x.

Step 1. There exists $K > 0$ such that, for any j and any ϵ, it holds $\lambda_{\epsilon,j} < K$. This follows from (15) together with the monotonicity of $\{\lambda_{\epsilon,j}\}_j$ (see [20, Introduction]).

Step 2. Let $x \geq 1$. Picking $k = 2$ and $m = 0$ in [20, Theorem 2, bound (a)], we have

$$\frac{I_{j+1}(x)}{I_j(x)} < \frac{x}{j + 1/2 + x}. \tag{24}$$

We show that the inequality

$$\frac{x}{j + 1/2 + x} \leq 1 - \frac{1}{x^\alpha} \tag{25}$$

holds when

$$j \geq C x^\beta, \tag{26}$$

$$x \geq \overline{x} = \overline{x}(\alpha, \beta) > 0, \tag{27}$$

for suitable $\overline{x}(\alpha, \beta) > 0$ and $C > 0$ to be discussed below. Simple algebraic rearrangements imply that (25) is equivalent to

$$0 \leq j(x^\alpha - 1) + \frac{1}{2}x^\alpha - x - \frac{1}{2}, \tag{28}$$

which is in turn satisfied (taking (26) into account), at least under the sufficient condition

$$0 \le Cx^{\alpha+\beta} - Cx^{\beta} + \frac{1}{2}x^{\alpha} - x - \frac{1}{2}. \tag{29}$$

Take $C > 0$ in (26) if $\alpha + \beta > 1$, otherwise take $C > 1$ if $\alpha + \beta = 1$. Then, for x large enough (i.e., for $\overline{x}$ large enough in (27)), inequality (29) is satisfied, and therefore so is inequality (25).

Step 3. By symmetry of $\lambda_{\epsilon,j}$ with respect to j, seen in (15), we only need consider non-negative indexes j. We define $A_1 := \{0, 1, 2, \dots, \lceil Cx^{\beta}\rceil\}$ and $A_2 := \mathbb{N}_0 \setminus A_1$. We split the sum in the left-hand side of (23) over these two sets. We use *Step 1* to deduce

$$\begin{aligned}\sum_{j=0}^{\lceil Cx^{\beta}\rceil} \{\lambda_{\epsilon,j}\}(1+j^2)^s &\le K \sum_{j=0}^{\lceil Cx^{\beta}\rceil} (1+j^2)^s \le K \sum_{j=0}^{\lceil Cx^{\beta}\rceil} (1+\lceil Cx^{\beta}\rceil^2)^s \le K \sum_{j=0}^{\lceil Cx^{\beta}\rceil} (1+\lceil Cx^{\beta}\rceil)^{2s} \\ &= K(1+\lceil Cx^{\beta}\rceil)^{2s+1} \le C(s)K(\lceil Cx^{\beta}\rceil)^{2s+1}.\end{aligned} \tag{30}$$

For the sum over A_2, we use the geometric decay $\lambda_{\epsilon,j+1} \le (1 - 1/x^{\alpha})\lambda_{\epsilon,j}$, which is implied by (24) and (25) combined with (15). We use *Step 1* to obtain

$$\begin{aligned}&\sum_{j=\lceil Cx^{\beta}\rceil+1}^{\infty} \{\lambda_{\epsilon,j}\}(1+j^2)^s \\ &\le C(s)K \sum_{j=\lceil Cx^{\beta}\rceil+1}^{\infty} \left(1-\frac{1}{x^{\alpha}}\right)^{j-(\lceil Cx^{\beta}\rceil+1)} j^{2s} \\ &= C(s)K \sum_{j=0}^{\infty} \left(1-\frac{1}{x^{\alpha}}\right)^{j} \left(j+\lceil Cx^{\beta}\rceil+1\right)^{2s} \\ &\le C(s)K \sum_{j=0}^{\infty} \left(1-\frac{1}{x^{\alpha}}\right)^{j} j^{2s} + C(s)K \sum_{j=0}^{\infty} \left(1-\frac{1}{x^{\alpha}}\right)^{j} \left(\lceil Cx^{\beta}\rceil+1\right)^{2s} \\ &\le C(s)K \left(\frac{1}{x^{\alpha}}\right)^{-(2s+1)} + C(s)Kx^{\alpha}\left(\lceil Cx^{\beta}\rceil+1\right)^{2s},\end{aligned} \tag{31}$$

where we have also used estimates on the polylogarithmic function $\mathrm{Li}_{\gamma}(z) := \sum_{j=1}^{\infty} z^j j^{-\gamma}$ for the first term in the last line, namely

$$\mathrm{Li}_{\gamma}(z) \le \frac{C(s)}{(1-z)^{-\gamma+1}}. \tag{32}$$

In our case, $\gamma = -2s$. Inequality (32) applies for negative integers γ as a simple consequence of differentiation of the geometric power series. Furthermore, (32) also applies for negative non-integers γ, provided that $z \in (1-\nu, 1)$ for some small $\nu = \nu(s)$. This is a consequence of the trivial bound $\mathrm{Li}_{\gamma}(z) \le \sum_{j=0}^{\infty} z^j(1+j)^{-\gamma}$, and of [15, (9.550) and (9.557)].

As in the case of [7, Lemma 4.3], we pick $x := (2\epsilon^2)^{-1}$, with ϵ small enough so that (27) holds (and that $z = 1 - 1/x^{\alpha} \in (1-\nu, 1)$, with this requirement only demanded if $\gamma = -2s \notin \mathbb{Z}$). Combining (30) and (31) gives (23).

Finally, it is easy to see that the choice $(\alpha, \beta) = (1/2, 1/2)$, which makes the right-hand side of (23) proportional to $\epsilon^{-\theta_c(s)} = \epsilon^{-(2s+d)}$, also minimises it among all admissible pairs (α, β). $\square$

Proof of Lemma 3.2 in dimension $d > 1$. The result promptly follows from the bound

$$\sum_{\boldsymbol{j}\in\mathbb{Z}^d} \{\lambda_{\boldsymbol{j},\epsilon}\}(1+|\boldsymbol{j}|^2)^s = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \prod_{\ell=1}^{d} \lambda_{j_\ell,\epsilon} \left(1+\sum_{k=1}^{d} j_k^2\right)^s$$

$$\leq C(s) \sum_{k=1}^{d} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \prod_{\ell=1}^{d} \lambda_{j_\ell,\epsilon}(1+j_k^2)^s \leq C(s) \sum_{k=1}^{d} \sum_{j_k\in\mathbb{Z}} \lambda_{j_k,\epsilon}(1+j_k^2)^s \sum_{j_\ell\in\mathbb{Z},\, \ell\neq k} \prod_{\ell=1,\ell\neq k}^{d} \lambda_{j_\ell,\epsilon}$$

$$= C(s,d) \left(\sum_{j\in\mathbb{Z}} \lambda_{j,\epsilon}(1+j^2)^s\right) \left(\sum_{j\in\mathbb{Z}} \lambda_{j,\epsilon}\right)^{d-1}$$

and the validity of (23) for $d = 1$. The optimality of the scaling under $(\alpha, \beta) = (1/2, 1/2)$ has already been dealt with in the one-dimensional case. $\square$

Remark 3.3. For $d = 1$, we have improved the scaling of [7, Lemma 4.3] in two points. Firstly, the bound on $\{\lambda_{j,\epsilon}\}_j$ is now uniform in ϵ and j (i.e., we no longer bound $\lambda_{j,\epsilon}$ using ϵ^{-1}). Secondly, the exponential decay of the eigenvalues 'kicks in' earlier, namely around $C\epsilon^{-2\beta}$ rather than around ϵ^{-2}. This leads to a sharper estimate concerning the sum on the region A_1.

These improvements bring the threshold θ_0 down from 7 to 3 for the suboptimal choice $s = 1$ (see [7, Lemma 4.3]). In addition, the switch to fractional Sobolev spaces, i.e., the choice $s = 1/2 + \eta$ instead of $s = 1$ as in [7], where η can be chosen arbitrarily small, grants a further decrease of θ_0 from 3 to 2.

3.3. *Regularity of the stochastic integrand $B_{N,\delta}$*

Lemma 3.4. *With the same notation as in Theorem 1.1, let $s = d/2 + \eta$, where $\eta > 0$ is such that $\eta < C(d) < 1/2$, where $C(d)$ is small enough (see Lemma B.4). Then*

(i) *$B_{N,\delta}$ is a map from $\mathcal{W}^s$ to $L(\mathcal{W}^s)$.*
(ii) *$B_{N,\delta}$ is* locally Lipschitz *with respect to the $L_2^0(\mathcal{W}^s)$-norm.*
(iii) *$B_{N,\delta}$ is* sublinear *with respect to the $L_2^0(\mathcal{W}^s)$-norm if $d = 1$, and* locally bounded *with respect to the same norm if $d > 1$.*

Proof of Lemma 3.4 for $d = 1$. We limit ourselves to proving Statements (ii) and (iii).

Statement (ii). Take $(u_1, v_1), (u_2, v_2) \in \mathcal{W}^s$, such that $\|(u_1, v_1)\|_{\mathcal{W}^s} \leq k$, $\|(u_2, v_2)\|_{\mathcal{W}^s} \leq k$. From (16) and (18) we have that

$$\left\| B_{N,\delta}((u_1, v_1)) - B_{N,\delta}((u_2, v_2)) \right\|^2_{L_2^0(\mathcal{W}^s)}$$
$$= \sum_{j\in\mathbb{Z}} \left\| \sqrt{\alpha_{j,s,\epsilon}} \left\{ B_{N,\delta}((u_1, v_1)) - B_{N,\delta}((u_2, v_2)) \right\} (0, f_{j,s}) \right\|^2_{\mathcal{W}^s}$$

$$= \frac{\sigma^2}{N} \sum_{j \in \mathbb{Z}} \alpha_{j,s,\epsilon} \left\| \left(0, \{h_\delta(u_1) - h_\delta(u_2)\} f_{j,s}\right) \right\|^2_{\mathcal{W}^s}. \tag{33}$$

We use the fact that $\{f_{j,s}\}_j$ are orthonormal in H^s, the equivalence of the norms $\|\cdot\|_{H^s}$ and $\|\cdot\|_{W^{s,2}}$ (see Subsection 1.3), the boundedness of h'_δ, and Lemma B.1 to write

$$\begin{aligned}
&\left\| \left(0, \{h_\delta(u_1) - h_\delta(u_2)\} f_{j,s}\right) \right\|^2_{\mathcal{W}^s} \\
&\quad \le C \| \{h_\delta(u_1) - h_\delta(u_2)\} f_{j,s} \|^2_{W^{s,2}} \le C K^2_{B.1} \| h_\delta(u_1) - h_\delta(u_2) \|^2_{W^{s,2}} \\
&\quad = C K^2_{B.1} \left\{ \int_{\mathbb{T}} |h_\delta(u_1(x)) - h_\delta(u_2(x))|^2 \mathrm{d}x \right. \\
&\qquad \left. + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|h_\delta(u_1(x)) - h_\delta(u_2(x)) - \{h_\delta(u_1(y)) - h_\delta(u_2(y))\}|^2}{|x-y|^{1+2s}} \mathrm{d}x\mathrm{d}y \right\} \\
&\quad \le C(\delta) K^2_{B.1} \left\{ \int_{\mathbb{T}} |u_1(x) - u_2(x)|^2 \mathrm{d}x \right. \\
&\qquad \left. + \int_{\mathbb{T}} \int_{\mathbb{T}} \frac{|h_\delta(u_1(x)) - h_\delta(u_2(x)) - \{h_\delta(u_1(y)) - h_\delta(u_2(y))\}|^2}{|x-y|^{1+2s}} \mathrm{d}x\mathrm{d}y \right\}.
\end{aligned} \tag{34}$$

We bound the numerator of (34). If either $u_1(x) = u_2(x)$ or $u_1(y) = u_2(y)$, then simply

$$\begin{aligned}
&|h_\delta(u_1(x)) - h_\delta(u_2(x)) - \{h_\delta(u_1(y)) - h_\delta(u_2(y))\}|^2 \\
&\quad \le C(\delta) |u_1(x) - u_2(x) - \{u_1(y) - u_2(y)\}|^2.
\end{aligned} \tag{35}$$

Otherwise, we use the embedding $H^s \subset C^0$ and write

$$\begin{aligned}
&|h_\delta(u_1(x)) - h_\delta(u_2(x)) - \{h_\delta(u_1(y)) - h_\delta(u_2(y))\}|^2 \\
&\le 2 \left| \frac{h_\delta(u_1(x)) - h_\delta(u_2(x))}{u_1(x) - u_2(x)} \right|^2 |u_1(x) - u_2(x) - \{u_1(y) - u_2(y)\}|^2 \\
&\quad + 2 \left| \frac{h_\delta(u_1(x)) - h_\delta(u_2(x))}{u_1(x) - u_2(x)} - \frac{h_\delta(u_1(y)) - h_\delta(u_2(y))}{u_1(y) - u_2(y)} \right|^2 |u_1(y) - u_2(y)|^2 \\
&\le 2C(\delta) |u_1(x) - u_2(x) - \{u_1(y) - u_2(y)\}|^2 \\
&\quad + 2K^2_{H^s \to C^0} \left| \frac{h_\delta(u_1(x)) - h_\delta(u_2(x))}{u_1(x) - u_2(x)} - \frac{h_\delta(u_1(y)) - h_\delta(u_2(y))}{u_1(y) - u_2(y)} \right|^2 \|u_1 - u_2\|^2_{H^s} \\
&=: T_1 + T_2.
\end{aligned} \tag{36}$$

We now focus on T_2. We define the auxiliary function

$$r(\alpha,\beta)=\begin{cases}\{h_\delta(\alpha)-h_\delta(\beta)\}/(\alpha-\beta), & \text{if } \alpha\neq\beta,\\ h_\delta'(\alpha), & \text{if } \alpha=\beta.\end{cases}$$

We write

$$\begin{aligned}&\left|\frac{h_\delta(u_1(x))-h_\delta(u_2(x))}{u_1(x)-u_2(x)}-\frac{h_\delta(u_1(y))-h_\delta(u_2(y))}{u_1(y)-u_2(y)}\right|^2\\ &\quad\le 2\left|r(u_1(x),u_2(x))-r(u_1(y),u_2(x))\right|^2\\ &\qquad+2\left|r(u_1(y),u_2(x))-r(u_1(y),u_2(y))\right|^2:=T_3+T_4.\end{aligned}$$

In the above, we perform a first-order Taylor expansion (with respect to the first variable of r only for T_3, and with respect to the second variable of r only for T_4). This is possible because r has partial derivatives defined everywhere (as a consequence of h_δ being $C^2(\mathbb{R})$). In addition, the partial derivatives of r are uniformly bounded by $\sup_{z\in\mathbb{R}}|h_\delta''(z)|\le C(\delta)$. This implies

$$T_3+T_4\le C(\delta)|u_1(x)-u_1(y)|^2+C(\delta)|u_2(x)-u_2(y)|^2. \tag{37}$$

We plug (35), (36) and (37) into (34) and take into account the assumption $\|(u_1,v_1)\|_{\mathcal{W}^s}\le k$, $\|(u_2,v_2)\|_{\mathcal{W}^s}\le k$ to obtain

$$\begin{aligned}&\left\|\left(0,\{h_\delta(u_1)-h_\delta(u_2)\}f_{j,s}\right)\right\|^2_{\mathcal{W}^s}\\ &\le C(\delta)K^2_{B.1}\left\{\int_{\mathbb{T}}|u_1(x)-u_2(x)|^2\mathrm{d}x\right.\\ &\qquad\left.+\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|h_\delta(u_1(x))-h_\delta(u_2(x))-\{h_\delta(u_1(y))-h_\delta(u_2(y))\}|^2}{|x-y|^{1+2s}}\mathrm{d}x\mathrm{d}y\right\}\\ &\le C(\delta)K^2_{B.1}\left\{\int_{\mathbb{T}}|u_1(x)-u_2(x)|^2\mathrm{d}x\right.\\ &\qquad+C(\delta)\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|u_1(x)-u_2(x)-\{u_1(y)-u_2(y)\}|^2}{|x-y|^{1+2s}}\mathrm{d}x\mathrm{d}y\\ &\qquad\left.+C(\delta)K^2_{H^s\to C^0}\|u_1-u_2\|^2_{H^s}\int_{\mathbb{T}}\int_{\mathbb{T}}\frac{|u_1(x)-u_1(y)|^2+|u_2(x)-u_2(y)|^2}{|x-y|^{1+2s}}\mathrm{d}x\mathrm{d}y\right\}\\ &\le C(\delta,k,K_{B.1},K_{H^s\to C^0})\|u_1-u_2\|^2_{H^s}\\ &\le C(\delta,k,K_{B.1},K_{H^s\to C^0})\|(u_1,v_1)-(u_2,v_2)\|^2_{\mathcal{W}^s}.\end{aligned}$$

We can go back to (33) and deduce the local Lipschitz property

$$\left\| B_{N,\delta}((u_1,v_1)) - B_{N,\delta}((u_2,v_2)) \right\|^2_{L^0_2(\mathcal{W}^s)}$$
$$\leq \frac{\sigma^2}{N}\left(\sum_j \alpha_{j,s,\epsilon}\right) C(\delta,k,K_{B.1},K_{H^s\to C^0})\|(u_1,v_1)-(u_2,v_2)\|^2_{\mathcal{W}^s}.$$

Statement (iii). We write

$$\begin{aligned}
&\left\| B_{N,\delta}((u,v)) \right\|^2_{L^0_2(\mathcal{W}^s)}\\
&= \sum_{j\in\mathbb{Z}} \left\| \sqrt{\alpha_{j,s,\epsilon}} B_{N,\delta}((u,v))(0,f_{j,s}) \right\|^2_{\mathcal{W}^s} = \frac{\sigma^2}{N}\sum_{j\in\mathbb{Z}}\alpha_{j,s,\epsilon}\left\|\left(0,h_\delta(u)f_{j,s}\right)\right\|^2_{\mathcal{W}^s}\\
&= \frac{\sigma^2}{N}\left[\sum_{j\in\mathbb{Z}}\alpha_{j,s,\epsilon}\left\|h_\delta(u)f_{j,s}\right\|^2_{H^s}\right] \leq K^2_{B.1}\frac{\sigma^2}{N}\left[\sum_{j\in\mathbb{Z}}\alpha_{j,s,\epsilon}\|h_\delta(u)\|^2_{H^s}\right]\\
&\leq C(\delta)K^2_{B.1}\frac{\sigma^2}{N}\epsilon^{-(2s+1)}\left(1+\|(u,v)\|^2_{\mathcal{W}^s}\right),
\end{aligned} \tag{38}$$

where we have used Lemma B.1, the sublinearity of h_δ at infinity, the boundedness of h'_δ, and Lemma 3.2. This completes the proof. □

Proof of Lemma 3.4 for $d > 1$. In this proof, we need to analyse quantities associated with derivatives of the distinctive nonlinearity $h_\delta(u)$, $u \in H^{d/2+\eta}$. For this purpose, we make heavy use of the contents of Appendix B (integrability properties of the Faà di Bruno representation of derivatives of $h_\delta(u)$) and Appendix A (factorisation of differences of two distinct instances of the same derivative).

We again focus on points (ii) and (iii) only.

Statement (ii). Take $(u_1,\boldsymbol{v}_1),(u_2,\boldsymbol{v}_2)\in\mathcal{W}^s$, such that $\|(u_1,\boldsymbol{v}_1)\|_{\mathcal{W}^s}\leq k$, $\|(u_2,\boldsymbol{v}_2)\|_{\mathcal{W}^s}\leq k$. In order to bound

$$\left\| B_{N,\delta}((u_1,\boldsymbol{v}_1)) - B_{N,\delta}((u_2,\boldsymbol{v}_2)) \right\|^2_{L^0_2(\mathcal{W}^s)}$$

we only need to bound

$$\frac{\sigma^2}{N}\sum_{\boldsymbol{j}\in\mathbb{Z}}\alpha_{\boldsymbol{j},s,\epsilon}\left\|\left(0,\{h_\delta(u_1)-h_\delta(u_2)\}f_{\boldsymbol{j},s}\right)\right\|^2_{H^s\times H^s}.$$

Moreover, Lemma B.1 allows us to only focus on estimating $\|h_\delta(u_1)-h_\delta(u_2)\|^2_{H^s}$. We introduce the shorthand notations

$$\mathcal{P}_{\pi,\alpha}u(\boldsymbol{x}) := \prod_{j\in J(\pi)}\prod_{b\in B(\pi):\,|b|=j}\frac{\partial^{(j)}u(\boldsymbol{x})}{\prod_{z\in b}\partial x_{\ell_z}}, \qquad \mathcal{P}_\pi u := \mathcal{P}_{\pi,\lfloor d/2\rfloor}u$$

for every $\mathbb{N} \ni \alpha \le \lfloor d/2 \rfloor$ and $\pi \in \Pi_\alpha$, and $\mathcal{P}_{\pi,0}u(\boldsymbol{x}) := 1$. Due to the Faà di Bruno formula recalled in Lemma B.2 and the equivalence of the norms $\|\cdot\|_{H^s}$ and $\|\cdot\|_{W^{s,2}}$, the term $\|h_\delta(u_1) - h_\delta(u_2)\|^2_{H^s}$ can be controlled by providing a bound for

$$\left[\sum_{\alpha=0}^{\lfloor d/2 \rfloor} \left\| h_\delta^{(|\pi_\alpha|)}(u_1)\mathcal{P}_{\pi_\alpha,\alpha}u_1 - h_\delta^{(|\pi_\alpha|)}(u_2)\mathcal{P}_{\pi_\alpha,\alpha}u_2 \right\|^2_{L^2}\right]$$
$$+ \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{1}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s \rfloor)2}} \left| h_\delta^{(|\pi|)}(u_1(\boldsymbol{x}))\mathcal{P}_\pi u_1(\boldsymbol{x}) - h_\delta^{(|\pi|)}(u_2(\boldsymbol{x}))\mathcal{P}_\pi u_2(\boldsymbol{x}) + \right.$$
$$\left. - \left\{ h_\delta^{(|\pi|)}(u_1(\boldsymbol{y}))\mathcal{P}_\pi u_1(\boldsymbol{y}) - h_\delta^{(|\pi|)}(u_2(\boldsymbol{y}))\mathcal{P}_\pi u_2(\boldsymbol{y}) \right\} \right|^2 \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} := A_1 + A_2 \tag{39}$$

for any choice $\pi_\alpha \in \Pi_\alpha$, $\alpha \in \{0, \ldots, \lfloor d/2 \rfloor\}$, and $\pi \in \Pi_{\lfloor d/2 \rfloor}$.

Upon adding and subtracting terms of the type $h_\delta^{(|\pi_\alpha|)}(u_2)\mathcal{P}_{\pi_\alpha,\alpha}u_1$, for $\alpha \in \{0, \ldots, \lfloor d/2 \rfloor\}$, the term A_1 is bounded (up to a constant) by

$$\sum_{\alpha=0}^{\lfloor d/2 \rfloor} \left\| \left\{ h_\delta^{(|\pi_\alpha|)}(u_1) - h_\delta^{(|\pi_\alpha|)}(u_2) \right\} \mathcal{P}_{\pi_\alpha,\alpha}u_2 \right\|^2_{L^2} + \sum_{\alpha=0}^{\lfloor d/2 \rfloor} \left\| h_\delta^{(|\pi_\alpha|)}(u_2) \left\{ \mathcal{P}_{\pi_\alpha,\alpha}u_1 - \mathcal{P}_{\pi_\alpha,\alpha}u_2 \right\} \right\|^2_{L^2}$$
$$\le C(d,\delta) \left[K^2_{H^s \to C^0} \|u_1 - u_2\|^2_{H^s} \sum_{\alpha=0}^{\lfloor d/2 \rfloor} \left\| \mathcal{P}_{\pi_\alpha,\alpha}u_2 \right\|^2_{L^2} + \sum_{\alpha=0}^{\lfloor d/2 \rfloor} \left\| \mathcal{P}_{\pi_\alpha,\alpha}u_1 - \mathcal{P}_{\pi_\alpha,\alpha}u_2 \right\|^2_{L^2} \right] \tag{40}$$

$$\le C(d,\delta,k) K^2_{H^s \to C^0} K_{B.3} \|u_1 - u_2\|^2_{H^s} + C(d,\delta) \sum_{\alpha=0}^{\lfloor d/2 \rfloor} \left\| \mathcal{P}_{\pi_\alpha,\alpha}u_1 - \mathcal{P}_{\pi_\alpha,\alpha}u_2 \right\|^2_{L^2}, \tag{41}$$

where we have used a Taylor expansion for (and boundedness of) derivatives of h_δ and the Sobolev embedding $H^s \subset C^0$ in (40), and Lemma B.3 in (41). We may now apply Lemma A.1–(i) to factorise $\mathcal{P}_{\pi_\alpha,\alpha}u_1 - \mathcal{P}_{\pi_\alpha,\alpha}u_2$ into a sum of terms, each of which can then be dealt with using Lemma B.3. We obtain

$$A_1 \le (41) \le C(d,\delta,k) K^2_{H^s \to C^0} K_{B.3} \|u_1 - u_2\|^2_{H^s}. \tag{42}$$

More generally, each application of Lemma A.1 below is, at least conceptually, identical to the one illustrated above. Namely, it is used to factorise a difference of objects into a sum of terms which in turn can be estimated using either Lemma B.3 or Lemma B.4.

Following simple algebraic rewritings, the term A_2 can be bounded (up to a constant) by

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\left| h_\delta^{(|\pi|)}(u_1(\boldsymbol{x})) - h_\delta^{(|\pi|)}(u_2(\boldsymbol{x})) - \{h_\delta^{(|\pi|)}(u_1(\boldsymbol{y})) - h_\delta^{(|\pi|)}(u_2(\boldsymbol{y}))\} \right|^2 |\mathcal{P}_\pi u_1(\boldsymbol{x})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s \rfloor)2}} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$

$$
\begin{aligned}
&+\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\left|h_\delta^{(|\pi|)}(u_1(\boldsymbol{y}))-h_\delta^{(|\pi|)}(u_2(\boldsymbol{y}))\right|^2|\mathcal{P}_\pi u_1(\boldsymbol{x})-\mathcal{P}_\pi u_1(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&+\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\left|h_\delta^{(|\pi|)}(u_2(\boldsymbol{x}))\right|^2|\mathcal{P}_\pi u_1(\boldsymbol{x})-\mathcal{P}_\pi u_2(\boldsymbol{x})-\{\mathcal{P}_\pi u_1(\boldsymbol{y})-\mathcal{P}_\pi u_2(\boldsymbol{y})\}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&+\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\left|h_\delta^{(|\pi|)}(u_2(\boldsymbol{x}))-h_\delta^{(|\pi|)}(u_2(\boldsymbol{y}))\right|^2|\mathcal{P}_\pi u_1(\boldsymbol{y})-\mathcal{P}_\pi u_2(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&=: T_1+\cdots+T_4.
\end{aligned}\tag{43}
$$

Term T_1 is dealt with using (36) and (37) (with $h_\delta^{(|\pi|)}$ replacing h_δ), the embedding $H^s\subset C^0$, and Lemma B.4–(i). Its bound reads

$$
\begin{aligned}
T_1 &\le C(\delta)\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u_1(\boldsymbol{x})-u_2(\boldsymbol{x})-\{u_1(\boldsymbol{y})-u_2(\boldsymbol{y})\}|^2|\mathcal{P}_\pi u_1(\boldsymbol{x})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\quad+C(\delta)K^2_{H^s\to C^0}\|u_1-u_2\|^2_{H^s}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u_1(\boldsymbol{x})-u_1(\boldsymbol{y})|^2|\mathcal{P}_\pi u_1(\boldsymbol{x})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\quad+C(\delta)K^2_{H^s\to C^0}\|u_1-u_2\|^2_{H^s}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|u_2(\boldsymbol{x})-u_2(\boldsymbol{y})|^2|\mathcal{P}_\pi u_1(\boldsymbol{x})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\le C(\delta,k)K^2_{H^s\to C^0}K_{B.4}\|u_1-u_2\|^2_{H^s}.
\end{aligned}
$$

The embedding $H^s\subset C^0$ and Lemmas A.1–(i) and B.4–(ii) allow to bound T_2 as

$$
\begin{aligned}
T_2 &\le C(\delta)K^2_{H^s\to C^0}\|u_1-u_2\|^2_{H^s}\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|\mathcal{P}_\pi u_1(\boldsymbol{x})-\mathcal{P}_\pi u_1(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\le C(\delta,k)K^2_{H^s\to C^0}K_{B.4}\|u_1-u_2\|^2_{H^s}.
\end{aligned}
$$

Term T_3 is dealt with by relying on the boundedness of $h_\delta^{(|\pi|)}$ and using Lemmas A.1–(ii) and B.4–(ii), thus giving

$$
\begin{aligned}
T_3 &\le C(\delta)\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|\mathcal{P}_\pi u_1(\boldsymbol{x})-\mathcal{P}_\pi u_2(\boldsymbol{x})-\{\mathcal{P}_\pi u_1(\boldsymbol{y})-\mathcal{P}_\pi u_2(\boldsymbol{y})\}|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\\
&\le C(\delta,k)K_{B.4}\|u_1-u_2\|^2_{H^s}.
\end{aligned}
$$

Finally, term T_4 is dealt with using a Taylor expansion on $h_\delta^{(|\pi|)}$, and Lemmas A.1–(i) and B.4–(i). Its bounds reads

$$T_4 \le C(\delta) \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{|u_2(\boldsymbol{x}) - u_2(\boldsymbol{y})|^2 \, |\mathcal{P}_\pi u_1(\boldsymbol{y}) - \mathcal{P}_\pi u_2(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$\le C(\delta, k) K_{B.4} \|u_1 - u_2\|^2_{H^s}.$$

Putting together the bounds obtained for (43) and (42) into (39), and using Lemma B.1 and Lemma 3.2–(ii), we deduce

$$\left\| B_{N,\delta}((u_1, \boldsymbol{v}_1)) - B_{N,\delta}((u_2, \boldsymbol{v}_2)) \right\|^2_{L_2^0(\mathcal{W}^s)}$$
$$\le K^2_{B.1} C(\delta, k, d, K_{H^s\to C^0}, K_{B.3}, K_{B.4}) \sigma^2 N^{-1} \epsilon^{-(2s+d)} \|(u_1, \boldsymbol{v}_1) - (u_2, \boldsymbol{v}_2)\|^2_{\mathcal{W}^s}. \quad (44)$$

Statement (iii). The proof is similar to that of Statement (ii). Take $(u, \boldsymbol{v}) \in \mathcal{W}^s$, such that $\|(u, \boldsymbol{v})\|_{\mathcal{W}^s} \le k$. We only need to bound

$$\left[\sum_{\alpha=0}^{\lfloor d/2\rfloor} \left\| h_\delta^{(|\pi_\alpha|)}(u) \mathcal{P}_{\pi_\alpha,\alpha} u \right\|^2_{L^2} \right]$$
$$+ \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\left| h_\delta^{(|\pi|)}(u(\boldsymbol{x})) \mathcal{P}_\pi u(\boldsymbol{x}) - h_\delta^{(|\pi|)}(u(\boldsymbol{y})) \mathcal{P}_\pi u(\boldsymbol{y}) \right|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} := A_3 + A_4 \quad (45)$$

for any choice $\pi_\alpha \in \Pi_\alpha$, $\alpha \in \{0, \ldots, \lfloor d/2\rfloor\}$, and $\pi \in \Pi_{\lfloor d/2\rfloor}$. The term A_3 is easily settled using the boundedness of derivatives of h_δ and Lemma B.3. Furthermore, A_4 is bounded (up to a constant) by

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\left| h_\delta^{(|\pi|)}(u(\boldsymbol{x})) - h_\delta^{(|\pi|)}(u(\boldsymbol{y})) \right|^2 |\mathcal{P}_\pi u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}$$
$$+ \int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \frac{\left| h_\delta^{(|\pi|)}(u(\boldsymbol{x})) \right|^2 |\mathcal{P}_\pi u(\boldsymbol{x}) - \mathcal{P}_\pi u(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(s-\lfloor s\rfloor)2}} \mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y} := T_5 + T_6. \quad (46)$$

Term T_5 is bounded using a Taylor expansion of $h_\delta^{(\pi)}$, and Lemma B.4–(i). Term T_6 is bounded relying on the boundedness of $h_\delta^{(|\pi|)}$ and using Lemmas A.1–(i) and B.4–(ii).

Putting the bounds obtained for (46) into (45) and using Lemmas B.1 and 3.2–(ii), we deduce

$$\left\| B_{N,\delta}((u, \boldsymbol{v})) \right\|^2_{L_2^0(\mathcal{W}^s)} = \frac{\sigma^2}{N} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \alpha_{\boldsymbol{j},s,\epsilon} \left\| \left(0, h_\delta(u) f_{\boldsymbol{j},s}\right) \right\|^2_{\mathcal{W}^s}$$
$$\le C(d) K^2_{B.1} \frac{\sigma^2}{N} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \alpha_{\boldsymbol{j},s,\epsilon} \|h_\delta(u)\|^2_{H^s}$$

$$\le K_{B.1}^2 C(\delta, d, K_{B.3}, K_{B.4}) \frac{\sigma^2}{N} \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \alpha_{\boldsymbol{j},s,\epsilon} \left(1 + \|(u, \boldsymbol{v})\|_{\mathcal{W}^s}^{2(\lfloor d/2\rfloor+1)}\right)$$

$$\le K_{B.1}^2 C(\delta, s, d, K_{B.3}, K_{B.4}) \frac{\sigma^2}{N} \epsilon^{-(2s+d)} \left(1 + \|(u, \boldsymbol{v})\|_{\mathcal{W}^s}^{2(\lfloor d/2\rfloor+1)}\right). \tag{47}$$

The assumption $\|(u, \boldsymbol{v})\|_{\mathcal{W}^s} \le k$ gives the desired local boundedness property. The proof is complete. $\square$

4. Proof of Theorem 1.1

This is an adaptation of [8, Theorem 4.4], and we heavily rely on the tools developed in Section 3. The functional α_U is locally Lipschitz and locally bounded in the $\mathcal{W}^s$-norm. This is a consequence of the following simple bound for $u \in H^s$ and $\ell \in \{1, \dots, d\}$

$$\|\partial_{x_\ell} U * u\|_{H^s}^2 = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} (\widehat{\partial_{x_\ell} U * u})_{\boldsymbol{j}} \overline{(\widehat{\partial_{x_\ell} U * u})_{\boldsymbol{j}}} (1 + |\boldsymbol{j}|^2)^s = \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \left|(\widehat{\partial_{x_\ell} U})_{\boldsymbol{j}}\right|^2 \left|\widehat{u}_{\boldsymbol{j}}\right|^2 (1 + |\boldsymbol{j}|^2)^s$$

$$\le C(\|U\|_{C^1}, d) \sum_{\boldsymbol{j}\in\mathbb{Z}^d} \left|\widehat{u}_{\boldsymbol{j}}\right|^2 (1 + |\boldsymbol{j}|^2)^s = C(\|U\|_{C^1}, d) \|u\|_{H^s}^2.$$

These properties of α_U, together with Lemmas 3.1 and 3.4, allow us to use [23, Theorem 4.5] and deduce the existence and uniqueness of a local $\mathcal{W}^s$-valued mild solution to (4) in the sense of [9, Chapter 7]. Specifically, there is a stopping time $\tau > 0$ and a unique $\mathcal{W}^s$-valued predictable process $X_{\epsilon,\delta} = (\tilde{\rho}_{\epsilon,\delta}, \tilde{\boldsymbol{j}}_{\epsilon,\delta})$ defined on $[0, \tau]$ such that $\mathbb{P}(\int_0^\tau \|X_{\epsilon,\delta}(z)\|_{\mathcal{W}^s}^2 \mathrm{d}z < \infty) = 1$, and satisfying, for each $t > 0$

$$X_{\epsilon,\delta}(t \wedge \tau) = S(t \wedge \tau)\, X_0 + \int_0^{t\wedge\tau} S(t \wedge \tau - s)\, \alpha_U(X_{\epsilon,\delta}(s))\, \mathrm{d}s$$

$$+ \int_0^{t\wedge\tau} S(t \wedge \tau - s)\, B_{N,\delta}(X_{\epsilon,\delta}(s))\, \mathrm{d}W_\epsilon, \qquad \mathbb{P}\text{-a.s.}, \tag{48}$$

where $\{S(t)\}_{t\ge 0}$ is the C_0-semigroup generated by A. Using [23, Theorem 4.5 and Remark 4.6], the continuous embedding $H^s \subset C^0$, and the assumption $\min_{\boldsymbol{x}\in\mathbb{T}^d} \tilde{\rho}_0(\boldsymbol{x}) > \delta$, we deduce that there exists $T = T(\tilde{\rho}_0)$ and a unique deterministic $\mathcal{W}^s$-valued mild solution $Z_\delta = (\rho_Z, \boldsymbol{j}_Z)$ to the noise-free equivalent of (4) up to T, such that $\min_{\boldsymbol{x}\in\mathbb{T}^d, s\in[0,T]} \rho_Z(\boldsymbol{x}, s) > \delta$. It is also obvious that there is $k > 0$ such that $\max_{s\in[0,T]} \|\rho_Z(\cdot, s)\|_{\mathcal{W}^s} < k$.

We compare $X_{\epsilon,\delta}$ and Z_δ. As $X_{\epsilon,\delta}$ is a local mild solution, it is well-defined up to the first exit time from the $\mathcal{W}^s$-ball of radius k. In particular, $X_{\epsilon,\delta}$ and Z_δ are well-defined up to the stopping time

$$\tau_{\delta,k} := \inf\left\{t > 0 : \|X_{\epsilon,\delta}(t)\|_{\mathcal{W}^s} \ge k\right\} \wedge \inf\left\{t > 0 : \min_{\boldsymbol{x}\in\mathbb{T}^d} \tilde{\rho}_{\epsilon,\delta}(\boldsymbol{x}, t) \le \delta\right\} \wedge T. \tag{49}$$

We consider the difference

$$X_{\epsilon,\delta}(t\wedge\tau_{\delta,k}) - Z(t\wedge\tau_{\delta,k}) = \int_0^{t\wedge\tau_{\delta,k}} S(t\wedge\tau_{\delta,k}-s)\Big[\alpha_U(X_{\epsilon,\delta}(s)) - \alpha_U(Z(s))\Big]\,\mathrm{d}s + \int_0^{t\wedge\tau_{\delta,k}} S(t\wedge\tau_{\delta,k}-s)\, B_{N,\delta}(X_{\epsilon,\delta}(s))\,\mathrm{d}W_\epsilon. \tag{50}$$

Let $q>2$. For some $C_1 = C_1\,(U,k,T,q,\eta,d,K_{B.1})$, we have

$$\begin{aligned}
&\mathbb{E}\left[\sup_{s\in[0,t]}\left\|X_{\epsilon,\delta}(s\wedge\tau_{\delta,k}) - Z(s\wedge\tau_{\delta,k})\right\|^q_{\mathcal{W}^s}\right]\\
&\le C_1\,\mathbb{E}\left[\int_0^t \left\|X_{\epsilon,\delta}(u) - Z(u)\right\|^q_{\mathcal{W}^s}\ \mathbf{1}_{[0,\tau_{\delta,k}]}(u)\,\mathrm{d}u\right]\\
&\quad + \mathbb{E}\left[\sup_{s\in[0,T]}\left\|\int_0^s S(s\wedge\tau_{\delta,k}-u)B_{N,\delta}(X_{\epsilon,\delta}(u))\,\mathbf{1}_{[0,\tau_{\delta,k}]}(u)\,\mathrm{d}W_\epsilon\right\|^q_{\mathcal{W}^s}\right].
\end{aligned}\tag{51}$$

We use [9, Proposition 7.3] and Lemma 3.4-(iii), inequality (47), to provide the bound

$$\begin{aligned}
(51) &\le C_1\int_0^t \mathbb{E}\left[\sup_{s\in[0,u]}\left\|X_{\epsilon,\delta}(s\wedge\tau_{\delta,k}) - Z(s\wedge\tau_{\delta,k})\right\|^q_{\mathcal{W}^s}\right]\mathrm{d}u\\
&\quad + C(\sigma,\delta,T,q,\eta,d,K_{B.1},K_{B.3},K_{B.4})\\
&\quad \times \left(N^{-1}\epsilon^{-(2s+d)}\right)^{q/2}\mathbb{E}\left[\int_0^T \left(1+\|X_{\epsilon,\delta}(u)\|^{(\lfloor d/2\rfloor+1)q}_{\mathcal{W}^s}\right)\mathbf{1}_{[0,\tau_{\delta,k}]}(u)\,\mathrm{d}u\right]\\
&\le C_1\int_0^t \mathbb{E}\left[\sup_{s\in[0,u]}\left\|X_{\epsilon,\delta}(s\wedge\tau_{\delta,k}) - Z(s\wedge\tau_{\delta,k})\right\|^q_{\mathcal{W}^s}\right]\mathrm{d}u\\
&\quad + C_2\left(N^{-1}\epsilon^{-(2s+d)}\right)^{q/2},
\end{aligned}\tag{52}$$

for some $C_2 = C_2(\sigma,\delta,T,q,k,\eta,d,K_{B.1},K_{B.3},K_{B.4})$. Crucially, the last inequality is not affected by the superlinear nature of the noise for $d>1$, as $X_{\epsilon,\delta}$ lives on a bounded set of $\mathcal{W}^s$ up to $\tau_{\delta,k}$. Applying the Gronwall Lemma to (51)–(52) gives

$$\mathbb{E}\left[\sup_{s\in[0,T]}\left\|X_{\epsilon,\delta}(s\wedge\tau_{\delta,k}) - Z(s\wedge\tau_{\delta,k})\right\|^q_{\mathcal{W}^s}\right] \le C_2\left(N^{-1}\epsilon^{-(2s+d)}\right)^{q/2} e^{TC_1}. \tag{53}$$

The choice of θ for (5) given in the assumption and a Chebyshev-type argument imply that

$$\lim_{N\to\infty} \mathbb{P}\left(\sup_{s\in[0,T]} \left\| X_{\epsilon,\delta}(s\wedge\tau_{\delta,k}) - Z(s\wedge\tau_{\delta,k})\right\|_{\mathcal{W}^s} \geq \beta\right) = 1$$

for any $\beta \in (0,1)$. It is now a standard routine (see [8, Theorem 4.4]) to pick β small enough, N big enough, and deduce the existence of a set F_ν such that $\mathbb{P}(F_\nu) > 1-\nu$, on which $\tau_{\delta,k} \equiv T$, on which $\tilde{\rho}_\epsilon \geq \delta$, and on which (4) is satisfied by $X_{\epsilon,\delta}$ the sense of mild solutions. Going back to (48), this implies

$$\begin{aligned} X_{\epsilon,\delta}(t) &= S(t)\,X_0 + \int_0^t S(t-s)\,\alpha_U(X_{\epsilon,\delta}(s))\,\mathrm{d}s + \int_0^t S(t-s)\,B_{N,\delta}(X_{\epsilon,\delta}(s))\,\mathrm{d}W_\epsilon \\ &= S(t)\,X_0 + \int_0^t S(t-s)\,\alpha_U(X_{\epsilon,\delta}(s))\,\mathrm{d}s + \int_0^t S(t-s)\,B_N(X_{\epsilon,\delta}(s))\,\mathrm{d}W_\epsilon \quad \text{on } F_\nu, \end{aligned}$$

and this concludes the proof.

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Appendix A. Factorisation of products

We recall the following simple factorisation for differences of products.

Lemma A.1. *Let* $\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{c}, \boldsymbol{d} \in \mathbb{R}^N$.

(i) We have

$$\prod_{i=1}^N a_i - \prod_{i=1}^N b_i = \sum_{k=1}^N b_{<k}(a_k - b_k)a_{>k}, \tag{54}$$

where we have used the shorthand notations $b_{<k} := \prod_{j=1}^{k-1} b_j$ *and* $a_{>k} := \prod_{j=k+1}^N a_j$ *(with the usual convention of the product over an empty set being unitary).*

(ii) For each $k = 1, \ldots, N$, *consider the families*

$$\{\alpha_j^k\}_{j=1}^{N-1} := (b_1, \ldots, b_{k-1}, a_{k+1}, \cdots, a_N), \qquad \{\beta_j^k\}_{j=1}^{N-1} := (d_1, \ldots, d_{k-1}, c_{k+1}, \cdots, c_N).$$

We have

$$\prod_{i=1}^N a_i - \prod_{i=1}^N b_i - \left(\prod_{i=1}^N c_i - \prod_{i=1}^N d_i\right) = \sum_{k=1}^N b_{<k}(a_k - b_k - (c_k - d_k))a_{>k} + \sum_{k=1}^N \sum_{j=1}^{N-1} (c_k - d_k)(\beta^k_{<j}(\alpha^k_j - \beta^k_j)\alpha^k_{>j}). \quad (55)$$

Proof. Point (i) is easily proven by induction. As for Point (ii), we use Point (i) twice and obtain

$$\begin{aligned}
&\prod_{i=1}^N a_i - \prod_{i=1}^N b_i - \left(\prod_{i=1}^N c_i - \prod_{i=1}^N d_i\right)\\
&= \sum_{k=1}^N \{b_{<k}(a_k - b_k)a_{>k} - d_{<k}(c_k - d_k)c_{>k}\}\\
&= \sum_{k=1}^N \{b_{<k}(a_k - b_k - (c_k - d_k))a_{>k} + (c_k - d_k)(b_{<k}a_{>k} - d_{<k}c_{>k})\}\\
&= \sum_{k=1}^N b_{<k}(a_k - b_k - (c_k - d_k))a_{>k} + \sum_{k=1}^N \sum_{j=1}^{N-1} (c_k - d_k)(\beta^k_{<j}(\alpha^k_j - \beta^k_j)\alpha^k_{>j}),
\end{aligned}$$

and the proof is complete. □

Appendix B. Technical lemmas on fractional Sobolev spaces

We recall a useful lemma about the multiplication of functions in fractional Sobolev spaces, which is a direct consequence of the Sobolev embedding [3, Section 2.1] and of [4, Lemma 5, inequality (25)].

Lemma B.1. *Let $u, v \in H^s$, where $s = d/2 + \eta$, for some $\eta > 0$. Then $uv \in H^s$ and there exists $K_{B.1} = K_{B.1}(d, \eta)$ such that*

$$\|uv\|_{H^s} \le K_{B.1}\|u\|_{H^s}\|v\|_{H^s}.$$

The following lemma is an adaptation of the classical multivariate Faà Di Bruno's formula [6] in the context of weak rather than classical derivatives. We derive it under some restrictive assumptions, which are however satisfied by the nonlinearity h_δ in our regularised Dean–Kawasaki noise (4).

Lemma B.2. *Let $\alpha \in \{1, \dots, \lfloor d/2 \rfloor\}$ and $u \in H^{d/2+\eta}$ for sufficiently small $\eta > 0$. Pick $h_\delta \in C^{\lfloor d/2 \rfloor+1}(\mathbb{R})$ with all derivatives up to order $\lfloor d/2 \rfloor$ being bounded, and let $(x_{\ell_1}, \dots, x_{\ell_\alpha})$ be an arbitrary element of $\{x_1, \dots, x_d\}^\alpha$. Then*

$$\frac{\partial^{(\alpha)}}{\partial x_{\ell_1}\cdots\partial x_{\ell_\alpha}}h_\delta(u(\boldsymbol{x}))=\sum_{\pi\in\Pi_\alpha}h_\delta^{(|\pi|)}(u(\boldsymbol{x}))\prod_{j\in J(\pi)}\prod_{b\in B(\pi)\colon |b|=j}\frac{\partial^{(|b|)}u(\boldsymbol{x})}{\prod_{z\in b}\partial x_{\ell_z}},\tag{56}$$

where we recall the notations $J(\pi):=\{j\in\{1,\ldots,\alpha\}\colon \beta_j(\pi)>0\}$ *and* $\beta_j(\pi):=\#\{b\in B(\pi)\colon |b|=j\}$. *In particular,* $\sum_{b\in B(\pi)}|b|=\alpha$ *for every* $\pi\in\Pi_\alpha$.

Proof. We only need to show that (56) holds in the sense of weak derivatives. Fix a test function $\varphi\in C^\infty(\mathbb{T}^d)$. Consider a standard sequence of mollifiers $\varrho_n\colon \mathbb{T}^d\to\infty$, and set $u_n:=\varrho_n * u$. As $u_n\in C^\infty(\mathbb{T}^d)$, we can apply the classical multivariate Faà Di Bruno's formula [6] to $h_\delta(u_n)$ and perform integration by parts to obtain

$$\int_{\mathbb{T}^d}\sum_{\pi\in\Pi_\alpha}h_\delta^{(|\pi|)}(u_n(\boldsymbol{x}))\prod_{j\in J(\pi)}\prod_{b\in B(\pi)\colon |b|=j}\frac{\partial^{(|b|)}u_n(\boldsymbol{x})}{\prod_{z\in b}\partial x_{\ell_z}}\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}$$
$$=(-1)^\alpha\int_{\mathbb{T}^d}h_\delta(u_n(\boldsymbol{x}))\frac{\partial^{(\alpha)}}{\partial x_{\ell_1}\cdots\partial x_{\ell_\alpha}}\varphi(\boldsymbol{x})\mathrm{d}\boldsymbol{x}.\tag{57}$$

All we need to do is pass to the limit in (57) to replace u_n with u. Since u is continuous on $\mathbb{T}^d$, we have $u_n\to u$ uniformly as $n\to\infty$. Using the boundedness of h_δ', it is immediate to pass in the limit in the right-hand side of (57). Now fix $\pi\in\Pi_\alpha$. The embedding $H^{d/2+\eta-j}\subset L^{d/(j-\eta)}$ (see [3, Corollary 1.2]) implies that, for all blocks $b\in B(\pi)$ with length j,

$$\frac{\partial^{(|b|)}u}{\prod_{z\in b}\partial x_{\ell_z}}\in L^{d/(j-\eta)},$$

and, as a result,

$$\frac{\partial^{(|b|)}u_n}{\prod_{z\in b}\partial x_{\ell_z}}=\varrho_n*\frac{\partial^{(|b|)}u}{\prod_{z\in b}\partial x_{\ell_z}}\longrightarrow\frac{\partial^{(|b|)}u}{\prod_{z\in b}\partial x_{\ell_z}}\text{ in }L^{d/(j-\eta)}\text{ as }n\to\infty.$$

We can then settle the convergence of the left-hand side of (57) using the boundedness of derivatives of h_δ and a multi-factor Hölder inequality (for any fixed $\pi\in\Pi_\alpha$) on the $|\pi|+2$ terms making up the product. Specifically, the exponents we use are $q_{h_\delta}=q_\varphi=\infty$ (for the first and last term), $q_j:=d/(j-\eta)$ for the each of the β_j terms associated with the product over the set $\{b\in B(\pi)\colon |b|=j\}$, and $q:=d/(d-\alpha+\eta\sum_{j\in J}\beta_j)$ for the remaining (identically unitary) term. $\square$

The following two lemmas are concerned with integrability properties closed related to the product term appearing in the right-hand side of (56).

Lemma B.3. *Fix* $1\le\alpha\le\lfloor d/2\rfloor$. *Fix some partition* $\pi\in\Pi_\alpha$, *and abbreviate* $\beta_j(\pi)=\beta_j$ *and* $J(\pi)=J$. *Let* $\{u_b\}_{b\in B(\pi)}\in[H^{d/2+\eta}]^{|\pi|}$ *for some* $0<\eta<1/2$. *Let* $(x_{\ell_1},\ldots,x_{\ell_\alpha})$ *be an arbitrary element of* $\{x_1,\ldots,x_d\}^\alpha$. *Then*

$$\int_{\mathbb{T}^d} \left\{ \prod_{j\in J} \prod_{b\in B(\pi)\colon |b|=j} \left| \frac{\partial^{(j)} u_b(\boldsymbol{x})}{\prod_{z\in b} \partial x_{\ell_z}} \right|^2 \right\} d\boldsymbol{x} \leq K_{B.3} \prod_{j\in J} \prod_{b\in B(\pi)\colon |b|=j} \|u_b\|^2_{H^{d/2+\eta}} \tag{58}$$

holds for some $K_{B.3} = K_{B.3}(\pi, d, \eta) > 0$.

Proof. We use a multi-factor Hölder inequality on the $|\pi|$ terms making up the product in the left-hand side of (58). The exponents we use are $q_j := d/(2(j-\eta))$ for the each of the β_j terms associated with the product over the set $\{b \in B(\pi)\colon |b| = j\}$, and $q := d/(d - 2\alpha + 2\eta \sum_{j\in J} \beta_j)$ for the remaining (identically unitary) term. We obtain

$$\begin{aligned}(58) &\leq C(\alpha, \pi, d, \eta) \prod_{j\in J} \prod_{b\in B(\pi)\colon |b|=j} \left\{ \int_{\mathbb{T}^d} \left| \partial^{(j)} u_b(\boldsymbol{x}) / \prod_{z\in b} \partial x_{\ell_z} \right|^{2q_j} \mathrm{d}\boldsymbol{x} \right\}^{1/q_j} \\ &\leq K_{B.3} \prod_{j\in J} \prod_{b\in B(\pi)\colon |b|=j} \|u_b\|^2_{H^{d/2+\eta}},\end{aligned}$$

where we have used the Sobolev embeddings $H^{d/2+\eta-j} \subset L^{2q_j}$ (see [3, Corollary 1.2]) in the final inequality. □

Lemma B.4. *Fix some* $\pi \in \Pi_{\lfloor d/2 \rfloor}$, *and abbreviate* $\beta_j(\pi) = \beta_j$ *and* $J(\pi) = J$. *Let* $\{u_b\}_{b\in B(\pi)} \in [H^{d/2+\eta}]^{|\pi|}$, $v \in H^{d/2+\eta}$ *where* $0 < \eta < C(d) < 1/2$ *for some small enough* $C(d)$. *Let* $(x_{\ell_1}, \ldots, x_{\ell_{\lfloor d/2 \rfloor}})$ *be an arbitrary element of* $\{x_1, \ldots, x_d\}^{\lfloor d/2 \rfloor}$, *and let* $\boldsymbol{z}$ *be an arbitrary element of* $\{\boldsymbol{x}, \boldsymbol{y}\}^{|\pi|}$.

(i) The inequality

$$\begin{aligned}&\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \left\{ \prod_{j\in J} \prod_{b\in B(\pi)\colon |b|=j} \left| \frac{\partial^{(j)} u_b(\boldsymbol{z}_b)}{\prod_{z\in b} \partial x_{\ell_z}} \right|^2 \right\} \frac{|v(\boldsymbol{x}) - v(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(d/2-\lfloor d/2\rfloor+\eta)2}} d\boldsymbol{x} d\boldsymbol{y} \\ &\quad \leq K_{B.4} \left(\prod_{j\in J} \prod_{b\in B(\pi)\colon |b|=j} \|u_b\|^2_{H^{d/2+\eta}} \right) \|v\|^2_{H^{d/2+\eta}}\end{aligned} \tag{59}$$

holds for some positive $K_{B.4} = K_{B.4}(d, \eta)$.

(ii) Pick $\tilde{j} \in J$ *and* $\tilde{b} \in \{b \in B(\pi) : |b| = \tilde{j}\}$. *Then we have the inequality*

$$\begin{aligned}&\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \left\{ \prod_{j\in J\setminus \tilde{j}} \prod_{b\in B(\pi)\colon |b|=j} \left| \frac{\partial^{(j)} u_b(\boldsymbol{z}_b)}{\prod_{z\in b} \partial x_{\ell_z}} \right|^2 \right\} \left\{ \prod_{b\in B(\pi)\colon |b|=\tilde{j}, b\neq \tilde{b}} \left| \frac{\partial^{(\tilde{j})} u_b(\boldsymbol{z}_b)}{\prod_{z\in b} \partial x_{\ell_z}} \right|^2 \right\} \\ &\times \frac{\left|\partial^{(\tilde{j})} u_{\tilde{b}}(\boldsymbol{x}) / \prod_{z\in\tilde{b}} \partial x_{\ell_z} - \partial^{(\tilde{j})} u_{\tilde{b}}(\boldsymbol{y}) / \prod_{z\in\tilde{b}} \partial x_{\ell_z}\right|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(d/2-\lfloor d/2\rfloor+\eta)2}} d\boldsymbol{x} d\boldsymbol{y}\end{aligned}$$

$$\leq K_{B.4}\left(\prod_{j\in J}\prod_{b\in B(\pi)\colon |b|=j}\|u_b\|^2_{H^{d/2+\eta}}\right). \tag{60}$$

Proof. It is useful to remember $\sum_{j\in J} j\beta_j = \lfloor d/2\rfloor$.

Point (i). We rewrite (59) as

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\left\{\prod_{j\in J}\prod_{b\in B(\pi)\colon |b|=j}\frac{\left|\partial^{(j)}u_b(z_b)/\prod_{z\in b}\partial x_{\ell_z}\right|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma\alpha_j\beta_j^{-1}}}\right\}\frac{|v(\boldsymbol{x})-v(\boldsymbol{y})|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+2(d/2-\lfloor d/2\rfloor+\eta)-\gamma}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}, \tag{61}$$

for some appropriate $\gamma>0$ and $\{\alpha_j\}_{j\in J}$ such that

$$\alpha_j\in[0,1]\quad\forall j\in J,\qquad \sum_{j\in J}\alpha_j=1 \tag{62}$$

to be chosen later. We use a multi-factor Hölder inequality on the $|\pi|+1$ terms making up (61). The exponents we use are $q_j := d/(2(j-\eta))$ for the each of the β_j terms associated with the product over the set $\{b\in B(\pi)\colon |b|=j\}$, and $q := d/(d-2\lfloor d/2\rfloor+2\eta\sum_{j\in J}\beta_j)$ for the remaining term. We obtain

$$\begin{aligned}
(61) \leq & \prod_{j\in J}\prod_{b\in B(\pi)\colon |b|=j}\left\{\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{\left|\partial^{(j)}u_b(z_b)/\prod_{z\in b}\partial x_{\ell_z}\right|^{2q_j}}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma\alpha_j\beta_j^{-1}q_j}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right\}^{1/q_j}\\
&\times\left\{\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|v(\boldsymbol{x})-v(\boldsymbol{y})|^{2q}}{|\boldsymbol{x}-\boldsymbol{y}|^{\{d+2(d/2-\lfloor d/2\rfloor+\eta)-\gamma\}q}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right\}^{1/q}\\
= & \prod_{j\in J}\prod_{b\in B(\pi)\colon |b|=j}\left\{\int_{\mathbb{T}^d}\frac{1}{|\boldsymbol{y}|^{\gamma\alpha_j\beta_j^{-1}q_j}}\mathrm{d}\boldsymbol{y}\right\}^{1/q_j}\left\{\int_{\mathbb{T}^d}\left|\partial^{(j)}u_b(\boldsymbol{x})/\prod_{z\in b}\partial x_{\ell_z}\right|^{2q_j}\mathrm{d}\boldsymbol{x}\right\}^{1/q_j}\\
&\times\left\{\int_{\mathbb{T}^d}\int_{\mathbb{T}^d}\frac{|v(\boldsymbol{x})-v(\boldsymbol{y})|^{2q}}{|\boldsymbol{x}-\boldsymbol{y}|^{\{d+2(d/2-\lfloor d/2\rfloor+\eta)-\gamma\}q}}\mathrm{d}\boldsymbol{x}\mathrm{d}\boldsymbol{y}\right\}^{1/q}\\
:= & \left[\prod_{j\in J}\prod_{b\in B(\pi)\colon |b|=j}C_j^{1/q_j}D_{j,b}^{1/q_j}\right]\times E^{1/q}.
\end{aligned}$$

We now impose conditions on η and γ so that C_j, D_j, and E are suitably bounded. The integrals C_j may be dealt with using a standard change of variables in spherical coordinates, and they are bounded if and only if $-\gamma\alpha_j\beta_j^{-1}q_j+(d-1)>-1$, or equivalently if

$$\alpha_j < \frac{2\beta_j(j-\eta)}{\gamma}, \qquad \forall j \in J. \tag{63}$$

The terms $D_{j,b}^{1/q_j}$ are bounded, as in the case of Lemma B.3, by using the Sobolev embedding $H^{d/2+\eta-j} \subset L^{2q_j}$ [3, Corollary 1.2]. We now turn to E. We rewrite the exponent of $|\boldsymbol{x}-\boldsymbol{y}|$ according to the notation of the space $W^{r,2q}$, for some r to be determined. More precisely, the rewriting

$$\{d + 2(d/2 - \lfloor d/2 \rfloor + \eta) - \gamma\}q = d + r(2q)$$

is solved in r, giving $r = (d-\gamma)/2 + \eta(1 - \sum_{j\in J}\beta_j)$. The restriction $r \in (0,1)$ gives the condition

$$d - 2 + 2\eta(1 - \sum_{j\in J}\beta_j) < \gamma < d + 2\eta(1 - \sum_{j\in J}\beta_j). \tag{64}$$

The term E may be bounded using the Sobolev embedding $W^{d/2+\eta,2} \subset W^{r,2q}$, and this embedding is true under the condition [2, Theorem 5.1]

$$d/2 + \eta - d/2 > r - d/(2q),$$

which is equivalent to

$$\gamma > 2\lfloor d/2 \rfloor - 4\eta \sum_{j\in J}\beta_j. \tag{65}$$

If we pick $\gamma := 2\lfloor d/2 \rfloor - 3\eta \sum_{j\in J}\beta_j$ and η small enough, then (65) and (64) are satisfied. Furthermore, summing the right-hand side of (63) over j, we obtain

$$\sum_{j\in J} \frac{2\beta_j(j-\eta)}{2\lfloor d/2 \rfloor - 3\eta\sum_{j\in J}\beta_j} = \frac{2\lfloor d/2 \rfloor - 2\eta\sum_{j\in J}\beta_j}{2\lfloor d/2 \rfloor - 3\eta\sum_{j\in J}\beta_j} > 1. \tag{66}$$

The above inequality implies that the α_j's can be chosen so that (62) and (63) are satisfied. As a result of the bounds for $C_j, D_{j,b}, E$, the inequality (59) follows and *Point (i)* is settled.

Point (ii). The case $\sum_{j\in J}\beta_j = 1$ uniquely corresponds to having $\tilde{j} = \lfloor d/2 \rfloor$ and $\beta_{\tilde{j}} = 1$. Therefore, the only term surviving in the product of integrands in the left-hand side of (60) is the last term, and the result is trivial.

We consider all the other cases, where necessarily $\sum_{j\in J}\beta_j > 1$. We rewrite (60) as

$$\int_{\mathbb{T}^d}\int_{\mathbb{T}^d} \left\{ \prod_{j\in J\setminus\tilde{j}} \prod_{b\in B(\pi):\,|b|=j} \frac{\left|\partial^{(j)} u_b(\boldsymbol{z}_b)/\prod_{z\in b}\partial x_{\ell_z}\right|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma\alpha_j\beta_j^{-1}}} \right\}$$
$$\times \left\{ \prod_{b\in B(\pi):\,|b|=\tilde{j},\, b\neq\tilde{b}} \frac{\left|\partial^{(\tilde{j})} u_b(\boldsymbol{z}_b)/\prod_{z\in b}\partial x_{\ell_z}\right|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{\gamma\alpha_{\tilde{j}}(\beta_{\tilde{j}}-1)^{-1}}} \right\}$$

$$\times \frac{\left|\partial^{(\tilde{j})} u_{\tilde{b}}(\boldsymbol{x}) / \prod_{z \in \tilde{b}} \partial x_{\ell_z} - \partial^{(\tilde{j})} u_{\tilde{b}}(\boldsymbol{y}) / \prod_{z \in \tilde{b}} \partial x_{\ell_z}\right|^2}{|\boldsymbol{x}-\boldsymbol{y}|^{d+(d/2-\lfloor d/2\rfloor+\eta)2-\gamma}} \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y}, \tag{67}$$

where the second curly brackets is understood to be equal to 1 should $\beta_{\tilde{j}} = 1$, for some appropriate $\gamma > 0$ and $\{\alpha_j\}_{j \in J^\star}$ such that

$$\alpha_j \in [0,1] \quad \forall j \in J^\star, \qquad \sum_{j \in J^\star} \alpha_j = 1 \tag{68}$$

to be chosen later, where $J^\star = J$ if $\beta_{\tilde{j}} > 1$, and $J^\star = J \setminus \tilde{j}$ otherwise. We use a multi-factor Hölder inequality on the $|\pi|$ terms making up (67). The exponents we use are $q_j := d/(2(j-\eta))$ for the each of the β_j terms associated with the product over the set $\{b \in B(\pi) \colon |b| = j\}$ for $j \in J \setminus \tilde{j}$, then $q_{\tilde{j}} := d/(2(\tilde{j}-\eta))$ for the each of the $\beta_{\tilde{j}} - 1$ terms associated with the product over the set $\{b \in B(\pi) \colon |b| = \tilde{j}, b \neq \tilde{b}\}$, and finally $q := d/(d - 2\lfloor d/2 \rfloor + 2\eta \sum_{j \in J} \beta_j + 2(\tilde{j} - \eta))$ for the remaining term. We obtain

$$\begin{aligned}
(67) \leq & \prod_{j \in J \setminus \tilde{j}} \prod_{b \in B(\pi)\colon |b|=j} \left\{ \int_{\mathbb{T}^d} \frac{1}{|\boldsymbol{y}|^{\gamma \alpha_j \beta_j^{-1} q_j}} \mathrm{d}\boldsymbol{y} \right\}^{1/q_j} \left\{ \int_{\mathbb{T}^d} \left| \partial^{(j)} u_b(\boldsymbol{x}) / \prod_{z \in b} \partial x_{\ell_z} \right|^{2q_j} \mathrm{d}\boldsymbol{x} \right\}^{1/q_j} \\
& \times \prod_{b \in B(\pi)\colon |b|=\tilde{j}, b \neq \tilde{b}} \left\{ \int_{\mathbb{T}^d} \frac{1}{|\boldsymbol{y}|^{\gamma \alpha_{\tilde{j}} (\beta_{\tilde{j}}-1)^{-1} q_{\tilde{j}}}} \mathrm{d}\boldsymbol{y} \right\}^{1/q_{\tilde{j}}} \left\{ \int_{\mathbb{T}^d} \left| \partial^{(j)} u_b(\boldsymbol{x}) / \prod_{z \in b} \partial x_{\ell_z} \right|^{2q_{\tilde{j}}} \mathrm{d}\boldsymbol{x} \right\}^{1/q_{\tilde{j}}} \\
& \times \left\{ \int_{\mathbb{T}^d} \int_{\mathbb{T}^d} \frac{\left|\partial^{(\tilde{j})} u_{\tilde{b}}(\boldsymbol{x}) / \prod_{z \in \tilde{b}} \partial x_{\ell_z} - \partial^{(\tilde{j})} u_{\tilde{b}}(\boldsymbol{y}) / \prod_{z \in \tilde{b}} \partial x_{\ell_z}\right|^{2q}}{|\boldsymbol{x}-\boldsymbol{y}|^{\{d+2(d/2-\lfloor d/2\rfloor+\eta)-\gamma\}q}} \mathrm{d}\boldsymbol{x}\,\mathrm{d}\boldsymbol{y} \right\}^{1/q} \\
:= & \left[\prod_{j \in J \setminus \tilde{j}} \prod_{b \in B(\pi)\colon |b|=j} C_j^{1/q_j} D_{j,b}^{1/q_j} \right] \times \left[\prod_{b \in B(\pi)\colon |b|=j, b \neq \tilde{b}} C_{\tilde{j}}^{1/q_{\tilde{j}}} D_{\tilde{j},b}^{1/q_{\tilde{j}}} \right] \times E^{1/q}.
\end{aligned}$$

Bounding the above involves similar discussions as per *Point (i)*. More specifically, the boundedness of the spherical integrals (the C_j's above) is granted under the conditions

$$\alpha_j < \frac{2\beta_j (j-\eta)}{\gamma}, \qquad \forall j \in J^\star \setminus \tilde{j}, \qquad \alpha_{\tilde{j}} < \frac{2(\beta_{\tilde{j}}-1)(\tilde{j}-\eta)}{\gamma}, \tag{69}$$

with the last condition only imposed if $\tilde{j} \in J^\star$. The bound for the terms $D_{j,b}^{1/q_j}$ is settled exactly as in *Point (i)*. As for E, we solve the equation

$$\{d + 2(d/2 - \lfloor d/2 \rfloor + \eta) - \gamma\} q = d + r(2q)$$

in the variable r, thus getting $r := (d-\gamma)/2 + \eta(1 - \sum_{j \in J} \beta_j) - (\tilde{j} - \eta)$. The constraint $r \in (0,1)$ results in the requirement

$$d-2+2\eta(1-\sum_{j\in J}\beta_j)-2(\tilde{j}-\eta)<\gamma<d+2\eta(1-\sum_{j\in J}\beta_j)-2(\tilde{j}-\eta). \tag{70}$$

We control E using the embedding $H^{d/2+\eta}\subset W^{\tilde{j}+r,2q}$, which is valid under the constraint

$$d/2+\eta-d/2>\tilde{j}+r-d/(2q),$$

which is equivalent to

$$\gamma>2\lfloor d/2\rfloor+4\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}. \tag{71}$$

If we take $\gamma:=2\lfloor d/2\rfloor+3\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}$ and η small enough, then (70) and (71) are satisfied. Furthermore, summing all the right-hand sides in (69) gives

$$\begin{aligned}
&\mathbf{1}_{\beta_{\tilde{j}}>1}\frac{2(\beta_{\tilde{j}}-1)(\tilde{j}-\eta)}{2\lfloor d/2\rfloor+3\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}}+\sum_{j\in J\setminus\tilde{j}}\frac{2\beta_j(j-\eta)}{2\lfloor d/2\rfloor+3\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}}\\
&=\sum_{j\in J}\frac{2\beta_j(j-\eta)}{2\lfloor d/2\rfloor+3\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}}-\frac{2(\tilde{j}-\eta)}{2\lfloor d/2\rfloor+3\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}}\\
&=\frac{2\lfloor d/2\rfloor+2\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}}{2\lfloor d/2\rfloor+3\eta(1-\sum_{j\in J}\beta_j)-2\tilde{j}}>1,
\end{aligned}$$

where the last inequality is valid because $\sum_{j\in J}\beta_j>1$. Therefore the α_j's can be chosen so that (68) and (69) are satisfied. As a result of the bounds for $C_j, D_{j,b}, E$, the inequality (60) follows and *Point (ii)* is settled. □

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