Numerical Algorithms 14 (1997) 189–209

Inertial manifolds and linear multi-step methods

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We determine the existence and C^1 convergence of an inertial manifold for a strongly $A(\alpha)$ stable, pth order, $p \geqslant 1$, linear multi-step method approximating a sectorial evolution equation that satisfies a gap condition. This inertial manifold gives rise to a one-step method that C^1 approximates the inertial form of the evolution equation and yields further approximation properties of the multi-step method.

Keywords: inertial manifolds, linear multi-step methods.

AMS subject classification: 65M06.

1. Introduction and statement of results

Consider a sectorial evolution equation on a Hilbert space X that satisfies a gap condition and approximate this equation in time with a strongly $A(\alpha)$ stable linear multi-step method, which is at least first order accurate. The q-step method generates a dynamical system on X^q, the q times product of X. This approach was developed by Hill and Suli [4], to apply Hale et al.'s results [3] on attractors to multi-step methods. This paper uses Hill and Suli's theory, to apply inertial manifold theory to multi-step methods. We present two results: the first gives existence and C^1 convergence of an inertial manifold of the multi-step method; the second describes a one-step method that C^1 approximates the inertial form of the evolution equation. This second result stems from theory developed for ODEs in [10, 12] (see also [1]), and has an important consequence: one step theory can be applied to give existence and convergence of various invariant sets of the multi-step method.

Hypothesis 1.1. Let A be a closed, densely defined, sectorial operator on X. Thus, all eigenvalues of A are contained in $\{\lambda \neq \lambda_*\colon \arg(\lambda - \lambda_*) < \theta\}$, where $\lambda_* > 0$ and $0 < \theta < \pi/2$. And, for some $M > 1$,

$$\left\|(\lambda I - A)^{-1}\right\| \leqslant \frac{M}{\lambda_* + |\lambda - \lambda_*|}, \quad \lambda \neq \lambda_*,\ \theta \leqslant \arg(\lambda - \lambda_*) \leqslant \pi.$$

Under this condition, the fractional powers, A^γ for $\gamma \in \mathbb{R}$, are well defined, and A generates an analytic semigroup, $\exp(-At)$ where $t \geqslant 0$ (see [11]). Furthermore, the domain of A^γ for $\gamma \geqslant 0$ is a Banach space with norm $\|\cdot\|_\gamma := \|A^\gamma \cdot\|$ – denote this space by X_γ. We state properties of these objects as we need them, mainly in lemma A.1.

Consider the equation

$$u_t + Au = f(u), \quad u \in X, \tag{$*$}$$

where the nonlinear term $f \in C^1(X_\gamma, X)$, some $\gamma < 1$. In terms of derivatives, this restriction on γ means that f carries only terms of lower order than A.

The first theorem deals with inertial manifolds [2]. An inertial manifold is a finite dimensional, positively invariant C^1 manifold that attracts all trajectories exponentially fast (here we tighten the definition of [2] and consider C^1 manifolds rather than Lipschitz manifolds). All inertial manifold theories depend on the spectrum of A having large gaps between eigenvalues. With A self-adjoint and subject to a gap condition, we find an inertial manifold for every strongly $A(\alpha)$, $0 \leqslant \alpha \leqslant \pi/2$, stable pth order, $p \geqslant 1$, multi-step method and prove convergence to the true inertial manifold in the C^1 sense. The definitions of $A(\alpha)$ stable and of pth order are reviewed in section 3.

This work depends on both the PDE and the multi-step method being dissipative: that is, each trajectory must enter a certain absorbing ball after a finite time and stay there. In some cases, absorbing balls for numerical methods arise from examining the absorbing balls of the PDE. In general, this is not known; theorems of this type depend on the structure of f [5]. One way of assuring an absorbing ball for the method would be to apply the method to a modified equation, where the nonlinear term has support in a bounded set; then, to be dissipative, the method need only mimic the linear part of the PDE. In this paper, the approach is to make dissipativity of the method an assumption, as follows.

In this assumption and in what follows, we need to relate the dynamical systems on X^q and on X. This issue is detailed in section 3. Consider $\boldsymbol{L}_0^T : X_\gamma^q \to X_\gamma$, the tensor contraction with the dominant left eigenvector of the companion matrix of the multi-step method at $\Delta t = 0$; and, $\boldsymbol{R}_0 : X_\gamma \to X_\gamma^q$, the tensor product with the dominant right eigenvector of the companion matrix of the multi-step method at $\Delta t = 0$. The dominant eigenvectors appear because, under the stability assumption, it is the dominant eigenvalues that approximate the eigenvalues of A. All other eigenvectors decay under the stability assumption; in fact on the inertial manifold, these 'bad' directions, the vectors in the kernel of $\boldsymbol{L}_0^T$, are written in terms of the dominant directions.

Hypothesis 1.2. Suppose that a bounded set $\mathcal{B} \subset X_\gamma^q$ exists such that

1. $\mathcal{B}$ is a positively invariant, absorbing ball of the multi-step method;
2. $\boldsymbol{L}_0^T \mathcal{B}$ is a positively invariant, absorbing ball of $(*)$ (equivalently, $\mathcal{B}$ is a positively invariant, absorbing ball of the monoid defined in section 3).

With this assumption in force, we have some freedom to modify f outside the absorbing ball without affecting the long term behaviour. Denote the subordinate norm of an operator from X_α to X_β by $\|\cdot\|_{\alpha,\beta}$. Then, under the dissipativity condition, we can assume that f satisfies the following properties: for some $K > 0$,

$$\big\|f(u)\big\| \leqslant K, \qquad \big\|f(u) - f(\tilde{u})\big\| \leqslant K\|u - \tilde{u}\|_\gamma,$$

$$\big\|\mathrm{d}f(u)\big\| \leqslant K, \qquad \big\|\mathrm{d}f(u) - \mathrm{d}f(\tilde{u})\big\|_{\gamma,0} \leqslant K\|u - \tilde{u}\|_\gamma.$$

Theorem 1.3. Let hypotheses 1.1–1.2 hold and, in addition, let A be self adjoint and unbounded with eigenvalues, $\lambda_1 \leqslant \lambda_2 \leqslant \cdots$, satisfying the gap condition:

$$\limsup_{\ell\to\infty} \lambda_{\ell+1}^{-\gamma}(\lambda_{\ell+1} - \lambda_\ell) \to \infty.$$

There exists $\ell \in \mathbb{N}$ large such that the following holds.

1. Let P denote the subspace of X_γ spanned by the first ℓ eigenfunctions of A, and let Q be such that $X_\gamma = P \oplus Q$. There exists a $\phi \in C^1(P, Q)$ such that $\operatorname{Graph}(\phi) \cap \boldsymbol{L}_0^T \mathcal{B}$ is an inertial manifold for $(*)$.
2. $X_\gamma^q = \boldsymbol{R}_0 P \oplus \boldsymbol{R}_0 Q \oplus \operatorname{Ker} \boldsymbol{L}_0^T$ and $\operatorname{Ker} \boldsymbol{L}_0^T$ is homeomorphic to X_γ^{q-1}. There exists a $\Phi_{\Delta t} \in C^1(\boldsymbol{R}_0 P, \boldsymbol{R}_0 Q \oplus \operatorname{Ker} \boldsymbol{L}_0^T)$ such that $\operatorname{Graph}(\Phi_{\Delta t}) \cap \mathcal{B}$ is an inertial manifold for a strongly $A(\alpha)$ stable, $0 \leqslant \alpha \leqslant \pi/2$, pth order accurate, $p \geqslant 1$, linear multi-step method applied to $(*)$ for all Δt sufficiently small (with the multi-step method considered as a dynamical system on X_γ^q).

Further, the inertial manifolds C^1 converge: when $0 < \delta < 1 - \gamma$, a constant C exists with

$$\sup\big\{\big\|\phi(p) - \boldsymbol{L}_0^T \Phi_{\Delta t}(\boldsymbol{R}_0 p)\big\|_\gamma \colon\ p \in P\big\} \leqslant C\Delta t^\delta \quad \text{as } \Delta t \to 0,$$

$$\sup\big\{\big\|\mathrm{d}\phi(p) - \boldsymbol{L}_0^T \mathrm{d}\Phi_{\Delta t}(\boldsymbol{R}_0 p)\big\|_{\gamma,\gamma} \colon\ p \in P\big\} \to 0 \quad \text{as } \Delta t \to 0.$$

The proof of this theorem is given in section 4.

Many theorems of this nature, the effect of numerical approximation on inertial manifolds, appear in Jones and Stuart [8], including C^0 approximation of the inertial manifold by one-step methods in time and finite element and spectral Galerkin methods in space. In the light of Jones et al. [7], this C^0 convergence could be strengthened to C^1 convergence.

The next theorem describes the existence of a one-step method on P, the space spanned by the first ℓ eigenfunctions of A, that C^1 approximates the inertial form. Denote by $\mathbb{P}$ the projection from X_γ to P. Then, the inertial form is the ODE

$$p_t + Ap = \mathbb{P}f\big(p + \phi(p)\big),$$

(here ϕ defines the inertial manifold of $(*)$); the inertial form describes all the long term dynamics of $(*)$. Let $S(\cdot)$ denote the solution operator of $(*)$ on X_γ, so that

$S(t)u$ solves $(*)$ for initial condition u. Then, the solution operator of the inertial form is

$$S^{\text{inertial}}(t)p := \mathbb{P}S(t)\big(p+\phi(p)\big), \quad p \in P.$$

Let $\boldsymbol{S}_{\Delta t}(\cdot)$ denote the semigroup on X_γ^q arising from applying a strongly $A(\alpha)$, $\theta \leqslant \alpha \leqslant \pi/2$, stable pth order, $p \geqslant 1$, multi-step method to $(*)$ – then the components $\boldsymbol{S}_{\Delta t}(k)\boldsymbol{u}$ are the kth, $\dots, (k+q-1)$th steps of the multi-step method. This idea is studied in section 3.

For a semigroup $S(t)$ acting on u, let $\mathrm{d}S(t)u$ be the Fréchet derivative of $S(t)$ at u with respect to the u; this notation is used for all the semigroups and monoids in this paper.

Denote the ball of radius r in the space $\mathcal{Z}$ by $B(\mathcal{Z}, r)$.

Theorem 1.4. Let the hypotheses of the preceding theorem hold. Consider the following one-step method on P:

$$S_{\Delta t}^{\text{1-step}}(n)p := \mathbb{P}\boldsymbol{L}_0^T\boldsymbol{S}_{\Delta t}(n)\big(\boldsymbol{R}_0 p + \boldsymbol{\Phi}_{\Delta t}(\boldsymbol{R}_0 p)\big), \quad p \in P.$$

1. For $T, R > 0$ and for $0 < \delta < 1-\gamma$, there exist $\Delta t^*, M > 0$ such that, for $\Delta t \leqslant \Delta t^*$, for $0 \leqslant n\Delta t \leqslant T$, and for $p \in B(P, R)$,

$$\big\|S_{\Delta t}^{\text{1-step}}(n)p - S^{\text{inertial}}(n\Delta t)p\big\|_\gamma \leqslant M\left(\frac{R}{n^{1-\gamma}} + \Delta t^\delta\right),$$

$$\big\|\mathrm{d}S_{\Delta t}^{\text{1-step}}(n)p - \mathrm{d}S^{\text{inertial}}(n\Delta t)p\big\|_{\gamma,\gamma} \leqslant M\left(\frac{R}{n^{1-\gamma}} + \mathrm{o}(1)\right).$$

   ($\mathrm{o}(1)$ is a function that goes to zero as Δt goes to zero uniformly in n.)

2. Denote by $\mathcal{M}_{\Delta t}$ the inertial manifold of the multi-step method, so that $\mathcal{M}_{\Delta t} := \operatorname{Graph}\boldsymbol{\Phi}_{\Delta t} \cap \mathcal{B}$. Define a map $\pi_{\Delta t}: P \to X_\gamma^q$ by $\pi_{\Delta t}(p) := \boldsymbol{R}_0 p + \boldsymbol{\Phi}_{\Delta t}(\boldsymbol{R}_0 p)$. Then, the following diagram commutes:

$$\begin{tikzcd}
\mathcal{M}_{\Delta t} \arrow[r, "\boldsymbol{S}_{\Delta t}"] \arrow[d, shift right, "\mathbb{P}\boldsymbol{L}_0^T"'] & \mathcal{M}_{\Delta t} \arrow[d, shift right, "\mathbb{P}\boldsymbol{L}_0^T"'] \\
P \arrow[u, shift right, "\pi_{\Delta t}"'] \arrow[r, "S_{\Delta t}^{\text{1-step}}"] & P \arrow[u, shift right, "\pi_{\Delta t}"']
\end{tikzcd}$$

This theorem is proved in section 6.

The advantage here is the C^1 convergence, which brings one-step theory to bear on multi-step methods approximating evolution systems. We have a one-step method on a finite dimensional space that C^1 converges to the solution of the PDE at fixed times (i.e., with $n\Delta t$ fixed as $\Delta t \to 0$); this is weaker than the convergence found in

ODE theory, where the method converges over one step, and thus the approximation theory of ODEs does not directly apply. This type of estimate, however, is typical of PDEs, where estimates often blow up at $t = 0$ because of the irregularity of the initial data. Stuart [14] considers approximations of semigroups on a Hilbert space satisfying exactly this type of estimate. All his theorems apply to our problem: consequences are the existence and convergence of hyperbolic equilibria, of unstable manifolds of hyperbolic equilibria, of phase portraits near hyperbolic equilibria, the upper semi-continuity of attractors, and the lower semi-continuity of attractors made from the union of unstable manifolds of hyperbolic equilibria. Furthermore, hyperbolic periodic orbits will persist under perturbation by multi-step methods, because the proof used for ODEs [13] depends only on estimates on a fixed time interval [15]. This programme has been successfully applied to multi-step methods approximating ordinary differential equations [12].

This theory fails to address the order of convergence. Both theorems give order of convergence Δt^{δ} where $0 < \delta < 1 - \gamma$; with high order methods, the estimate does not improve. The derivation of the convergence rate has two weak points. Problem 1: the monoid (see section 3) is defined by multiplying the solution operator of the PDE with eigenvectors of the companion matrix (see section 3) evaluated at $\Delta t = 0$. With this approach, even estimates for the linear equation are order Δt. For higher order accuracy, the monoid should be made from the eigenvectors of the companion matrix at general Δt – for the linear equation in one dimension, estimating between the monoid and the companion matrix then corresponds to removing the dominant eigenvalue of the companion matrix and this suggests higher order convergence. The difficulty with this appoach lies in defining the monoid using two eigenvectors that are *not* orthogonal; when the dominant eigenvalue of the companion matrix is not simple, the eigenvectors may be orthogonal [6]. Problem 2: when deriving the nonlinear estimate from the linear estimate, we estimate $\|\boldsymbol{u}(t) - \boldsymbol{u}(s)\|$ where $|t - s| \leqslant \Delta t$ (cf. integrals 2, 3, 5 of the appendix) by using a Lipschitz property. This bears no relation to the order of the method and introduces Δt terms. Higher order convergence needs a better method of moving from linear to nonlinear estimates.

2. The PDE

We state two well-known results [11, 14] concerning the well posedness of $(*)$. The first describes $(*)$; the second the variation of solutions to $(*)$ with respect to the initial condition – that is, for solutions $u(t)$ of $(*)$, we discuss

$$v_t + Av = \mathrm{d}f(u)v, \quad v(0) = \xi. \tag{$**$}$$

This equation is important to understanding the C^1 estimate.

Theorem 2.1. Let hypothesis 1.1 hold and let $f \in C^1(X_\gamma, X)$, some $\gamma < 1$. For every $u_0 \in X_\gamma$, there exists a unique continuously differentiable solution, $u(t)$, of $(*)$

such that $u(0) = u_0$. Define $S(t)$, the solution operator of $(*)$ for initial data $u_0 \in X_\gamma$, by $u(t) = S(t)u_0$. For every $T, R > 0$, a constant C exists such that

$$\left\|\frac{\mathrm{d}}{\mathrm{d}t}S(t)u_0\right\|_\gamma \leqslant C\frac{1}{t}, \quad u_0 \in B(X_\gamma, R),\ 0 < t < T.$$

The next theorem needs a weaker definition of solution. We say that $u(t)$ is a *mild solution* of $u_t + Au = f(u)$ if u is continuous and if u satisfies the integral equation

$$u(t) = \exp(-At)u(0) + \int_0^t \exp\bigl(A(s-t)\bigr) f\bigl(u(s)\bigr)\,\mathrm{d}s.$$

Theorem 2.2. Let hypothesis 1.1 hold and let $f \in C^1(X_\gamma, X)$, some $\gamma < 1$. For every $\xi \in X^\gamma$, there exists a mild solution, $v(t)$, of $(**)$ such that $v(0) = \xi$. Furthermore, $v(t) = \mathrm{d}S(t)u_0[\xi]$, where $\mathrm{d}S(t)u_0$ is the Fréchet derivative of $S(t)$ with respect to the initial condition u_0. For every $T, R > 0$, a constant C exists such that, for $u_0, \tilde{u}_0 \in B(X^q_\gamma, R)$ and for $0 < t < T$,

$$\begin{aligned}
\bigl\|\mathrm{d}S(t)u_0\bigr\|_{\gamma,\gamma} &\leqslant C,\\
\bigl\|\mathrm{d}S(t)u_0 - \mathrm{d}S(t)\tilde{u}_0\bigr\|_{\gamma,\gamma} &\leqslant C\bigl\|u_0 - \tilde{u}_0\bigr\|_\gamma,\\
\bigl\|\mathrm{d}S(t+\Delta t)u_0 - \mathrm{d}S(t)u_0\bigr\|_{\gamma,\gamma} &\leqslant C\Delta t/t.
\end{aligned}$$

Proof. The first inequality is standard; the others are simple manipulations with the integral equation. □

3. The linear multi-step methods

The general linear multi-step method for $u_t = G(u)$ consists of solving

$$\sum_{i=0}^{q} \alpha_i u_{n+i} = \Delta t \sum_{i=0}^{q} \beta_i G(u_{n+i}),$$

where $\alpha_0, \ldots, \alpha_q, \beta_0, \ldots, \beta_q$ are constants. Given u_k for $k = n, \ldots, n+q-1$ and some regularity, this equation can be solved to give u_{n+q}. As a method for solving $u_t = G(u)$, it is incomplete – it requires more initial data than the differential equation, a problem often overcome by applying a one-step starting method. To ensure that solutions of the multi-step method converge to the solutions of the PDE, we need consistency and stability conditions. We now state standard definitions of consistency and of strong $A(\alpha)$ stability, the stability condition needed when approximating sectorial evolution equations in time.

Define polynomials ρ and σ by

$$\rho(z) := \alpha_0 + \alpha_1 z + \cdots + \alpha_q z^q, \qquad \sigma(z) := \beta_0 + \beta_1 z + \cdots + \beta_q z^q.$$

The multi-step method is *consistent* if both $\rho(1) = 0$ and $\sigma(1) = \rho'(1)$, and is pth order consistent if

$$\rho(\mathrm{e}^{-z}) + z\,\sigma(\mathrm{e}^{-z}) = \mathcal{O}(z^{p+1}).$$

To develop the stability condition, apply the multi-step method to the test equation $u_t + Au = 0$:

$$(\alpha_0, \dots, \alpha_q) \cdot (u_n, \dots, u_{n+q})^T + \Delta t(\beta_0, \dots, \beta_q)A(u_n, \dots, u_{n+q})^T = 0,$$

which can be written

$$(\alpha_q + \Delta t \beta_q A)u_{n+q} = -\big[(\alpha_0, \dots, \alpha_{q-1}) + \Delta t(\beta_0, \dots, \beta_{q-1})A\big](u_n, \dots, u_{n+q-1})^T.$$

In vector notation, $\boldsymbol{u}_n := (u_n, \dots, u_{n+q-1})^T$, this means

$$\boldsymbol{u}_{n+1} = \mathcal{C}(A\Delta t)\boldsymbol{u}_n,$$

where $\mathcal{C}(\cdot)$ is the companion matrix defined by

$$\mathcal{C}(z) := -\begin{bmatrix} 0 & 1 & & & & \\ 0 & 0 & 1 & & & \\ & & & 1 & & \\ & & & & 1 & \\ & & & & & 1 \\ c_0(z) & c_1(z) & \cdots & \cdots & \cdots & c_{q-1}(z) \end{bmatrix}, \quad c_i(z) := \frac{\alpha_i + \beta_i z}{\alpha_q + \beta_q z}.$$

Denote the eigenvalues of $\mathcal{C}(z)$ by $\eta_1(z), \dots, \eta_q(z)$ with $|\eta_q(z)| \leqslant \cdots \leqslant |\eta_1(z)|$. Then, the multi-step method is *strongly* $A(\alpha)$ *stable* if $\mathcal{C}(z)$ satisfies the following: there exists $\mu < 1$ such that

$$\eta_1(0) = 1;$$

$$\big|\eta_1(z)\big| < \begin{cases} 1, & z \neq 0, \\ \mu, & \|z\| \geqslant 1, \end{cases} \quad \arg z < \alpha;$$

$$\big|\eta_i(z)\big| < \mu, \quad i = 2, \dots, q, \ \arg z < \alpha.$$

In particular, strongly $A(\alpha)$ stable methods are implicit ($\beta_q \neq 0$) – for otherwise, the norm of $\mathcal{C}(z)$ would be unbounded as $z \to \infty$ and therefore also the eigenvalues would be unbounded.

Normalise the multi-step method to have $\alpha_q = 1$ and apply it to $(*)$; this yields the following implicit equation for $\boldsymbol{u}_{n+1} \in X^q$, the q times product of X:

$$\boldsymbol{u}_{n+1} = \mathcal{C}(A\Delta t)\boldsymbol{u}_n + \Delta t(I + \Delta t \beta_q A)^{-1} F(\boldsymbol{u}_n, \boldsymbol{u}_{n+1}), \qquad (***)$$

where

$$F(\boldsymbol{u}_n, \boldsymbol{u}_{n+1}) := \left(0, \dots 0, \sum_{i=0}^{q-1} \beta_i f(u_{n+i}) + \beta_q f(u_{n+q})\right)^T.$$

The next result guarantees a unique solution. We restrict the stability region of the multi-step method by taking $\theta \leqslant \alpha < \pi/2$, so that it includes all the eigenvalues of A. The theorems of this section apply to non self adjoint A, though we later restrict attention to self adjoint operators.

Lemma 3.1. Suppose that hypothesis 1.1 holds, that $f \in C^1(X_\gamma, X)$, and that the linear multi-step method is pth order consistent, $p \geqslant 1$, and strongly $A(\alpha)$ stable, $\theta \leqslant \alpha < \pi/2$. Then, for all $R > 0$, there exists Δt^* such that (∗∗∗), applied with $\Delta t \leqslant \Delta t^*$, has a unique solution, $\boldsymbol{u}_{n+1} \in B(X_\gamma^q, R)$, for each $\boldsymbol{u}_n \in B(X_\gamma^q, R)$.

Proof. See Hill and Suli [4]. □

Let $\eta_1(z)$ denote the largest eigenvalue of $\mathcal{C}(z)$, and denote by $L(z)$, $R(z)$ the corresponding left, right eigenvectors normalised so that $\langle L(z), R(z)\rangle = 1$. Then $\mathcal{C}(z) - R(z)\eta_1(z)L(z)^T$ has the same spectrum as $\mathcal{C}(z)$, except that the largest eigenvalue, $\eta_1(z)$, has been replaced by zero. The strong $A(\alpha)$ stability restricts the remaining eigenvalues to have magnitude less than μ, some $\mu < 1$. Consequently,

$$\left\|\mathcal{C}(z)^n - R(z)\eta_1(z)^n L^T(z)\right\| \to 0 \quad \text{as } n \to \infty \text{ uniformly in } z.$$

Hill and Suli [4] have proved an analogous property for $\mathcal{C}(A\Delta t)$. Write $L(z) = (L_1(z), \ldots, L_q(z))$ and $R(z) = (R_1(z), \ldots, R_q(z))$, to define $\boldsymbol{R}_0$, $\boldsymbol{L}_0^T$ between X^q and X by

$$\boldsymbol{L}_0^T : X^q \to X; \quad (x_1, \ldots, x_q) \mapsto L_1(0)x_1 + \cdots + L_q(0)x_q;$$
$$\boldsymbol{R}_0 : X \to X^q; \quad x \mapsto \big(R_1(0)x, \ldots, R_q(0)x\big).$$

Hill and Suli show that $R(0)$ can be scaled to make $\boldsymbol{R}_0(x) = (x, x, \ldots, x)$.

Lemma 3.2. Under the hypotheses of proposition 3.1, there exists a constant M such that

$$\left\|\mathcal{C}(A\Delta t)^n - \boldsymbol{R}_0 \exp(-An\Delta t)\boldsymbol{L}_0^T\right\|_{\gamma,\gamma} \leqslant M\frac{1}{n}, \quad n \geqslant 0.$$

Proof. Hill and Suli [4]. □

This result is fundamental to what follows. Consequence 1: As $n \to \infty$, $\mathcal{C}(A\Delta t)^n$ tends to $\boldsymbol{R}_0 \exp(-An\Delta t)\boldsymbol{L}_0^T$, a rank one matrix. Therefore, in the limit, $\mathcal{C}(A\Delta t)^n$ has one dimensional image space. Section 6 interprets this as convergence to a one-step method. Indeed in the case $f = 0$, we find from (∗∗∗) that

$$\boldsymbol{L}_0^T \boldsymbol{u}_{n+k} = \Big(\mathrm{e}^{-Ak\Delta t} + \frac{1}{k}\mathcal{O}(1)\Big)\boldsymbol{L}_0^T \boldsymbol{u}_n,$$

and thus

$$\frac{1}{k\Delta t}\big(\boldsymbol{L}_0^T \boldsymbol{u}_{n+k} - \boldsymbol{L}_0^T \boldsymbol{u}_n\big) = A\boldsymbol{L}_0^T \boldsymbol{u}_n + \text{small terms},$$

which is a forward Euler method over k steps.

Consequence 2: Lemma 3.2 motivates lifting the semigroup of the differential equation up to an operator on the product space X_γ^q. In fact, this lemma estimates the error between the multi-step method and a lifting of the true solution operator for the homogeneous PDE ($f = 0$). We now study the general case.

Let $S(\cdot)$ denote the solution operator for $(*)$, so that $u(t) = S(t)u_0$ solves $(*)$ at time t. Define

$$\boldsymbol{S}(t)\boldsymbol{u} := \boldsymbol{R}_0 S(t)\boldsymbol{L}_0^T\boldsymbol{u}, \quad \boldsymbol{u} \in X_\gamma^q.$$

This operator is called a *monoid* and was first introduced by Hill and Suli [4]. A monoid is an operator that satisfies only the time autonomous property, $\boldsymbol{S}(t_1+t_2) = \boldsymbol{S}(t_1)\boldsymbol{S}(t_2)$ for any $t_1, t_2 \geqslant 0$, of the definition of a semigroup (the monoid fails the condition $\boldsymbol{S}(0) = I$).

Denote the solution operator for the numerical method by $\boldsymbol{S}_{\Delta t}(\cdot)$, so that $\boldsymbol{u}_n = \boldsymbol{S}_{\Delta t}(n)\boldsymbol{u}_0$ solves $(***)$ when n, Δt, and $\|\boldsymbol{u}_0\|_\gamma$ are such that a solution exists (proposition 3.1). The following estimate holds between $\boldsymbol{S}$ and $\boldsymbol{S}_{\Delta t}$. Again, d is used to denote the Fréchet derivative of the semigroup with respect to the initial condition.

Theorem 3.3. For any $R > 0$, there exist $\Delta t^*, T > 0$ such that, for $0 \leqslant n\Delta t \leqslant T$ and $\Delta t \leqslant \Delta t^*$ and $\boldsymbol{u} \in B(X_\gamma^q, R)$,

$$\|\boldsymbol{S}_{\Delta t}(n)\boldsymbol{u} - \boldsymbol{S}(n\Delta t)\boldsymbol{u}\|_\gamma \leqslant \frac{MR}{n^{1-\gamma}} + L\Delta t^\delta,$$

$$\|\mathrm{d}\boldsymbol{S}_{\Delta t}(n)\boldsymbol{u} - \mathrm{d}\boldsymbol{S}(n\Delta t)\boldsymbol{u}\|_{\gamma,\gamma} \leqslant \frac{MR}{n^{1-\gamma}} + L\Delta t^\delta,$$

where $0 < \delta < 1-\gamma$ and the constant M depends on δ and R.

Proof. Hill and Suli [4] gives the first estimate. The second estimate, the C^1 bound, is proved in the appendix.

We can modify the semigroups outside the absorbing ball without effecting the long term behaviour, and do so now to gain a modified numerical solution operator that uniformly approximates the monoid, and has the same long term behaviour as the multi-step method. Take R so that $\mathcal{B} \subset B(X_\gamma^q, R)$ and define $\boldsymbol{S}^*_{\Delta t}(\cdot)$ by

$$\boldsymbol{S}^*_{\Delta t}(n)\boldsymbol{u} := \theta\left(\frac{\|\boldsymbol{u}\|}{R}\right)\boldsymbol{S}_{\Delta t}(n)\boldsymbol{u} + \left(1 - \theta\left(\frac{\|\boldsymbol{u}\|}{R}\right)\right)\boldsymbol{S}(n\Delta t)\boldsymbol{u},$$

where $\theta \in C^\infty(\mathbb{R}, [0,1])$ and satisfies $\theta(x) = 0$ for $|x| > 2$ and $\theta(x) = 1$ for $|x| < 1$. $\boldsymbol{S}^*_{\Delta t}(\cdot)$ is not a semigroup. However, the monoid inherits a positively invariant, absorbing ball, $\mathcal{B}$, from the PDE, and therefore this operator satisfies the semigroup properties for initial conditions inside $\mathcal{B}$.

Corollary 3.4. For all Δt small enough, we have, for $\boldsymbol{u} \in X_\gamma^q$ and $n\Delta t = T$ and $0 < \delta < 1-\gamma$,

$$\|\boldsymbol{S}^*_{\Delta t}(n)\boldsymbol{u} - \boldsymbol{S}(n\Delta t)\boldsymbol{u}\|_\gamma \leqslant M\Delta t^\delta,$$

$$\|\mathrm{d}\boldsymbol{S}^*_{\Delta t}(n)\boldsymbol{u} - \mathrm{d}\boldsymbol{S}(n\Delta t)\boldsymbol{u}\|_{\gamma,\gamma} \leqslant M\Delta t^\delta,$$

where M is a constant depending on γ and δ.

4. The invariant manifold theorem

Let $\mathcal{X}$ be a Banach space, L be a linear operator from $\mathcal{X}$ to itself, and N belong to $C^1(\mathcal{X},\mathcal{X})$. Let $\mathcal{X}$ be the direct sum of $\mathcal{X}_1$ and $\mathcal{X}_2$, two subspaces that are invariant under L, and denote by π_1, π_2 the projections to these spaces. Consider

$$x \mapsto Lx + N(x), \quad x \in \mathcal{X}. \tag{†}$$

We look for a manifold that is attractive and invariant under this mapping. Of the many invariant manifold theorems [16], it is the theory of Jones and Stuart that solves our problem. This theory gives attractive invariant manifolds that can be written as a graph from $\mathcal{X}_1$ to $\mathcal{X}_2$ under a certain set of conditions on L and N. These conditions are essentially those needed for Hadamard's graph transform. We state their theory in a context suitable for inertial manifolds; the paper [9] deals with unstable manifolds.

Hypothesis 4.1. Let $0 < a < b < c < 1$ be such that

$$b\|x_1\|_{\mathcal{X}} \leqslant \|Lx_1\|_{\mathcal{X}} \leqslant c\|x_1\|_{\mathcal{X}}, \quad x_1 \in \mathcal{X}_1;$$

$$\|Lx_2\|_{\mathcal{X}} \leqslant a\|x_2\|_{\mathcal{X}}, \quad x_2 \in \mathcal{X}_2.$$

Consider the following semi-norms:

$$|N|_0 := \sup\bigl\{\bigl\|\pi N(x)\bigr\|_{\mathcal{X}}\colon\ x \in \mathcal{X},\ \pi = I, \pi_2\bigr\},$$

$$|N|_1 := \sup\bigl\{\bigl\|\mathrm{d}\pi N(x)\bigr\|_{\mathcal{X},\mathcal{X}}\colon\ x \in \mathcal{X},\ \pi = I, \pi_1, \pi_2\bigr\}.$$

Theorem 4.2. Let hypothesis 4.1 hold. For some $r, r^* > 0$ and some $\mu < 1$, suppose that

$$(1 + r^*)|N|_1 < \mu\, b, \tag{c1}$$

$$|N|_0 < r(1 - a), \tag{c2}$$

$$(1 + r^*)^2|N|_1 < r^*(b - a), \tag{c3}$$

$$a + (1 + r^*)|N|_1 < \mu. \tag{c4}$$

Then, there exists a unique $\Phi \in C^0(\mathcal{X}_1, \mathcal{X}_2)$ such that

1. $\mathcal{M} := \operatorname{Graph}\Phi$ is an attractive, invariant manifold for (†) in the sense that

$$x \in \mathcal{M} \iff Lx + N(x) \in \mathcal{M},$$

and

$$\operatorname{dist}\bigl(Lx + N(x), \mathcal{M}\bigr) \leqslant \mu \operatorname{dist}(x, \mathcal{M}), \quad \|\pi_2 x\|_{\mathcal{X}} \leqslant r.$$

2. $\|\Phi(x)\|_{\mathcal{X}} \leqslant r$ for all $x \in \mathcal{X}_1$.
3. $\|\Phi(x) - \Phi(y)\|_{\mathcal{X}} \leqslant r^* \|x - y\|_{\mathcal{X}}$ for $x, y \in \mathcal{X}_1$.

Furthermore, if we replace (c1) by

$$4(1 + r^*)|N|_1 < b - a \tag{c1$'$}$$

then $\Phi \in C^1(\mathcal{X}_1, \mathcal{X}_2)$.

Proof. The C^0 part of the theorem originates from [8], but we state the theorem as presented in [14]. The C^1 regularity is discussed by [7]. □

Corollary 4.3. Let hypothesis 4.1 and also (c1–c4) of the preceding theorem hold for some $r, r^* > 0$ and $\mu < 1$. Consider the mapping

$$x \mapsto Lx + N(x) + E(x, \varepsilon), \tag{$\dagger_\varepsilon$}$$

where E is a C^1 perturbation converging to zero uniformly:

$$\sup\bigl\{\bigl\|E(x,\varepsilon)\bigr\|_{\mathcal{X}}, \bigl\|\mathrm{d}E(x,\varepsilon)\bigr\|_{\mathcal{X},\mathcal{X}} \colon\ x \in \mathcal{X}\bigr\} \to 0 \quad \text{as } \varepsilon \to 0.$$

Then, there exists $\mathcal{M}_\varepsilon$, a graph of a continuous function $\Phi_\varepsilon : \mathcal{X}_1 \to \mathcal{X}_2$, such that $\mathcal{M}_\varepsilon$ is attractive and invariant and such that $\mathcal{M}_\varepsilon$ converges to $\mathcal{M}$: for a constant C,

$$\bigl\|\Phi(y) - \Phi_\varepsilon(y)\bigr\|_{\mathcal{X}} \leqslant C \sup\bigl\{\bigl\|E(x,\varepsilon)\bigr\|_{\mathcal{X}} \colon\ x \in \mathcal{X}\bigr\}, \quad y \in \mathcal{X}_1.$$

Moreover, when (c1$'$) holds and N has uniformly continuous Fréchet derivative, the convergence of Φ_ε to Φ occurs in the C^1 norm:

$$\sup_{x \in \mathcal{X}_1} \bigl\|\mathrm{d}\Phi(x) - \mathrm{d}\Phi_\varepsilon(x)\bigr\|_{\mathcal{X},\mathcal{X}} \to 0 \quad \text{as } \varepsilon \to 0.$$

Proof. See the references cited for the previous theorem. □

5. Inertial manifolds: theorem 1.3

Proof. We prepare the monoid $\boldsymbol{S}$ for the Invariant Manifold Theorem. Consider the time T mapping of the monoid: for $\boldsymbol{u} \in X^q_\gamma$,

$$\boldsymbol{S}(T)\boldsymbol{u} = L\boldsymbol{u} + N(\boldsymbol{u}),$$

where

$$L\boldsymbol{u} := \boldsymbol{R}_0 \exp(-AT)\boldsymbol{L}_0^T \boldsymbol{u},$$

$$N(\boldsymbol{u}) := \int_0^T \boldsymbol{R}_0 \exp\bigl(-A(T-s)\bigr) f\bigl(\boldsymbol{L}_0^T \boldsymbol{S}(s)\boldsymbol{u}\bigr)\, \mathrm{d}s.$$

The eigenvalues of A are $\lambda_1, \lambda_2, \ldots$; therefore the eigenvalues of L are $\mathrm{e}^{-\lambda_1 T}$, $\mathrm{e}^{-\lambda_2 T}$, $\ldots$ and 0, the 0 having multiplicity equal to $q-1$ copies of X_γ. Let $\ell \in \mathbb{N}$, and let P denote the space spanned by the first ℓ eigenfunctions of A, and let Q denote the space spanned by the remaining eigenfunctions. Then, X_γ^q is the direct sum of $\boldsymbol{R}_0 P$, $\boldsymbol{R}_0 Q$, and $\operatorname{Ker} \boldsymbol{L}_0^T$, which are written as invariant spaces of L as follows:

$$\begin{aligned}
\boldsymbol{R}_0 P &= \text{eigenspace associated to eigenvalues } \mathrm{e}^{-\lambda_1 T}, \ldots, \mathrm{e}^{-\lambda_\ell T};\\
\boldsymbol{R}_0 Q &= \text{eigenspace associated to eigenvalues } \mathrm{e}^{-\lambda_{\ell+1} T}, \ldots;\\
\operatorname{Ker} \boldsymbol{L}_0^T &= \text{eigenspace associated to the 0 eigenvalue.}
\end{aligned}$$

The following properties of L hold:

$$\mathrm{e}^{-\lambda_\ell T}\|\boldsymbol{u}\|_\gamma \leqslant \|L\boldsymbol{u}\|_\gamma \leqslant \mathrm{e}^{-\lambda_1 T}\big\|\boldsymbol{u}\big\|_\gamma, \quad \boldsymbol{u} \in \boldsymbol{R}_0 P;$$

$$\|L\boldsymbol{u}\|_\gamma \leqslant \mathrm{e}^{-\lambda_{\ell+1} T}\|\boldsymbol{u}\|_\gamma, \quad \boldsymbol{u} \in \boldsymbol{R}_0 Q \oplus \operatorname{Ker} \boldsymbol{L}_0^T.$$

With $\mathcal{X}_1 = \boldsymbol{R}_0 P$ and $\mathcal{X}_2 = \boldsymbol{R}_0 Q \oplus \operatorname{Ker} \boldsymbol{L}_0^T$ in hypothesis 4.1, this gives

$$a = \mathrm{e}^{-\lambda_{\ell+1} T}, \qquad b = \mathrm{e}^{-\lambda_\ell T}, \qquad c = \mathrm{e}^{-\lambda_1 T}.$$

Lemma A.1(i), the smoothing property, yields a constant K such that

$$\left\|\boldsymbol{R}_0 \int_0^T \exp\big(-A(T-s)\big) f\big(\boldsymbol{L}_0^T \boldsymbol{S}(s)\boldsymbol{u}\big)\, \mathrm{d}s\right\|_\gamma \leqslant KT^{1-\gamma}, \quad \boldsymbol{u} \in X_\gamma^q;$$

viz. $\|N(\boldsymbol{u})\|_\gamma \leqslant KT^{1-\gamma}$. Similarly,

$$\big\|\mathrm{d}N(\boldsymbol{u})\big\|_{\gamma,\gamma} \leqslant KT^{1-\gamma}, \quad \boldsymbol{u} \in X_\gamma^q.$$

We establish conditions (c1′) and (c2–c4) of theorem 4.2 by choosing T small and ℓ large. Let σ be a fixed positive constant, and pick R so that $\mathcal{B} \subset B(X_\gamma^q, R)$. Fix $r, r^* > 0$. Below, we take limits as $T \to 0$ and $\ell \to \infty$ subject to $\sigma \leqslant \lambda_\ell T \leqslant 2\sigma$.

(c1′) Verify $4|N|_1(1+r^*) < b - a$:

$$\begin{aligned}
\frac{|N|_1}{b-a} &= \frac{KT^{1-\gamma}}{\mathrm{e}^{-\lambda_\ell T} - \mathrm{e}^{-\lambda_{\ell+1} T}}\\
&= \frac{KT(T\lambda_{\ell+1})^{-\gamma}}{\mathrm{e}^{-\lambda_{\ell+1} T}\big(\mathrm{e}^{(-\lambda_\ell + \lambda_{\ell+1})T} - 1\big)} \cdot \frac{1}{\lambda_{\ell+1}^{-\gamma}}
\end{aligned}$$

(using $\mathrm{e}^x - 1 \geqslant x$ for $x \geqslant 0$)

$$\leqslant \frac{K(T\lambda_{\ell+1})^{-\gamma}}{\mathrm{e}^{-\lambda_{\ell+1} T}} \cdot \frac{1}{\lambda_{\ell+1}^{-\gamma}(\lambda_{\ell+1} - \lambda_\ell)},$$

which can be made arbitrarily small by using the gap condition as $\ell \to \infty$ and $T \to 0$ with $\sigma \leqslant \lambda_\ell T \leqslant 2\sigma$.

(c2) Verify $|N|_0 < r(1-a)$:

$$\frac{|N|_0}{1-a} = \frac{KT^{1-\gamma}}{1-\mathrm{e}^{-\lambda_{\ell+1}T}} = \frac{K(T\lambda_{\ell+1})^{1-\gamma}}{(1-\mathrm{e}^{-\lambda_{\ell+1}T})\lambda_{\ell+1}^{1-\gamma}} \to 0$$

as $\ell \to \infty$ and $T \to 0$ with $\sigma \leqslant \lambda_\ell T \leqslant 2\sigma$.

(c3) Verify $(1+r^*)^2|N|_1 < (b-a)r^*$: As in (i),

$$(b-a)^{-1}|N|_1 \to 0$$

as $\ell \to \infty$ and $T \to 0$ with $\sigma \leqslant \lambda_\ell T \leqslant 2\sigma$.

(c4) Verify $a + (1+r^*)|N|_1 < \mu$:

$$\begin{aligned} a + (1+r^*)|N|_1 &\leqslant \mathrm{e}^{-\lambda_{\ell+1}T} + (1+r^*)KT^{1-\gamma} \\ &\leqslant \mathrm{e}^{-\lambda_{\ell+1}T} + (1+r^*)\frac{K(\lambda_{\ell+1}T)^{1-\gamma}}{\lambda_{\ell+1}^{1-\gamma}}. \end{aligned}$$

Here, as we take the limit, the second term becomes negligible. For $\mathrm{e}^{-\sigma} < \mu < 1$, (c4) holds by taking ℓ large and T small.

The derivative $\mathrm{d}N$ is uniformly continuous in $\boldsymbol{u}$ because of the Lipschitz properties of $\mathrm{d}f$ (from section 1) and of $\mathrm{d}S(\cdot)\boldsymbol{u}$ with respect to $\boldsymbol{u}$ (theorem 2.2).

The hypotheses of the Invariant Manifold Theorem hold for $\boldsymbol{S}(T)$ by taking ℓ large and $\sigma \leqslant T\lambda_\ell \leqslant 2\sigma$. Consequently, there exist C^1 manifolds, $\mathcal{M}_T$, invariant and uniformly exponentially attractive for all initial conditions in $\mathcal{B}$. In fact, $\mathcal{M}_T$ is independent of T and is an invariant of the flow [8, 14]. Let $\mathcal{M}$ denote the intersection of $\mathcal{M}_T$ with the absorbing ball $\mathcal{B}$. Then, $\mathcal{M}$ is positively invariant. Further, the uniform exponential attraction of $\mathcal{M}_T$ on $\mathcal{B}$ combined with the attractivity of $\mathcal{B}$ gives exponential attraction of $\mathcal{M}$ on the whole of X^q_γ [2]. Thus, $\mathcal{M}$ is positively invariant and exponentially attractive (though not uniformly), and therefore is an inertial manifold for the monoid.

By construction, $\mathcal{M} = \mathrm{Graph}\,\Phi \cap \mathcal{B}$, some $\Phi \in C^1(\boldsymbol{R}_0 P, \boldsymbol{R}_0 Q \oplus \mathrm{Ker}\,\boldsymbol{L}_0^T)$. Let $\phi(\cdot) := \boldsymbol{L}_0^T \Phi(\boldsymbol{R}_0 \cdot)$; then $\phi \in C^1(P, Q)$ and $\mathrm{Graph}\,\phi \cap \boldsymbol{L}_0^T\mathcal{B}$ is an inertial manifold of the PDE.

The k step mapping of the modified numerical operator reads, for $\boldsymbol{u} \in X^q_\gamma$,

$$\boldsymbol{S}^*_{\Delta t}(k)\boldsymbol{u} = L\boldsymbol{u} + N(\boldsymbol{u}) + E(\boldsymbol{u}, k, \Delta t),$$

where

$$E(\boldsymbol{u}, k, \Delta t) := \boldsymbol{S}^*_{\Delta t}(k)\boldsymbol{u} - \boldsymbol{S}(k\Delta t)\boldsymbol{u}.$$

According to corollary 3.4, E goes to zero uniformly in C^1 as $\Delta t \to 0$ with $k\Delta t$ fixed. Thus, corollary 4.3 applies and gives manifolds, $\mathcal{M}_k$, invariant and attractive on $\mathcal{B}$ for k steps of the modified numerical operator, $\sigma \leqslant k\Delta t\lambda_\ell \leqslant 2\sigma$. These manifolds are each one-step invariant: To see this, hold ℓ fixed and let Δt be small enough that inertial manifolds $\mathcal{M}_k$, $\mathcal{M}_{k+1}$ exist for the k, $k+1$ steps of the multi-step method. Because $\mathcal{M}_k \cap \mathcal{B}$ lies inside the domain of uniform attraction of $\mathcal{M}_{k+1}$,

$$\operatorname{dist}\big(\boldsymbol{S}_{\Delta t}\big(n(k+1)\big)(\mathcal{M}_k \cap \mathcal{B}), \mathcal{M}_{k+1}\big) \leqslant \mu^n \operatorname{dist}(\mathcal{M}_k \cap \mathcal{B}, \mathcal{M}_{k+1}).$$

Substitute $n = km$ and apply the k-step invariance of $\mathcal{M}_k$, to gain

$$\operatorname{dist}(\mathcal{M}_k \cap \mathcal{B}, \mathcal{M}_{k+1}) \leqslant \mu^{km} \operatorname{dist}(\mathcal{M}_k \cap \mathcal{B}, \mathcal{M}_{k+1}).$$

Then, as $\mu < 1$ and as $\operatorname{dist}(\mathcal{M}_k \cap \mathcal{B}, \mathcal{M}_{k+1})$ is finite, $\operatorname{dist}(\mathcal{M}_k \cap \mathcal{B}, \mathcal{M}_{k+1}) = 0$; i.e., $\mathcal{M}_k \cap \mathcal{B} \subset \mathcal{M}_{k+1}$. This argument can be repeated to gain $\mathcal{M}_{k+1} \cap \mathcal{B} \subset \mathcal{M}_k$. Thus $\mathcal{M}_{k+1} \cap \mathcal{B} = \mathcal{M}_k \cap \mathcal{B}$, and we can write $\mathcal{M}_{\Delta t} := \mathcal{M}_k \cap \mathcal{B}$. $\mathcal{M}_{\Delta t}$ is an inertial manifold of the multi-step method; the exponential attraction for every initial condition comes, as before, by exploiting the absorbing ball.

By construction, the inertial manifold $\mathcal{M}_{\Delta t} = \operatorname{Graph} \Phi_{\Delta t} \cap \mathcal{B}$, some $\Phi_{\Delta t} \in C^1(\boldsymbol{R}_0 P, \boldsymbol{R}_0 Q \oplus \operatorname{Ker} \boldsymbol{L}_0^T)$. The convergence of $\Phi_{\Delta t}$ to Φ comes directly from corollary 4.3. □

6. An approximate one-step method: theorem 1.4

Proof. Let Φ, $\Phi_{\Delta t}$ be the functions constructed in the preceding theorem, so that the graphs of Φ and $\Phi_{\Delta t}$ are the inertial manifolds of the monoid and the multistep method. We now define a one-step method on P, the span of the first ℓ eigenfunctions of A, that approximates the inertial form. First, define π, $\pi_{\Delta t}$ by

$$\pi : P \to \mathcal{M}; \qquad p \mapsto \boldsymbol{R}_0 p + \Phi(\boldsymbol{R}_0 p);$$

$$\pi_{\Delta t} : P \to \mathcal{M}_{\Delta t}; \qquad p \mapsto \boldsymbol{R}_0 p + \Phi_{\Delta t}(\boldsymbol{R}_0 p).$$

Then, the solution operator of the inertial form may be written,

$$S^{\text{inertial}}(t)p = \mathbb{P}\boldsymbol{L}_0^T \boldsymbol{S}(t)\pi(p), \quad p \in P.$$

This motivates defining a one-step method by

$$S_{\Delta t}^{\text{1-step}}(n)p := \mathbb{P}\boldsymbol{L}_0^T \boldsymbol{S}_{\Delta t}(n)\pi_{\Delta t}(p), \quad p \in P.$$

We derive the error estimates: consider $T, R > 0$, and let Δt^*, M, δ be the constants arising from theorem 3.3. Take n, Δt such that $0 \leqslant n\Delta t \leqslant T$ and $\Delta t \leqslant \Delta t^*$.

$S_{\Delta t}^{\text{1-step}}$ approximates the inertial form as follows: for $p \in B(P, R)$,

$$\begin{aligned}
&\left\|S^{\text{inertial}}(n\Delta t)p - S_{\Delta t}^{\text{1-step}}(n)p\right\|_\gamma \\
&\quad \leqslant \left\|\boldsymbol{L}_0^T\right\| \cdot \left\|\boldsymbol{S}(n\Delta t)\bigl(\pi(p)\bigr) - \boldsymbol{S}(n\Delta t)\bigl(\pi_{\Delta t}(p)\bigr)\right\|_\gamma \\
&\qquad + \left\|\boldsymbol{L}_0^T\right\| \cdot \left\|\boldsymbol{S}(n\Delta t)\bigl(\pi_{\Delta t}(p)\bigr) - \boldsymbol{S}_{\Delta t}(n)\bigl(\pi_{\Delta t}(p)\bigr)\right\|_\gamma \\
&\quad \leqslant \left\|\boldsymbol{L}_0^T\right\| \left(C\left\|\Phi_{\Delta t} - \Phi\right\|_\gamma + \frac{MR}{n^{1-\gamma}} + M\Delta t^\delta \right) \\
&\quad \leqslant \left\|\boldsymbol{L}_0^T\right\| \left(\frac{MR}{n^{1-\gamma}} + (M + C^2)\Delta t^\delta \right).
\end{aligned}$$

C^1 estimates follow similarly:

$$\begin{aligned}
&\left\|\mathrm{d}S^{\text{inertial}}(n\Delta t)p - \mathrm{d}S_{\Delta t}^{\text{1-step}}(n)p\right\|_{\gamma,\gamma} \\
&\quad \leqslant \left\|\boldsymbol{L}_0^T\right\| \cdot \left\|\mathrm{d}\boldsymbol{S}(n\Delta t)\boldsymbol{w}|_{\boldsymbol{w}=\pi(p)}\bigl(\mathrm{d}\pi(p) - \mathrm{d}\pi_{\Delta t}(p)\bigr)\right\|_{\gamma,\gamma} \\
&\qquad + \left\|\boldsymbol{L}_0^T\right\| \cdot \left\|\bigl(\mathrm{d}\boldsymbol{S}(n\Delta t)\boldsymbol{w}|_{\boldsymbol{w}=\pi(p)} - \mathrm{d}\boldsymbol{S}(n\Delta t)\boldsymbol{w}|_{\boldsymbol{w}=\pi_{\Delta t}(p)}\bigr)\mathrm{d}\pi_{\Delta t}(p)\right\|_{\gamma,\gamma} \\
&\qquad + \left\|\boldsymbol{L}_0^T\right\| \cdot \left\|\bigl(\mathrm{d}\boldsymbol{S}(n\Delta t)\boldsymbol{w} - \mathrm{d}\boldsymbol{S}_{\Delta t}(n)\boldsymbol{w}\bigr)|_{\boldsymbol{w}=\pi_{\Delta t}(p)}\mathrm{d}\pi_{\Delta t}(p)\right\|_{\gamma,\gamma} \\
&\quad \leqslant C\left\|\boldsymbol{L}_0^T\right\| \cdot \left\|\mathrm{d}\Phi_{\Delta t} - \mathrm{d}\Phi\right\|_{\gamma,\gamma} \\
&\qquad + \left\|\boldsymbol{L}_0^T\right\| \left(C\|\Phi_{\Delta t} - \Phi\|\gamma,\gamma + \frac{MR}{n^{1-\gamma}} + M\Delta t^\delta \right) \cdot \left\|\bigl(I + \mathrm{d}\Phi_{\Delta t}(\boldsymbol{R}_0 u)\bigr)\right\|_{\gamma,\gamma} \\
&\quad \leqslant \mathrm{o}(1) + \left\|\boldsymbol{L}_0^T\right\| \left(\frac{MR}{n^{1-\gamma}} + M\Delta t^\delta \right)\bigl(1 + \|\mathrm{d}\Phi\|_{\gamma,\gamma} + \mathrm{o}(1)\bigr).
\end{aligned}$$

That the diagram of theorem 1.4 commutes is a simple consequence of the definition of the one-step method. □

Appendix: The C^1 estimate: theorem 3.3

Proof. The C^1 bound is derived by applying Gronwall's inequality to an integral representation of the error. Fix $\xi \in X_\gamma^q$ and $\boldsymbol{u}_0 \in B(X_\gamma^q, R)$, and let

$$D_n := \boldsymbol{v}(n\Delta t) - \boldsymbol{v}_n,$$

where

$$\boldsymbol{v}(t) := \mathrm{d}\boldsymbol{S}(t)\boldsymbol{u}_0[\xi] \quad \text{and} \quad \boldsymbol{v}_n := \mathrm{d}\boldsymbol{S}_{\Delta t}(n)\boldsymbol{u}_0[\xi].$$

Applying the Variation of Constants formula to ($**$) and to the derivative of ($***$) (with respect to initial condition) and subtracting gives the following expression for D_n.

$$\begin{aligned}D_n &= \boldsymbol{R}_0 \mathrm{e}^{-An\Delta t} \boldsymbol{L}_0^T \xi - \mathcal{C}^n(A\Delta t)\xi \\ &\quad + \int_0^{n\Delta t} \boldsymbol{R}_0 \mathrm{e}^{-A(n\Delta t - s)} \boldsymbol{L}_0^T \, \mathrm{d}F\big(\boldsymbol{u}(s), \boldsymbol{u}(s)\big)\big[\boldsymbol{v}(s), \boldsymbol{v}(s)\big] \; \mathrm{d}s \\ &\quad - \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} \mathcal{C}^{n-1-i}(A\Delta t)(I + \beta_q \Delta t A)^{-1} \mathrm{d}F(\boldsymbol{u}_i, \boldsymbol{u}_{i+1})[\boldsymbol{v}_i, \boldsymbol{v}_{i+1}] \; \mathrm{d}s.\end{aligned}$$

We split the right-hand side into the sum of eight integrals, aiming to make each integral either small or bounded in terms of D_n. By small we mean "goes to zero as $\Delta t \to 0$ uniformly in n" and quantify this by saying $f(\Delta t) = \mathcal{O}(g(\Delta t))$ if $f(\Delta t)/g(\Delta t)$ is bounded by a constant independent of n.

The following properties of fractional powers are used during the proof. The notation $\|\cdot\|_{\alpha,\beta}$ denotes the subordinate norm of an operator from X_α to X_β.

Lemma A.1. For each $\gamma \geqslant 0$, a constant C depending only on γ exists such that

(i) $\|\exp(-At)\|_{0,\gamma} \leqslant Ct^{-\gamma}\mathrm{e}^{-at},\ t > 0$;

(ii) $\|(I + \beta_q \Delta t A)^{-1}\|_{0,\gamma} \leqslant C\Delta t^{-\gamma}$;

(iii) $\|I - \mathrm{e}^{-A\Delta t}{}_{0,-\gamma}\|_{0,-\gamma} \leqslant C\Delta t^{\gamma}$.

Integral 1

$$I_1 := \boldsymbol{R}_0 \mathrm{e}^{-An\Delta t} \boldsymbol{L}_0^T \xi - \mathcal{C}^n(A\Delta t)\xi.$$

The linear estimate, lemma 3.2, gives

$$\|I_1\|_\gamma = \mathcal{O}(1) \cdot \frac{1}{n} \cdot \|\xi\|_\gamma.$$

Integral 2

$$\begin{aligned}I_2 := \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} &\boldsymbol{R}_0 \mathrm{e}^{-A(n\Delta t - s)} \boldsymbol{L}_0^T \\ &\times \big(\mathrm{d}F\big(\boldsymbol{u}(s), \boldsymbol{u}(s)\big) - \mathrm{d}F\big(\boldsymbol{u}(i\Delta t), \boldsymbol{u}\big((i+1)\Delta t\big)\big)\big) \cdot \big[\boldsymbol{v}(s), \boldsymbol{v}(s)\big] \; \mathrm{d}s.\end{aligned}$$

We bound this integral by applying the smoothing property and the inequalities (see theorems 2.1–2.2)

$$\big\|\boldsymbol{v}(s)\big\|_\gamma = \mathcal{O}(1) \cdot \|\xi\|_\gamma,$$

$$\big\|\mathrm{d}F\big(\boldsymbol{u}(s), \boldsymbol{u}(s)\big) - \mathrm{d}F\big(\boldsymbol{u}(t), \boldsymbol{u}(t + \Delta t)\big)\big\|_{\gamma,0} = \mathcal{O}(\Delta t) \cdot \frac{1}{t}, \quad t \leqslant s \leqslant t + \Delta t.$$

Then,

$$\begin{aligned}
\|I_2\|_\gamma &\leqslant \sum_{i=0}^{n-1}\int_{i\Delta t}^{(i+1)\Delta t} \|\boldsymbol{R}_0 \exp\big(-A(n\Delta t - s)\big)\boldsymbol{L}_0^T\|_{0,\gamma}\cdot\|\mathrm{d}F\big(\boldsymbol{u}(s),\boldsymbol{u}(s)\big) \\
&\qquad - \mathrm{d}F\big(\boldsymbol{u}(i\Delta t),\boldsymbol{u}\big((i+1)\Delta t\big)\big)\|_{\gamma,0}\cdot\|\boldsymbol{v}(s)\|_\gamma \,\mathrm{d}s \\
&= \mathcal{O}\bigg(\int_0^{\Delta t}(n\Delta t - s)^{-\gamma}\mathrm{e}^{-a(n\Delta t - s)}\,\mathrm{d}s\bigg)\cdot\|\xi\|_\gamma \\
&\quad + \mathcal{O}\bigg(\int_{\Delta t}^{n\Delta t}(n\Delta t - s)^{-\gamma}\mathrm{e}^{-a(n\Delta t - s)}\frac{\Delta t}{s}\,\mathrm{d}s\bigg)\cdot\|\xi\|_\gamma \\
&= \mathcal{O}\bigg(\int_0^{\Delta t}(n\Delta t - s)^{-\gamma}\,\mathrm{d}s + \int_{\Delta t}^{n\Delta t/2}\frac{(n\Delta t - s)^{-\gamma}}{s}\Delta t\,\mathrm{d}s \\
&\quad + \int_{n\Delta t/2}^{n\Delta t}\frac{(n\Delta t - s)^{-\gamma}}{s}\Delta t\,\mathrm{d}s\bigg)\cdot\|\xi\|_\gamma
\end{aligned}$$

and, for any $\varepsilon > 0$,

$$= \mathcal{O}(\Delta t^{1-\gamma-\varepsilon})\cdot\|\xi\|_\gamma.$$

Integral 3

$$\begin{aligned}
I_3 := \sum_{i=0}^{n-1}\int_{i\Delta t}^{(i+1)\Delta t} &\boldsymbol{R}_0\mathrm{e}^{-A(n\Delta t - s)}\big(I - (I + \beta_q A\Delta t)^{-1}\big)\boldsymbol{L}_0^T \\
&\times \mathrm{d}F\big(\boldsymbol{u}(i\Delta t),\boldsymbol{u}\big((i+1)\Delta t\big)\big)\big[\boldsymbol{v}(s),\boldsymbol{v}(s)\big]\,\mathrm{d}s.
\end{aligned}$$

Note that

$$\int_{(i+1)\Delta t}^{i\Delta t}\mathrm{e}^{-A(n\Delta t - s)}\,\mathrm{d}s = A^{-1}\big(\mathrm{e}^{-A(n-i)\Delta t} - \mathrm{e}^{-A(n-i-1)\Delta t}\big)$$

and that

$$I - (I + \beta_q\Delta t A)^{-1} = \beta_q\Delta t(I + \beta_q\Delta t A)^{-1}A.$$

These equalities and lemma A.1(ii) bound I_3:

$$\begin{aligned}
\|I_3\|_\gamma \leqslant \sum_{i=0}^{n-2}&\|\mathrm{e}^{-A(n-1-i)\Delta t} - \mathrm{e}^{-A(n-i)\Delta t}\|_{0,\gamma}\cdot\|\big(I - (I + \beta_q\Delta t A)^{-1}\big)A^{-1}\|_{0,0} \\
&\times\|\mathrm{d}F\big(\boldsymbol{u}(i\Delta t),\boldsymbol{u}\big((i+1)\Delta t\big)\big)\|_{\gamma,0}\cdot\|\xi\|_\gamma\Delta t \\
&+ \|I - \mathrm{e}^{-A\Delta t}\|_{0,0}\cdot\|\big(I - (I + \beta_q\Delta t A)^{-1}\big)A^{-1}\|_{0,\gamma} \\
&\times\|\mathrm{d}F\big(\boldsymbol{u}((n-1)\Delta t),\boldsymbol{u}(n\Delta t)\big)\|_{\gamma,0}\cdot\|\xi\|_\gamma\Delta t
\end{aligned}$$

$$
\begin{aligned}
&= \mathcal{O}\left(\sum_{i=0}^{n-2} \frac{\Delta t}{((n-1-i)\Delta t)^{\gamma}} \Delta t\right) \cdot \|\xi\|_{\gamma} + \mathcal{O}(\Delta t^{1-\gamma}) \cdot \|\xi\|_{\gamma} \\
&= \mathcal{O}(\Delta t^{1-\gamma}).
\end{aligned}
$$

Integral 4

$$
\begin{aligned}
I_4 := \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} & \boldsymbol{R}_0 \mathrm{e}^{-A(n\Delta t - s)} \boldsymbol{L}_0^T (I + \beta_q A \Delta t)^{-1} \\
& \times \big(\mathrm{d}F\big(\boldsymbol{u}(i\Delta t), \boldsymbol{u}\big((i+1)\Delta t\big)\big) - \mathrm{d}F(\boldsymbol{u}_i, \boldsymbol{u}_{i+1})\big)\big[\boldsymbol{v}(s), \boldsymbol{v}(s)\big] \,\mathrm{d}s.
\end{aligned}
$$

Therefore, by the nonlinear C^0 estimate and lemma A.1(i) and (ii),

$$
\begin{aligned}
\big\|I_4\big\|_{\gamma} \leqslant & \int_0^{\Delta t} \big\|\boldsymbol{R}_0 \exp\big(-A(n\Delta t - s)\big)\boldsymbol{L}_0^T\big\|_{\gamma,\gamma} \cdot \big\|(I + \beta_q A\Delta t)^{-1}\big\|_{0,\gamma} \\
& \times \big\|\mathrm{d}F\big(\boldsymbol{u}(0), \boldsymbol{u}(\Delta t)\big) - \mathrm{d}F(\boldsymbol{u}_0, \boldsymbol{u}_1)\big\|_{\gamma,0} \cdot \big\|\boldsymbol{v}(s)\big\|_{\gamma} \,\mathrm{d}s \\
& + \sum_{i=1}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} \big\|\boldsymbol{R}_0 \exp\big(-A(n\Delta t - s)\big)\boldsymbol{L}_0^T\big\|_{0,\gamma} \cdot \big\|(I + \beta_q \Delta t A)^{-1}\big\|_{0,0} \\
& \times \big\|\mathrm{d}F\big(\boldsymbol{u}(i\Delta t), \boldsymbol{u}\big((i+1)\Delta t\big)\big) - \mathrm{d}F(\boldsymbol{u}_i, \boldsymbol{u}_{i+1})\big\|_{\gamma,0} \cdot \big\|\boldsymbol{v}(s)\big\|_{\gamma} \,\mathrm{d}s \\
= {} & \mathcal{O}(\Delta t^{1-\gamma}) \cdot \big\|\xi\big\|_{\gamma} + \mathcal{O}\left(\sum_{i=1}^{n-1} \Delta t^{1-\gamma}\left(\frac{LR}{i^{1-\gamma}} + L\Delta t^{\delta}\right)\right) \cdot \|\xi\|_{\gamma} \\
= {} & \mathcal{O}(\Delta t^{1-\gamma}) \cdot \|\xi\|_{\gamma}.
\end{aligned}
$$

Integral 5

$$
\begin{aligned}
I_5 := \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} & (I + \beta_q A\Delta t)^{-1} \boldsymbol{R}_0 \big(\mathrm{e}^{-A(n\Delta t - s)} - \mathrm{e}^{-A(n-1-i)\Delta t}\big) \boldsymbol{L}_0^T \\
& \times \mathrm{d}F(\boldsymbol{u}_i, \boldsymbol{u}_{i+1})\big[\boldsymbol{v}(s), \boldsymbol{v}(s)\big] \,\mathrm{d}s.
\end{aligned}
$$

Here we employ lemma A.1(iii):

$$
\begin{aligned}
\big\|I_5\big\|_{\gamma} \leqslant \sum_{i=0}^{n-2} \int_{i\Delta t}^{(i+1)\Delta t} & \big\|(I + \beta_q \Delta t A)^{-1}\big\|_{0,\gamma} \cdot \big\|\mathrm{e}^{-A(n-1-i)\Delta t}\big\|_{1-\gamma,0} \\
& \times \big\|\boldsymbol{R}_0 A^{-1+\gamma}\big(I - \mathrm{e}^{A((n-1-i)\Delta t - (n\Delta t - s))}\big)\boldsymbol{L}_0^T\big\|_{0,0} \\
& \times \big\|\mathrm{d}F(\boldsymbol{u}_i, \boldsymbol{u}_{i+1})\big\|_{\gamma,0} \cdot \big\|\boldsymbol{v}(s)\big\|_{\gamma} \,\mathrm{d}s
\end{aligned}
$$

$$+\int_{(n-1)\Delta t}^{n\Delta t} \big\|(I+\beta_q \Delta t A)^{-1}\big\|_{0,\gamma} \cdot \big\|\boldsymbol{R}_0(\mathrm{e}^{-A(n\Delta t-s)} - \mathrm{e}^{-A(n-1-i)\Delta t})\boldsymbol{L}_0^T\big\|_{0,0}$$

$$\times \big\|\mathrm{d}F(\boldsymbol{u}_i,\boldsymbol{u}_{i+1})\big\|_{\gamma,0} \cdot \big\|\boldsymbol{v}(s)\big\|_\gamma \,\mathrm{d}s$$

$$= \mathcal{O}\left(\sum_{i=0}^{n-2} \frac{1}{(n-1-i)} \Delta t^{1-\gamma}\right) \cdot \|\xi\|_\gamma + \mathcal{O}(\Delta t^{1-\gamma}) \cdot \|\xi\|_\gamma.$$

Now, as $\sum_{i=0}^{n} 1/i = \mathcal{O}(\Delta t^{-\varepsilon})$ for any $\varepsilon > 0$,

$$\|I_5\|_\gamma = \mathcal{O}(\Delta t^{1-\gamma-\varepsilon}) \cdot \|\xi\|_\gamma, \quad 0 < \varepsilon < 1-\gamma.$$

Integral 6

$$I_6 := \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} (I+\beta_q A\Delta t)^{-1} \big(\boldsymbol{R}_0 \mathrm{e}^{-(n-1-i)A\Delta t} \boldsymbol{L}_0^T - \mathcal{C}^{n-1-i}(A\Delta t)\big)$$

$$\times \mathrm{d}F(\boldsymbol{u}_i,\boldsymbol{u}_{i+1})[\boldsymbol{v}(s),\boldsymbol{v}(s)] \,\mathrm{d}s.$$

This term is bounded by another application of the linear estimate:

$$\big\|I_6\big\|_\gamma \leqslant \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} \big\|(I+\beta_q \Delta t A)^{-1}\big\|_{0,\gamma}$$

$$\times \big\|\boldsymbol{R}_0 \exp\big(-A(n-1-i)\Delta t\big) \boldsymbol{L}_0^T - \mathcal{C}^{n-1-i}(A\Delta t)\big\|_{0,0}$$

$$\times \big\|\mathrm{d}F(\boldsymbol{u}_i,\boldsymbol{u}_{i+1})\big\|_{\gamma,0} \cdot \big\|\boldsymbol{v}(s)\big\|_\gamma \,\mathrm{d}s$$

$$= \mathcal{O}\left(\sum_{i=0}^{n-2} \Delta t^{1-\gamma} \frac{1}{n-1-i}\right) \cdot \|\xi\|_\gamma + \mathcal{O}(\Delta t^{1-\gamma}) \cdot \|\xi\|_\gamma$$

$$= \mathcal{O}(\Delta t^{1-\gamma-\varepsilon}) \cdot \|\xi\|_\gamma, \quad 0 < \varepsilon < 1-\gamma.$$

Integral 7

$$I_7 := \sum_{i=0}^{n-1} \int_{i\Delta t}^{(i+1)\Delta t} \mathcal{C}^{n-1-i}(A\Delta t) \cdot (I+\beta_q A\Delta t)^{-1}$$

$$\times \mathrm{d}F(\boldsymbol{u}_i,\boldsymbol{u}_{i+1})\big[\boldsymbol{v}(i\Delta t) - \boldsymbol{v}_i, \boldsymbol{v}\big((i+1)\Delta t\big) - \boldsymbol{v}_{i+1}\big] \,\mathrm{d}s.$$

Using the linear estimate, lemma 3.2, we have

$$\big\|\mathcal{C}^n(A\Delta t)(I+\beta_q \Delta t A)^{-1}\big\|_{0,\gamma} \leqslant \big\|\exp(-An\Delta t) - \mathcal{C}^n(A\Delta t)\big\|_{\gamma,\gamma} \cdot \big\|(I+\beta_q \Delta t A)^{-1}\big\|_{0,\gamma}$$

$$+ \big\|\exp(-An\Delta t)\big\|_{0,\gamma} \cdot \big\|(I+\beta_q A\Delta t)^{-1}\big\|_{0,0}$$

$$= \mathcal{O}(\Delta t^{-\gamma})/n + \mathcal{O}((n\Delta t)^{-\gamma})$$

$$= \mathcal{O}(\Delta t^{-\gamma}) n^{-\gamma}.$$

Consequently,

$$\begin{aligned}\|I_7\|_\gamma &\leqslant \sum_{i=0}^{n-1}\int_{i\Delta t}^{(i+1)\Delta t} \|\mathcal{C}^{n-1-i}(A\Delta t)(I+\beta_q\Delta tA)^{-1}\|_{0,\gamma}\cdot\|\mathrm{d}F(\boldsymbol{u}_i,\boldsymbol{u}_{i+1})\|_{\gamma,0}\\ &\qquad\times\big[\|D_i\|_\gamma+\|D_{i+1}\|_\gamma\big]\,\mathrm{d}s\\ &=\mathcal{O}(\Delta t^{1-\gamma})\cdot\left[\sum_{i=0}^{n-1}\frac{1}{(n-i)^\gamma}\|D_i\|_\gamma+\|D_n\|_\gamma\right].\end{aligned}$$

Integral 8

$$\begin{aligned}I_8 := \sum_{i=0}^{n-1}\int_{i\Delta t}^{(i+1)\Delta t} &\mathcal{C}^{n-1-i}(A\Delta t)(I+\beta_qA\Delta t)^{-1}\\ &\times \mathrm{d}F(\boldsymbol{u}_i,\boldsymbol{u}_{i+1})\big[\boldsymbol{v}(s)-\boldsymbol{v}(i\Delta t),\boldsymbol{v}(s)-\boldsymbol{v}\big((i+1)\Delta t\big)\big]\,\mathrm{d}s.\end{aligned}$$

By applying theorem 2.2,

$$\begin{aligned}\|I_8\|_\gamma &\leqslant \sum_{i=0}^{n-1}\int_{i\Delta t}^{(i+1)\Delta t} \|\mathcal{C}^{n-1-i}(A\Delta t)\|_{\gamma,\gamma}\cdot\|(I+\beta_q\Delta tA)^{-1}\|_{0,\gamma}\cdot\|\mathrm{d}F(\boldsymbol{u}_i,\boldsymbol{u}_{i+1})\|_{\gamma,0}\\ &\qquad\times\big[\|\boldsymbol{v}(s)-\boldsymbol{v}(i\Delta t)\|_\gamma+\|\boldsymbol{v}(s)-\boldsymbol{v}\big((i+1)\Delta t\big)\|_\gamma\big]\,\mathrm{d}s\\ &=\mathcal{O}(\Delta t^{1-\gamma})\cdot\|\xi\|_\gamma+\mathcal{O}\left(\sum_{i=1}^{n-1}\Delta t^{-\gamma}\cdot\frac{\Delta t}{i\Delta t}\cdot\Delta t\right)\cdot\|\xi\|_\gamma\\ &=\mathcal{O}(\Delta t^{1-\gamma-\varepsilon})\|\xi\|_\gamma,\quad 0<\varepsilon<1-\gamma.\end{aligned}$$

Summing the terms, we find that, for $0<\delta<1-\gamma$,

$$\|D_n\|_\gamma=\mathcal{O}(1)\cdot\frac{1}{n}+\mathcal{O}(\Delta t^\delta)+\mathcal{O}(\Delta t^{1-\gamma})\left[\sum_{i=0}^{n-1}\frac{1}{(n-i)^\gamma}\|D_i\|_\gamma+\|D_n\|_\gamma\right].$$

An application of the Gronwall lemma, as in [4], completes the proof. □

Acknowledgements

I thank Andrew Stuart for suggesting this problem and for reading the paper in detail. I also thank Don Jones and Adrian Hill for discussions about their work.

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