Interaction of waves in a one dimensional stochastic PDE model of excitable media

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1 Introduction

The phenomenon of waves in excitable media, such as travelling pulses and spiral waves, has received a significant amount of attention due to their occurrence in a wide range of natural systems [26, 17]. The propagation of travelling waves along nerve fibres [22, 28] and a chemically-active medium with the Belousov–Zhabotinsky reaction [27] are typical examples; see [19, 28] for further examples of excitable media. An excitable medium is a non-linear, spatially-extended system, characterized by three states: the rest, excited, and refractory or recovery states [28]. Under a sufficiently strong stimulus, the excitable system switches from the rest state to the excited state and then falls into the refractory state after a short time, before returning to the rest state. Subsequent excitation cannot be generated until a suitable amount of time, known as refractory time, has passed. Spatial coupling causes a rest state to become excited when neighbouring an excited state and waves to be produced [17, 28]. A generic excitable medium can be represented simply by a two-variable system of reaction-diffusion equations, such as the Barkley model [1] or the FitzHugh Nagumo system [5, 20]. Adding noise to these systems increases the production of spatio-temporal patterns, such as spiral waves and travelling pulses [11, 10]. Noise, generally, can affect the excitable media in various ways; see [13] for an extensive review and the references given there.

In this paper, we are concerned with the Barkley model in one dimension subject to space-time additive noise [23],

$$\begin{aligned} du &= \Big(D\frac{\partial^2 u}{\partial x^2} + f(u,v)\Big)dt + \sigma dw(t,x) \\ dv &= g(u,v)dt, \end{aligned} \tag{1.1}$$

with initial conditions $u(0,x) = u_0(x)$, $v(0,x) = v_0(x)$ and periodic boundary conditions on domain $[0,L]$. D is a diffusion coefficient and $\sigma > 0$ is a small noise parameter. $w(t,x)$ is a Wiener process on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, white in time and correlated in space as [24, 7]

$$\mathbf{E}w(t,x)w(s,y) = \min\{t,s\}C(x-y), \quad C(x-y) = \frac{1}{2\xi}\exp\Big(\frac{-\pi(x-y)^2}{4\xi^2}\Big), \tag{1.2}$$

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where C, a function of $x-y$, describes the spatial correlation, the parameter ξ is the spatial correlation length, and $\mathbf{E}$ denotes expectation with respect to the probability measure $\mathbf{P}$. This type of covariance is known as the squared exponential covariance function, and is stationary and thus invariant to translations [21].

We assume the reaction terms take the form:

$$\tilde{f}(u,v)=\begin{cases} f(u,v), & u\leq 1,\\ -|f(u,v)|, & u\geq 1,\end{cases}, \qquad \tilde{g}(u,v)=\begin{cases} g(u,v), & v\geq 0,\\ |g(u,v)|, & v<0,\end{cases}$$

where

$$\begin{aligned} f(u,v) =& \frac{1}{\epsilon}u(1-u)\Big(u-\frac{v+b}{a}\Big),\\ g(u,v) =& u-v, \end{aligned}$$

for $a,b>0$. Following [1], we set $a=0.75$, and $b=0.01$. The small parameter $0<\epsilon\ll 1$ represents the time-scale separation of the fast variable u and the slow variable v. We choose parameters in the sub-excitable regime, where the system can produce waves from the homogeneous state when appropriate amounts of noise are added though no structure can be nucleated under purely deterministic conditions [8]. The choice of $\tilde{f},\tilde{g}$ in place of f,g avoids an instability not present in the deterministic model for initial data in $(u(x),v(x))\in[0,1]^2$, which may be stimulated by additive noise [24].

The nucleation and dynamics of patterns in spatially-extended systems have been studied extensively, in particular for the ϕ^4- equation in one-dimension with additive space-time white noise [3]. The patterns in this case are defined by kinks, and their nucleation, propagation and eventual annihilation are worth studying. A kink is defined here as a boundary with a region close to 0 to its left and a region close to 1 to its right; the opposite case is called an anti-kink [15].

Figure 1: Nucleation of the (a) front wave and (b) back wave of a travelling wave. (c) Illustration of the kinks and anti-kinks of the front and back waves. The parameters used are $a=0.75$, $b=0.01$, $\epsilon=0.02$, $D=1$ and $L=40$ with noise of correlation length $\xi=2$ and intensity $\sigma=0.09$.

We study kinks and anti-kinks, their nucleation and dynamics, in the Barkley model with additive noise on the periodic interval $[0,L]$. A nucleation event due to noise consists of the following dynamics illustrated in Figure 1: (a) the noise causes the fast variable u to cross an excitability threshold in some part of phase space and two wave fronts appear propagating in opposite directions; (b) the excitation endures for a period of time before entering a refractory state, causing a wave back to appear some fixed distance behind the wave front; (c) two separate solitary structures have formed, comprising two pairs of wave fronts and wave backs travelling in opposite directions. The left(right) sides of front waves and the right(left) sides of back waves are examples of kink(anti-kink) structures. The nucleation time can be defined as the first time the maximum value of the fast variable u passes the threshold of excitability, which we treat as a first exit problem. The kinks and anti-kinks are nucleated at random times and in random positions. As in Figure 2–3, they travel in opposite directions and due to the periodic structure are annihilated in collision [9]. We will study excitation, nucleation times, and kink lifetimes.

Figure 2: Plots of the nucleation and annihilation of a wave for the Barkley model(1.1). The parameters of the system are $a = 0.75$, $b = 0.01$, $\epsilon = 0.02$, $D = 1$ and $L = 40$ with noise of correlation length $\xi = 2$ and intensity $\sigma = 0.09$.

Figure 3: (a) Space-time contour plot of dynamical behaviour of the Barkley model (1.1) with parameters values used in Figure 2. (b) Simulation of the dynamical behaviour of the Barkley model using the reduced model. N_1 and N_2 are nucleation points of the front and back waves, respectively. A_1 and A_2 are corresponding points of annihilation.

The paper is arranged as follows. In §2, we review the dynamics of the Barkley model without noise and in §3 we describe a numerical method for the Barkley model with noise (i.e., the stochastic PDE (1.1)) with a view to determining the mean value for lifetime of a wave. However, making accurate calculations over long time intervals requires a large amount of computer time and this approach becomes computationally impractical in the case of interacting waves. In §4, a reduced model is given that can deal with interacting waves adequately. Numerical experiments with the stochastic PDE and the reduced model are described in §5, where we calculate the mean number of kinks at a specific time and the probability of a given part of the phase space of the Barkley system being excited.

2 Dynamics of the Barkley model

Figure 4: The Barkley model in the absence of diffusion ($D = 0$), for $a = 0.75$, $b = 0.01$, $\epsilon = 0.02$. The u and v nullclines intersect at the stable fixed point $O = (0, 0)$ with excitation threshold $u = \frac{v+b}{a}$. For initial conditions near O and to the left of the threshold $u = \frac{v+b}{a}$, the system decays directly to the fixed point O as shown for $P_2 = (0.4, 0.3)$. For the initial conditions located to the right of the excitation threshold such as $P_1 = (0.25, 0.15)$, the system undergoes a large excursion before returning to the fixed point O. The small boundary layer δ is plotted as well.

For the deterministic Barkley model, the dynamics of the reaction kinetics (or, in other words, the dynamics of the model in the absence of diffusion) is illustrated in Figure 4, with nullclines shown for u and v. The u-nullclines ($f(u, v) = 0$) are represented by three straight lines: $u = 0, u = 1, u = \frac{v+b}{a}$, whereas v-nullclines ($g(u, v) = 0$) is the line $u = v$. By setting a small boundary, say δ, bordering the line $u = 0$, a given point (u, v) is said to be excited if $u > \delta$. For $u \leq \delta$, we say the system is recovering if $v \geq \delta$ and at rest if $v < \delta$. The physical parameters ϵ, a and b specify the details of the local dynamics. ϵ is selected to be very small, so that the activator u is much faster than the inhibitor v within the excited region. However, $u \approx 0$ within the recovery region and therefore the exponential decay of the inhibitor v affects only the local dynamics [1]. Larger a would increase the duration of the excitation and a larger value of $\frac{b}{a}$ would raise the threshold of the excitation [2].

The intersection of all the nullclines yields the fixed points $(0, 0)$ and $(1, 1)$. The origin $(0, 0)$ is the stable and excitable fixed point of the model, with excitation threshold $u_{\text{th}} = \frac{v+b}{a}$. To be precise, when the starting point is located to the left of the threshold u_{th} and close to

the origin $(0,0)$, the solution to the ordinary differential equations (ODEs), $\frac{du}{dt}=0$, $\frac{dv}{dt}=0$, converges directly to the origin fixed point. In contrast, when the starting point is to the right of u_{th} , the solution moves away initially, before finally shrinking to $(0,0)$ [1].

Adding spatial diffusion to these reaction kinetics leads to the propagation of waves for certain initial data. It will be necessary to understand the wave speed and width (the separation between wave front and back) for the deterministic Barkley system,

$$\begin{aligned}\frac{\partial u}{\partial t} &= D\frac{\partial^2 u}{\partial x^2}+\frac{1}{\epsilon}u(1-u)\Big(u-\frac{v+b}{a}\Big)\\ \frac{\partial v}{\partial t} &= u-v. \end{aligned} \tag{2.1}$$

To this end, we set $z=x-ct$, and so $u(t,x)=U(x-ct)=U(z)$ and $v(t,x)=V(x-ct)=V(z)$. Consequently, we get

$$u_t=-c\frac{dU}{dz}=-cU' \quad , \quad \frac{\partial^2 u}{\partial x^2}=\frac{d^2U}{dz^2}=U'' \quad \text{and} \quad v_t=-cV'.$$

Substituting these into (2.1) and setting $D=1$ yields

$$\begin{aligned} U''+cU'+\frac{1}{\epsilon}U(1-U)\Big(U-\frac{V+b}{a}\Big) &= 0,\\ cV'-V+U &= 0, \end{aligned}$$

with boundary conditions $U\to 0$, $U'\to 0$ and $V\to 0$ as $|z|\to\infty$. Now, supposing $\epsilon\to 0$, which implies $\frac{\partial v}{\partial t}\to 0$, and consequently, $V\approx \text{const}=0$, we obtain

$$L(U)=U''+cU'+\frac{1}{\epsilon}U(1-U)\Big(U-\frac{b}{a}\Big)=0. \tag{2.2}$$

Let us suppose that $U'=AU(U-1)$, where A is a constant that needs to be determined [25]. It follows that

$$U''=A^2U(U-1)(2U-1).$$

Substituting these expressions for U' and U''into (2.2) gives [19]

$$\begin{aligned} L(U)=A^2U(U-1)(2U-1)+cAU(U-1)+\frac{1}{\epsilon}U(1-U)\Big(U-\frac{b}{a}\Big) &= 0\\ U(1-U)\Big[A^2(1-2U)-cA+\frac{1}{\epsilon}\Big(U-\frac{b}{a}\Big)\Big] &= 0\\ U(1-U)\Big[\Big(-2A^2+\frac{1}{\epsilon}\Big)U-cA+A^2-\frac{b}{a\epsilon}\Big] &= 0. \end{aligned}$$

Now, setting $-2A^2+\frac{1}{\epsilon}=0$ and $-cA+A^2-\frac{b}{a\epsilon}=0$ yields $L(U)=0$. This gives $A=\frac{1}{\sqrt{2\epsilon}}$ and $cA=A^2-\frac{b}{a\epsilon}$. The wave speed can then be approximated as

$$c=\frac{1}{\sqrt{2\epsilon}}\Big(1-\frac{2b}{a}\Big). \tag{2.3}$$

Following [6], the width $W=cT_{\text{diff}}$, where T_{diff} is the time the system (2.1) spends in the excited state and c is the wave speed. As $\epsilon\to 0$, we have [6]

$$T_{\text{diff}}=\int_0^{v_m}\frac{dv}{g(1,v)}=\int_0^{v_m}\frac{dv}{1-v},$$

where v_m represents the maximum value that the inhibitor v over x reaches before the activator u leaves the excited level [6]. Consequently,

$$T_{\text{diff}} = \int_0^{v_m} \frac{dv}{(1-v)} = \Big[-\ln(1-v) \Big]_0^{v_m} = -\ln(1-v_m) = \ln\Big(\frac{1}{1-v_m}\Big). \tag{2.4}$$

Hence, the width can be written as [6]

$$W = c \ln\Big(\frac{1}{1-v_m}\Big). \tag{2.5}$$

We will assume that $1 = (v_m + b)/a = u_{\text{th}}$ so that $v_m = a - b$.

3 Numerical approximation of the stochastic PDE

To simulate the stochastic PDE (1.1) numerically, we approximate the Laplacian and the Wiener process $w(t,x)$. We define approximations u_k^n,v_k^n, for $k = 0, 1, 2, \cdots, N-1$ and $n \in \mathbf{N}$ at time $t_n = n\Delta t$ to $u(t_n, x_k)$, $v(t_n, x_k)$ where Δt is the time step, $x_k = \frac{kL}{N}$ is the spatial grid, and L is the length of spatial domain.

Approximating the Wiener process We approximate $w(x,t)$ and the increment $w_k^n = w(t_{n+1}, x_k) - w(t_n, x_k)$ using a spectral method. Assume $w(t,x)$ has the expansion [24, 23]

$$w(t,x) = \sum_{j=0}^{\infty} \alpha_j \Big[e_j(x)\beta_j(t) + \tilde{e}_j(x)\tilde{\beta}_j(t) \Big], \tag{3.1}$$

for independent standard Wiener processes $\beta_j(t)$, $\tilde{\beta}_j(t)$. Here,

$$e_j(x) = \sqrt{\frac{2}{L}} \cos\Big(\frac{2\pi j x}{L}\Big), \quad j = 1, 2, 3, \cdots, \quad e_0(x) = \sqrt{\frac{1}{L}},$$

and

$$\tilde{e}_j(x) = \sqrt{\frac{2}{L}} \sin\Big(\frac{2\pi j x}{L}\Big), \quad j = 0, 1, 2, 3, \cdots$$

are orthonormal eigenfunctions of the Laplacian on $[0, L]$ with periodic boundary conditions. The coefficients α_j are determined as [24]

$$\alpha_j = \frac{1}{2} \exp\Big(\frac{-\lambda_j \xi^2}{2\pi}\Big), \quad j = 1, 2, 3, \cdots, \quad \text{for} \quad \lambda_j = \Big(\frac{2\pi j}{L}\Big)^2 \quad \text{and} \quad \alpha_0 = 1 \quad \text{for} \quad \lambda_0 = 0.$$

It follows that

$$w(t,x) = \alpha_0 \sqrt{\frac{1}{L}} \beta_0(t) + \sqrt{\frac{2}{L}} \sum_{j=1}^{\infty} \alpha_j \Big[\cos\Big(\frac{2\pi j x}{L}\Big)\beta_j(t) + \sin\Big(\frac{2\pi j x}{L}\Big)\tilde{\beta}_j(t) \Big]. \tag{3.2}$$

For numerical purposes, we need to calculate the approximation $w_k^n = w(t_{n+1}, x_k) - w(t_n, x_k)$ at $t_n = n\Delta t$, where Δt is the time step, and at $x_k = \frac{kL}{N}$, where x_k is the spatial grid, and

$k = 0, 1, 2, \cdots, N-1$. By substituting x_k for x in (3.2) and truncating the series, σw_k^n is given by [12]

$$\sigma w_k^n = \alpha_0 \eta_0^n + \sqrt{2} \sum_{j=1}^{\frac{N}{2}} \alpha_j \Big[\cos\Big(\frac{2\pi jk}{N}\Big) \eta_{j,1}^n + \sin(\frac{2\pi jk}{N}\Big) \eta_{j,2}^n \Big], \tag{3.3}$$

where $k = 0, 1, 2, \cdots, N-1$, σ is the noise intensity, and $\eta_0^n, \eta_{j,1}^n, \eta_{j,2}^n \sim N(0, \frac{\sigma^2 \Delta t}{L})$ are identical and independent normally distributed random variables. The exponential decay in spatial correlation provides very rapid decay in the Fourier coefficient in expansion (3.1) and therefore the Wiener process can be generated efficiently using FFTs[24]. Expressed in terms of complex exponentials: [24]

$$\cos\Big(\frac{2\pi jk}{N}\Big) \eta_{j,1}^n + \sin\Big(\frac{2\pi jk}{N}\Big) \eta_{j,2}^n = \frac{1}{2} \Big[z_j^n e^{\frac{2i\pi jk}{N}} + \bar{z}_j^n e^{\frac{-2i\pi jk}{N}} \Big], \tag{3.4}$$

where $z_j^n = \eta_{j,1}^n - i\eta_{j,2}^n$, $\bar{z}_j^n = \eta_{j,1}^n + i\eta_{j,2}^n$ and $i = \sqrt{-1}$. Now (3.3) becomes

$$\sigma w_k^n = \alpha_0 \eta_0^n + \frac{\sqrt{2}}{2} \sum_{j=1}^{\frac{N}{2}} \alpha_j \Big[z_j^n e^{\frac{2i\pi jk}{N}} + \bar{z}_j^n e^{\frac{-2i\pi jk}{N}} \Big]. \tag{3.5}$$

This is easily evaluated using a FFT by writing [24]

$$\sigma w_k^n = \sum_{j=0}^{N-1} \hat{\alpha}_j Z_j^n e^{\frac{2i\pi jk}{N}}, \tag{3.6}$$

where $\hat{\alpha_0} = 1$, $\hat{\alpha_j} = \hat{\alpha}_{N-j} = \frac{1}{\sqrt{2}} \alpha_j$ for $j = 1, 2, \cdots, N/2$, $Z_0^n = \eta_0^n$ and

$$Z_j^n = \begin{cases} z_j^n, & j = 1, 2, \cdots, \frac{N}{2}, \\ \bar{z}_{N-j}^n, & j = \frac{N}{2}+1, \frac{N}{2}+2, \cdots, N-1. \end{cases}$$

Time stepping A geometric integrator is used for the time discretisation. First, look at the one-dimensional diffusion equation:

$$\frac{\partial u}{\partial t} = D \frac{\partial^2 u}{\partial x^2}, \quad u(0, x) = f(x), \tag{3.7}$$

on the interval $[0, L]$ with boundary conditions $u(t, 0) = u(t, L)$, where $t \geq 0$ and the assumption that $f \in L^2[0, L]$. For the Fourier expansion, an approximation of $u(t, x)$ can be obtained as [4]

$$u_N(t, x) = \sum_{k=0}^{N-1} \tilde{u}_k(t) e^{\frac{-2\pi ikx}{L}}. \tag{3.8}$$

From

$$\frac{\partial u_N}{\partial t} = D \frac{\partial^2 u_N}{\partial x^2}, \tag{3.9}$$

we have

$$\sum_{k=0}^{N-1} \frac{d\tilde{u}_k}{dt} e^{\frac{-2\pi ikx}{L}} = \sum_{k=0}^{N-1} D \frac{-4\pi^2 k^2}{L^2} \tilde{u}_k(t) e^{\frac{-2\pi ikx}{L}}, \tag{3.10}$$

which leads to the system of ODEs

$$\frac{d\tilde{u}_k}{dt} = -D\frac{4\pi^2k^2}{L^2}\tilde{u}_k(t), \quad \tilde{u}_k(0) = B_k, \quad k = 0, 1, 2, \cdots, N-1, \tag{3.11}$$

where $B_k = \frac{1}{L}\int_0^L f(x)e^{\frac{2\pi ikx}{L}}dx$ are the Fourier coefficients of the initial function $f(x)$. Setting $\lambda_k = (\frac{4\pi^2k^2}{L^2})$ yields

$$\frac{d\tilde{u}_k}{dt} = -D\lambda_k\tilde{u}_k(t), \quad \tilde{u}_k(0) = B_k, \quad k = 0, 1, 2, \cdots, N-1. \tag{3.12}$$

The exact solution of this system is given by

$$\tilde{u}_k(t) = B_k e^{-D\lambda_k t}, \quad t \geq 0, \quad k = 0, 1, 2, \cdots, N-1. \tag{3.13}$$

Numerically, we use the the exponential Euler method

$$\tilde{u}_k^{n+1} = e^{-D\lambda_k\Delta t}\tilde{u}_k^n, \quad \tilde{u}_k^0 = B_k, \quad k = 0, 1, 2, \cdots, N-1, \quad n = 0, 1, 2, \cdots \tag{3.14}$$

which gives the exact solutions to the linear system (3.12) at mesh points t_n(i.e. $\tilde{u}_k(t_n) = \tilde{u}_k^n$) and is thus considered a linearization-preserving (geometric) integrator. For more details of exponential integrators, see for instance [18], or for more on linearization-preserving methods, see [16] and the further references given there.

Now we use the above discretisations in the case of (1.1), following [24, 14] in the following manner:

- Calculate the Fourier coefficient $\hat{u}_k^n$, where $u_k^n = \sum_{j=0}^{N-1}\hat{u}_j^n[e_j(x_k) + \tilde{e}_j(x_k)]$ and $k = 0, 1, \cdots, N-1$ by the FFT.
- $\hat{u}_k^{n+\frac{1}{2}} = e^{-D\lambda_k\Delta t}\hat{u}_k^n + \hat{\alpha_k}Z_k^n$, where $\hat{\alpha_k}$ and Z_k^n are as defined in (3.6).
- $u_k^{n+\frac{1}{2}} = \text{ifft}(\hat{u}_k^{n+\frac{1}{2}})$ where ifft is the inverse FFT.
- Apply the modified reaction terms:

$$\begin{aligned} u_k^{n+1} &= u_k^{n+\frac{1}{2}} + \Delta t\tilde{f}(u_k^{n+\frac{1}{2}}, v_k^n) \\ v_k^{n+1} &= v_k^n + \Delta t\tilde{g}(u_k^{n+\frac{1}{2}}, v_k^n). \end{aligned}$$

Using this numerical approach, we will calculate the mean first exit time or the mean of nucleation time of a travelling wave and its mean lifetime, for the Barkley system (1.1).

Figure 5: Illustration of the threshold of nucleation of waves for the Barkley model using maximum value of activator u over x. The parameters used are as in Figure 2 except for ϵ.

Nucleation times We now study the time for a wave to be nucleated when started from the homogeneous rest state $u = v = 0$. We use the infinity norm to formulate the problem as a first exit time problem for a threshold θ, so that the nucleation of a wave occurs when the infinity norm, or precisely, the maximum value $u_{\max}$, of the activator variable u over x as a function of time, exceeds θ. As in the case illustrated in Figure 5, the threshold level is assumed to be $\theta = 0.275$. The nucleation time of a wave is, therefore, the first time the threshold level is exceeded. Thus, the wave is nucleated when $u_{\max}$ goes above θ for the first time and then is annihilated when $u_{\max}$ becomes smaller than the small boundary layer $\delta = 0.008$. The lifetime is then the difference between the time of nucleation and the time of annihilation.

The lifetime is a random variable and therefore its mean is of great interest. The mean lifetime is simply calculated as the average over a sample of random lifetimes produced using the SPDE (1.1) over the spatial interval $[0, L]$. This technique can be applied when L is small and consequently a single wave is nucleated at time T as shown in Figure 2. This technique is used in §5. However, in the case of large L and when many waves are nucleated at almost the same time, this approach becomes computationally impracticable. To overcome this drawback, we develop a simple model of the dynamics of the underlying SPDE (1.1), which allows us to compute the mean lifetime of the generated waves, even for a large domain.

4 The reduced model

Figure 6: The wave speed as a function of the noise parameter σ with initial condition $u_0(x) = 1_{[1,5]}(x)$ (the indicator function on $[1, 5] \subset [0, L]$) and $v_0 = 0$. The parameters $\epsilon = 0.02$, $L = 40$, (a) $N = 512$, $\Delta t = 0.01$ and (b) $N = 1024$, $\Delta t = 0.001$.

We derive a reduced model for the dynamics of the kinks and anti-kinks. For this purpose, we need the wave speed and width of the wave. Unlike under multiplicative noise, the characteristics of propagated waves are not affected, generally, by small additive noise [7, 6]. In Figures 6, the wave speed and width obtained from simulating the SPDEs (1.1) are plotted as functions of the noise parameter σ, for $\epsilon = 0.02$. The results indicate that both remain approximately constant over different values of σ. Since we are looking at small additive noise in our work, we expect the wave speed and width to be close to the deterministic case. Numerical evidence given in Figures 6 supports this conclusion and theoretical values of wave speed and width can be considered good approximations to the corresponding quantities obtained in the simulation of the SPDEs (1.1), in particular for high resolution N and small time step Δt.

In order to construct the reduced model, we first model the wave nucleation positions and times. The Wiener process $w(t, x)$, with covariance given by (1.2), is a spatially stationary Gaussian process, since the covariance is a function of the increment $x-y$, where $0 \leq x < y \leq L$. The process is thus invariant to shifts in space and, therefore, the distribution of nucleation events, which are caused by the input noise, is uniform in space. However, no wave can be nucleated within a wave, as is observed from the dynamic behaviour of the SPDE (1.1). Thus the position of a nucleation at time t is assumed to be uniformly distributed over $[0, L] - \mathcal{W}$, where $\mathcal{W}$ presents the excited region or region bounded by the union of all front and back waves at time t.

We now model the nucleation times of the front and back waves under the Barkley model. Suppose $T_i, i = 1$ to N_1 are the nucleation times of front waves. Since the nucleation events occur with a constant probability per unit of time, the exponential distribution with rate λ is appropriate for modelling the nucleation times. The rate of nucleation, λ, is calculated simply as the inverse of the mean first exit time, $\lambda = 1/T_m$, obtained from the simulation of the SPDE (1.1). The nucleation times of the back waves, $\bar{T}_i$ are given by $\bar{T}_i = T_i + T_{\text{diff}}$, where T_{diff} is given by (2.4).

Figure 7: Plots of the nucleation and annihilation of a wave for the Barkley model (1.1) with $\epsilon = 0.02$ and using the reduced model.

We are now in a position to simulate the spatio-temporal behaviour of the front and back waves under the Barkley model (1.1). To this end, suppose Y_L and Y_R represent the position of front waves towards the left and right, respectively as shown in Figure 7. Thus,

$$\begin{aligned} Y_L(t) &= x_i - c(t - T_i), \quad t \geq T_i, \\ Y_R(t) &= x_i + c(t - T_i), \end{aligned} \tag{4.1}$$

where x_i is the nucleation position of the front wave, T_i is its nucleation time, c is the constant wave speed and $t > 0$ is the time variable. Similarly,

$$\begin{aligned} \bar{Y}_L(t) &= x_i - c(t - \bar{T}_i), \quad t \geq \bar{T}_i, \\ \bar{Y}_R(t) &= x_i + c(t - \bar{T}_i), \end{aligned}$$

are the corresponding expressions for the back waves.

As observed from the numerical simulation of the SPDE (1.1), a kink (anti-kink) at the spatial position, i say, will keep moving until it meets an anti-kink (kink) at position, j say, usually from a different nucleation process. Both the kink and the anti-kink then annihilate on collision. To simulate such an annihilation event using the reduced model, we first solve the equations for the front waves (4.1) (a similar process is then used for the back waves), in order to find all the crossing points (x^c_{ij}, t^c_{ij}) for i moving right and j moving left, where $i \neq j$ and, $i, j = 1, 2, \cdots, N_1$. We then set the minimum value of t^c_{ij}, $T^n_{i_1 j_1}$ say, to be the annihilation time, and the corresponding value of x^c_{ij}, $x^n_{i_1 j_1}$ say, to be the annihilation position of i_1 moving to the right and j_1 moving to the left. We then continue in this fashion to compute all the annihilation points of other i right waves and j left waves, where $i \neq i_1$ and $j \neq j_1$. However, as a result of the chosen finite time interval in the simulation, a few so-called survivor particles (kinks or anti-kinks) will still be alive. A survivor particle with nucleation position x_k and nucleation time T_k will never meet another particle during the finite time interval of the simulation. The particle then will move according to (4.1) as

$$\begin{aligned} Y_L(t) &= x_k - c(t - T_k) \quad t \geq T_k, \\ Y_R(t) &= x_k + c(t - T_k), \end{aligned}$$

with the possibility that t can take the value infinity. However, for the purposes of plotting the values, we assume that the particle will be killed at time $T_k + T_{extra}$ where T_{extra} is chosen to be a large value. The corresponding surviving back wave can be treated in a similar way.

It remains only to include the periodic boundary conditions of the Barkley system in order to complete the reduced model algorithm. To do this, we duplicate the spatio-temporal

dynamic behaviour of the underlying SPDE on $[-L, 0]$ and $[L, 2L]$ as reflections of its dynamics on $[0, L]$. This simple technique guarantees that the periodic boundary conditions, as illustrated in Figure 8 are fulfilled.

Figure 8: Simulation of the Barkley model using the reduced model when L is large and many waves are nucleated. The parameters values are $a = 0.75$, $b = 0.01$, $\epsilon = 0.02$, $D = 1$ and $L = 400$.

5 Computations

Figure 9: The mean first exit time (MFET) as a function of (a) Δt and (b) N, with parameters used are as in Figure 2.

We start by computing the mean first exit time (MFET) using numerical simulations of the stochastic PDE. Using the numerical methods of §3 with $\theta = 0.0275$, we approximate the MFET and now describe two numerical experiments.

MFET We first plot the MFET, denoted by T_m, as a function of the time step Δt (Figure 9(a)) and as a function of the resolution N, where the values of N are powers of 2: 128, 256, 512 and 1024, as shown in Figure 9(b). The parameters used are $a = 0.75$,$b = 0.01$,$\epsilon = 0.02$, $D = 1$, and the averages are taken over $M = 10^4$ realizations. The noise has a spatial correlation of length $\xi = 2$ and with intensity $\sigma = 0.09$ over the domain $[0, L]$, where $L = 40$. In Figure 9(a), the resolution is $N = 512$ and in Figure 9(b), the time step Δt is 0.01. The results reveal that the MFET T_m is approximately independent of the value of Δt and of the value of $\Delta x = \frac{L}{N}$, and therefore, approximately constant values of T_m are obtained in both cases.

In the second case, T_m is plotted against the small parameter ϵ, in Figure 10(a), and the spatial correlation length ξ, that is, the range of similarity between nearby points, in Figure 10(b), with other parameters as for Figure 9. Figure 10(a) shows that just a slight increase in ϵ produces a dramatic increase in the MFET. In Figure 10(b), meanwhile, we observe that the MFET grows as ξ increases.

Figure 10: The mean first exit time (MFET) as a function of (a) ϵ and (b) ξ, with parameters used are as in Figure 2.

When the spatial domain becomes much larger, one may expect that the MFET will be reduced considerably. To investigate this, we raise L to 400, and set $N = 1024$, with other parameters remaining as they were for Figure 9(a). For $L = 40$, the MFET is $T_{m1} = 22.7521$, while for $L = 400$, it is $T_{m2} = 17.7819$. Although the length of the spatial domain in the second case ($L = 400$) is ten times that of the first ($L = 40$), the ratio $\tau = \frac{T_{m1}}{T_{m2}}$ between the corresponding values of MFET is only $\tau = 1.2795$. One possible key reason for this unexpected result is due to the effect of spatial correlation length ξ, which is chosen as $\xi = 2$ in both cases. In order to examine the effect of ξ, we calculate the ratio $\tau = \frac{T_{m1}}{T_{m2}}$ for both

cases ($L = 40$ and $L = 400$), for $\xi = 1, 1.5, 2$. To be more precise, it increases from $\tau = 1.2795$ for $\xi = 2$ to $\tau = 2.1574$ for 1.5 to $\tau = 2.8020$ for $\xi = 1$.

We continue, now computing the mean lifetime of waves, and consider two cases, the mean lifetime of a single wave and the mean lifetime of interacting waves.

Single wave Consider the computation of the mean lifetime of a single wave by direct numerical simulation of the stochastic PDE in comparison to the reduced model. To apply the reduced model, we first need the rate of nucleation times λ, the difference in nucleation times T_{diff}, and the wave speed c. The rate λ is calculated as the inverse of the mean first exit time, obtained from the simulation of the underlying SPDE, as shown in Figure 9. T_{diff} and c are given by (2.3)–(2.4).

Figure 11 displays the mean lifetime of a wave, using the simulation of the SPDE (1.1) and the reduced model discussed in §4, as a function of the small noise parameter σ, for (a) $\epsilon = 0.02$ and (b) $\epsilon = 0.025$. The results obtained from the simulation of the SPDE (1.1), represented by stars symbols, with parameters as used for Figure 2 and using $M = 200$ simulations, indicate that the additive noise does not affect the mean lifetime of the Barkley wave, and this should be taken into account when we design the corresponding reduced model. However, a comparison of the results from the reduced model, represented as solid lines, with those obtained from the simulation of the SPDE, shows excellent agreement, in particular for smaller ϵ. This is due to the asymptotic constant speed c used for the reduced model, as $\epsilon \to 0$.

In Figure 12, we plot the mean lifetime of a wave versus the small parameter ϵ for (a) $\sigma = 0.0825$ and (b) $\sigma = 0.09$, where the shaded circles and red stars represent the results obtained with the reduced model and simulation of the SPDE (1.1), respectively. Both graphs indicate that the mean lifetime increases as ϵ increases, and there is close agreement between the two methods. The harmony shown in Figures 11–12 between the two approaches led us to use the reduced model to study the mean lifetime of the interacting waves, since applying the simulation of the underlying SPDE directly becomes computationally impractical in this case.

Interacting waves When the domain is large, many waves co-exist due to more frequent nucleations, as shown in Figure 8, and the waves interact through annihilation events. In Figure 8, a simulation of the reduced model is shown with $a = 0.75$, $b = 0.01$, $\epsilon = 0.02$, $D = 1$, $L = 400$ and $\lambda = \frac{1}{T_m} = \frac{1}{17.78}$, where λ is the rate of nucleation time and T_m is the mean first exit time obtained by simulating the SPDE (1.1). The nucleation points and corresponding annihilation points are represented by shaded circles and stars, respectively.

Due to the interaction of the waves with each other, we need to calculate the mean lifetime of each kink and anti-kink, individually, as demonstrated in Figure 13. To be precise, Figure 13 shows the mean lifetime, computed using the reduced model, of (a) the first 10 front waves and (b) the first 10 back waves, with number of trials $M = 5000$, $\epsilon = 0.02$ and a total of $N_1 = 30$ front waves nucleated during the overall period of simulation on the space domain $[0, 400]$, with the same number of corresponding back waves. There are thus $2N_1 = 60$ kinks and 60 anti-kinks. In the Figure 13, right-moving waves are represented by circles and left-moving waves by stars. The results reveal that each kink has approximately the same mean lifetime as its corresponding anti-kink, for a large number of trials M. Furthermore, the first kink and anti-kink have the longest mean lifetimes. The lifetime falls sharply but then

remains fairly constant for the rest of the particles.

The simplicity of the reduced model encourages us to explore further the dynamics of kinks and anti-kinks under the Barkley model (1.1). For instance, the average number of kinks and anti-kinks alive at a specific time, t, can be computed easily using the reduced model, as shown in Figure 14(a). The figure describes the mean number of kinks at time t, denoted by N_k, using the same parameters as used for Figure 13, except for the number of trials, which is $M = 1000$. Firstly, we observe that no particles are nucleated at $t = 0$. The mean number of kinks (anti-kinks) at time t then increases as t increases, until $t = 30$. At this time, the mean number of kinks begins to level off, with the maximum being just above 2.3.

Our calculations of the mean number of kinks at time t can be used to obtain the probability of a part of the phase space of the Barkley model being excited at (t, x), where $t > 0$ and $0 \leq x \leq L$. Precisely, this is calculated simply by multiplying the mean number of kinks by the constant width W given in (2.5) and dividing the result by the length of the domain L. Thus

$$\mathbf{P}(\text{a part of the phase space being excited at}(t, x)) = \frac{N_K \times W}{L}. \tag{5.1}$$

These calculations are demonstrated as shaded circles in Figure 14(b). The results provide important information about the excitability of the Barkley system (1.1). For instance, there is an approximately 4% chance of the phase space of the system (1.1) being excited at $t = 50$, whereas the chance is only around 0.8% at time $t = 5$. To check the validity of our calculations, we also compute the probabilities using the simulation of the SPDE (1.1), with parameters $L = 400$, $\Delta t = 0.01$, $N = 1024$, $\epsilon = 0.02$, $\sigma = 0.09$ and $M = 2000$. The probability obtained from the simulation of the SPDE increases gradually until $t = 30$ when it begins to become stable at around 0.03. In spite of this similarity, there are slight differences between the results. Besides the effects of statistical errors and other errors which can arise due to the use of numerical approximations, the wave speed c also affects the results. The wave speed used in the reduced model is a symptomatic value, derived for $\epsilon \to 0$, whereas the corresponding wave speed for the underlying SPDE is simulated for $\epsilon = 0.02$.

Figure 11: Mean lifetime of a wave for the Barkley model as a function of noise parameter σ is shown for (a) $\epsilon = 0.02$ and (b) $\epsilon = 0.025$, computed by simulation of (1.1) (stars) by the reduced model (solid lines). There is good agreement between these results, in particular for $\epsilon = 0.02$.

Figure 12: Mean lifetime of a wave for the Barkley model plotted versus ϵ for (a) $\sigma = 0.0825$ and (b) $\sigma = 0.09$, computed by simulation of the underlying SPDE (stars) and by the reduced model (shaded circles).

Figure 13: The mean lifetime of (a) the first 10 front waves and (b) the first 10 back waves. The parameters of the system are $a = 0.75$, $b = 0.01$, $\epsilon = 0.02$, $D = 1$ and $L = 400$.

Figure 14: (a) The mean number of kinks (anti-kinks) at time t on space domain $[0, 400]$ with $\epsilon = 0.02$, $N_1 = 30$ and $M = 1000$. (b) The probability that a part of the phase space is excited at (x, t) .

References

[1] D. Barkley, *A model for fast computer simulation of waves in excitable media*, Physica D: Nonlinear Phenomena, 49 (1991), pp. 61–70.

[2] R. Bürger, R. Ruiz-Baier, and K. Schneider, *Adaptive multiresolution methods for the simulation of waves in excitable media*, Journal of Scientific Computing, 43 (2010), pp. 261–290.

[3] M. Büttiker and T. Christen, *Nucleation of weakly driven kinks*, Physical Review Letters, 75 (1995), pp. 1895–1898.

[4] B. Costa, *Spectral methods for partial differential equations*, CUBO,A Mathematical Journal, 6 (2004), pp. 1–32.

[5] R. FitzHugh, *Impulses and physiological states in theoretical models of nerve membrane*, Biophysical Journal, 1 (1961), pp. 445–466.

[6] J. García-Ojalvo, F. Sagués, L. Schimansky-Geier, and J. M. Sancho, *Noise-enhanced excitability in bistable activator-inhibitor media*, Physical Review E, 65 (2002), p. 011105.

[7] J. García-Ojalvo and J. M. Sancho, *Noise in spatially extended systems*, Springer Verlag, 1999.

[8] J. García-Ojalvo and L. Schimansky-Geier, *Noise-induced spiral dynamics in excitable media*, Europhysics Letters, 47 (1999), pp. 298–303.

[9] S. Habib and G. Lythe, *Dynamics of kinks: Nucleation, diffusion and annihilation*, Physical Review Letters, 84 (2000), pp. 1070–1073.

[10] P. Jung and G. Mayer-Kress, *Noise controlled spiral growth in excitable media*, Chaos, 5 (1995), pp. 458–462.

[11] S. Kádár, J. Wang, and K. Showalter, *Noise-supported travelling waves in sub-excitable media*, Nature, 391 (1998), pp. 770–772.

[12] P. Kloeden and E. Platen, *Numerical Solution of Stochastic Differential Equations*, Springer-Verlag, Berlin, 1999.

[13] B. Lindner, J. García-Ojalvo, A. Neimand, and L. Schimansky-Geier, *Effects of noise in excitable systems*, Physics Reports, 392 (2004), pp. 321–424.

[14] G. J. Lord and T. Shardlow, *Postprocessing for stochastic parabolic partial differential equations*, SIAM J. Numer. Anal., 45 (2007), pp. 870–889.

[15] G. Lythe and S. Habib, *Kink stochastics*, Computing in Science and Engineering, 8 (2006), pp. 10–15.

[16] R. McLachlan, G. Quispel, and P. Tse, *Linearization-preserving self-adjoint and symplectic integrators*, BIT Numerical Mathematics, 49 (2009), pp. 177–197.

[17] A. S. Mikhailov, *Foundations of Synergetics I*, Springer Verlag, Berlin, 1994.

[18] B. V. Minchev and W. M. Wright, *A review of exponential integrators for first order semi-linear problems*, tech. report, The Norwegian University of Science and Technology, Norway, 2005.

[19] J. D. Murray, *Mathematical Biology II*, Springer Verlag, Berlin, 1993.

[20] J. Nagumo, S. Arimoto, and S. Yoshizawa, *An active pulse transmission line simulating nerve axon*, Proceedings of the IRE, 50 (1962), pp. 2061–2070.

[21] C. E. Rasmussen and C. Williams, *Gaussian processes for machine learning*, The MIT Press,Cambridge, UK, 2006.

[22] A. C. Scott, *The electrophysics of a nerve fiber*, Reviews of Modern Physics, 47 (1975), pp. 487–533.

[23] T. Shardlow, *Nucleation of waves in excitable media by noise*, Multiscale Modeling and Simulation, 3 (2005), pp. 151–167.

[24] ——, *Numerical simulation of stochastic PDEs for excitable media*, Journal of Computational and Applied Mathematics, 175 (2005), pp. 429–446.

[25] H. C. Tuckwell and F. Y. Wan, *Time to first spike in stochastic Hodgkin-Huxley systems*, Physica A: Statistical Mechanics and its Applications, 351 (2005), pp. 427–438.

[26] A. T. Winfree, *Varieties of spiral wave behavior: An experimentalist's approach to the theory of excitable media*, Chaos, 1 (1991), pp. 303–334.

[27] A. M. Zhabotinsky, *A history of chemical oscillations and waves*, Chaos, 1 (1991), pp. 379–386.

[28] V. S. Zykov and A. T. Winfree, *Simulation of Wave Processes in Excitable Media*, Manchester University Press, Manchester, 1987.