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The exponential integrator scheme for stochastic partial differential equations: Pathwise error bounds[☆]

P.E. Kloeden[a], G.J. Lord[b], A. Neuenkirch[c,*], T. Shardlow[d]

[a] Institut für Mathematik, Goethe-Universität Frankfurt, D-60325 Frankfurt a.M., Germany
[b] Department of Mathematics, Heriot-Watt University, Edinburgh EH14 4AS, UK
[c] Fakultät für Mathematik, TU Dortmund, D-44227 Dortmund, Germany
[d] School of Mathematics, University of Manchester, Manchester M13 9PL, UK

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ABSTRACT

We present an error analysis for the pathwise approximation of a general semilinear stochastic evolution equation in d dimensions. We discretise in space by a Galerkin method and in time by using a stochastic exponential integrator. We show that for spatially regular (smooth) noise the number of nodes needed for the noise can be reduced and that the rate of convergence degrades as the regularity of the noise reduces (and the noise becomes rougher).

1. Introduction

We consider the pathwise numerical approximation of the stochastic evolution equation

$$\mathrm{d}u(t) = [-Au(t) + F(u(t))]\mathrm{d}t + \mathrm{d}W(t), \quad t \geq 0, \qquad (1)$$
$$u(0) = u_0,$$

on the Hilbert space $H = L^2([a, b]^d)$. Here, $-A$ is the generator of an analytic semigroup $(\mathrm{e}^{-At}, t \geq 0)$ on H, $u(0) \in D(A)$, $W = (W(t), t \geq 0)$ is a Q-Wiener process on $(\Omega, \mathcal{A}, \mathbf{P})$ with values in H and the mapping $F : H \to H$ is nonlinear; precise assumptions are given in Section 2.1. Finally, we assume that A and the covariance operator Q of the Wiener process have the same eigenfunctions ϕ_n, i.e.

$$A\phi_n = \alpha_n \phi_n, \qquad Q\phi_n = \lambda_n \phi_n, \quad n \in \mathbb{N}^d,$$

where $\alpha_n, \lambda_n \geq 0$ and $\phi_n, n \in \mathbb{N}^d$, is an orthonormal basis of H. In particular, we have the representation

$$W(t) = \sum_{n \in \mathbb{N}^d} \lambda_n^{1/2} \beta_n(t) \cdot \phi_n, \quad t \geq 0,$$

with independent scalar Brownian motions $\beta_n, n \in \mathbb{N}^d$.

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* Corresponding author.
E-mail addresses: kloeden@math.uni-frankfurt.de (P.E. Kloeden), gabriel@ma.hw.ac.uk (G.J. Lord), andreas.neuenkirch@math.tu-dortmund.de (A. Neuenkirch), shardlow@maths.man.ac.uk (T. Shardlow).

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Typical examples for equations of the above type are the stochastic cable equation

$$\mathrm{d}u(t) = [\Delta u(t) - u(t)]\mathrm{d}t + \mathrm{d}W(t)$$

and the stochastic Allen–Cahn equation

$$\mathrm{d}u(t) = [\nu \Delta u(t) + u(t) - u(t)^3]\mathrm{d}t + \mathrm{d}W(t)$$

with periodic boundary conditions, where Δ denotes the Laplace operator and $\nu > 0$ is a parameter. However, our assumptions cover also the case where A is a fractional power of the Laplacian. This paper builds on the error analysis for the exponential integrator method, introduced in [1,2] for Eq. (1) with A being the one-dimensional Laplacian. In [1] an H^1 error bound for smooth Gevrey noise, i.e. with exponential spatial correlation, was derived, and an L^2 and H^m error analysis for a post-processing variant of the exponential integrator scheme is given in [2], in the case of an arbitrary driving infinite dimensional Wiener process W. Here, we extend these results in the following way. We consider a general differential operator A in d dimensions instead of the one-dimensional Laplacian and we derive pathwise error bounds for this exponential integrator scheme. To do this we first derive error bounds in the p-th mean for all $p \geq 1$. Then by a Borel–Cantelli type of argument, which has been used in a similar way e.g. in [3–5], we obtain the pathwise convergence rates.

Up to the preparation of this article pathwise approximation of SPDEs with an infinite dimensional Wiener process has been considered so far mainly for stochastic parabolic PDEs with multiplicative space–time white noise, i.e. for equations with one space dimension; see e.g. [6–8]. *In this article the pathwise convergence rates of several finite difference schemes are determined.*

However, very recently, research on the pathwise approximation of SPDEs has intensified. Simultaneously with the preparation of this article, pathwise convergence rates for an approximation scheme for Eq. (1), *which uses linear functionals of the driving noise, were derived in* [9,10] *under weaker assumptions on the nonlinearity F. Moreover, pathwise approximation of SPDEs with coloured additive and multiplicative noise has also been considered in e.g.* [11,12].

2. The numerical scheme

We now describe our numerical scheme for the approximation of (1). For this, recall that ϕ_n are the eigenvectors of A, so $A\phi_n = \alpha_n\phi_n$, $n \in \mathbb{N}^d$, and moreover that the driving Wiener process is given by

$$W(t) = \sum_{n\in\mathbb{N}^d} \lambda_n^{1/2}\beta_n(t) \cdot \phi_n. \tag{2}$$

So, consider the mild solution of Eq. (1), i.e.

$$u(t) = \mathrm{e}^{-At}u(0) + \int_0^t \mathrm{e}^{-A(t-s)}F(u(s))\mathrm{d}s + \int_0^t \mathrm{e}^{-A(t-s)}\mathrm{d}W(s). \tag{3}$$

(*Existence, uniqueness and further properties of the mild solution are established in* Theorem 4.) *Expanding the solution with respect to the orthonormal basis ϕ_n, $n \in \mathbb{N}^d$, i.e. writing $u(t) = \sum_{n\in\mathbb{N}^d} u_n(t) \cdot \phi_n$,* we obtain the infinite system of coupled equations

$$u_n(t) = \mathrm{e}^{-t\alpha_n}u_n(0) + \int_0^t \mathrm{e}^{-(t-s)\alpha_n}F_n(u(s))\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-s)\alpha_n}\lambda_n^{1/2}\mathrm{d}\beta_n(s). \tag{4}$$

Here $F_n(u)$ denotes the n-th coefficient of $F(u)$, that is we have $F(u) = \sum_{n\in\mathbb{N}^d} F_n(u) \cdot \phi_n$.

Now let $\Delta t > 0$ denote the time step and N the size of the Galerkin truncation. Consider the discretisation of (1) at times $t_k = k\Delta t$ given by

$$\begin{aligned}\widehat{u}_n(t_{k+1}) &= \mathrm{e}^{-\Delta t\alpha_n}\left(\widehat{u}_n(t_k) + \Delta t F_n(\widehat{u}(t_k)) + \lambda_n^{1/2}\Delta B_{k,n}\right), \\ \widehat{u}_n(0) &= u_n(0),\end{aligned} \tag{5}$$

where $|n| \leq N$ and $\Delta B_{k,n} = \beta_n(t_{k+1}) - \beta_n(t_k)$. The time continuous version of this scheme is given by

$$\widehat{u}_n(t) = \mathrm{e}^{-t\alpha_n}u_n(0) + \int_0^t \mathrm{e}^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}F_n(\widehat{u}(\lfloor s\rfloor_{\Delta t}))\mathrm{d}s + \int_0^t \mathrm{e}^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\lambda_n^{1/2}\mathrm{d}\beta_n(s). \tag{6}$$

Here we use the notation $\lfloor s\rfloor_{\Delta t} = \max_{k\in\mathbb{N}}\{t_k : t_k \leq s\}$. We study a version of the post-processing method introduced in [2]:

$$\begin{aligned}\widehat{u}_n(t_{k+1}) &= \mathrm{e}^{-\Delta t\alpha_n}\left(\widehat{u}_n(t_k) + \Delta t F_n(\widehat{u}(t_k)) + 1_{\{|n|\leq N_w\}}\lambda_n^{1/2}\Delta B_{k,n}\right), \\ \widehat{u}_n(0) &= u_n(0),\end{aligned} \tag{7}$$

where $|n| \leq N$. The constant N_w describes the number of modes used to approximate the Wiener process W. If the noise is smooth, then fewer modes for the approximation of the noise than for the approximation of the nonlinearity can be used; see Corollary 3.4 in [2].

For the numerical analysis we use the following interpolant of $\widehat{u}_n(t_k)$ in time:

$$\widehat{u}_n(t) = \mathrm{e}^{-t\alpha_n} u_n(0) + \int_0^t \mathrm{e}^{-(t-\lfloor s \rfloor_{\Delta t})\alpha_n} F_n(\widehat{u}(\lfloor s \rfloor_{\Delta t}))\mathrm{d}s + 1_{\{|n|\le N_w\}} \int_0^t \mathrm{e}^{-(t-\lfloor s \rfloor_{\Delta t})\alpha_n} \lambda_n^{1/2} \mathrm{d}\beta_n(s). \quad (8)$$

So, finally our approximation of $u(t)$ is given by $\widehat{u}(t) = \sum_{|n|\le N} \widehat{u}_n(t) \cdot \phi_n$ for $t \ge 0$. Note that $\widehat{u}(t)$ depends on N, the size of the Galerkin truncation, on N_w, the number of the modes for the approximation of the noise W, and on the step size Δt.

2.1. Error bounds in the p-th mean

We make the following assumptions on the nonlinearity F and on the operators A and Q:

Assumption 1. Let $F \in \mathcal{C}^2(H; H)$, i.e. the mapping $F : H \to H$ is twice continuously Fréchet differentiable, and there exist constants $K_0, K_1, K_2 > 0$ such that

$$\|F(u)\|_H \le K_0(1 + \|u\|_H) \quad (9)$$

and

$$\|\mathrm{d}F(u)\|_{L(H;H)} \le K_1, \quad (10)$$

$$\|d^2F(u)\|_{L(H\times H;H)} \le K_2. \quad (11)$$

for all $u \in H$.

Note that the above assumption implies that F satisfies a global Lipschitz condition. Moreover, we have the following assumption on the eigenvalues of the covariance operator Q, which is by definition self-adjoint and positive.

Assumption 2. There exist $\gamma \ge 0$ and constants $C_1, C_2 > 0$ such that

$$C_1 \cdot |n|^{-\gamma} \le \lambda_n \le C_2 \cdot |n|^{-\gamma}$$

for $n \in \mathbb{N}^d$.

Note that for $\gamma > d$ we have the so called trace class noise and $Q = \mathrm{id}$ is included in the case $\gamma = 0$. For the eigenvalues of the operator A we assume that they are strictly positive and have a polynomial growth.

Assumption 3. The operator $A : H \to H$ is self-adjoint and positive. Moreover, $\alpha_n > 0$ for $n \in \mathbb{N}^d$, $\alpha_m \le \alpha_n$ for $|m| \le |n|$ and there exist $\kappa > 0$ and constants $C_3, C_4 > 0$ such that

$$C_3 \cdot |n|^{\kappa} \le \alpha_n \le C_4 \cdot |n|^{\kappa}$$

for $n \in \mathbb{N}^d$.

Thus $-A$ generates in particular an analytical semigroup $(\mathrm{e}^{-At}, t \ge 0)$ on H; see [13].
Under the above assumptions, we have the following theorem, which in particular describes the smoothness of the solution in terms of the parameters γ and κ. Its proof is given in the Appendix.

Theorem 4. *Let Assumptions 1–3 hold, $u(0) \in D(A)$ and let $\gamma + \kappa > d$ and $T > 0$. Then Eq. (1) has a unique mild solution $(u(t), t \in [0, T])$, which satisfies*

$$\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|_H^p < \infty \quad (12)$$

for all $p \ge 1$.

Moreover, let $\theta^* := \frac{\gamma+\kappa-d}{2\kappa}$. Then we have $u(t, \omega) \in D(A^\theta)$, $t \in [0, T]$, for all $\theta < \min\{1, \theta^*\}$ and almost all $\omega \in \Omega$. Finally, for all $p \ge 1$ and all $\theta < \min\{1, \theta^*\}$ we have

$$\sup_{t\in[0,T]} \mathbf{E}\|A^\theta u(t)\|_H^p < \infty \quad (13)$$

and there exist constants $K_{p,T,\theta} > 0$ such that

$$\left(\mathbf{E}\|u(t) - u(s)\|_H^p\right)^{1/p} \le K_{p,T,\theta} \cdot |t - s|^\theta \quad (14)$$

for all $s, t \in [0, T]$ and all $\theta < \min\{1/2, \theta^*\}$.

Our main result for the convergence rates in the p-th mean is as follows:

Theorem 5. *Let Assumptions 1–3 hold and let $\gamma + \kappa > d$ and $u_0 \in D(A)$. Then for all $\varepsilon > 0$, $T > 0$ and $p \geq 1$ there exists a constant $K_{\varepsilon,T,p} > 0$ such that*

$$\sup_{t\in[0,T]} \left(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\right)^{1/p} \leq K_{\varepsilon,T,p} \cdot \left(\Delta t^{\min\{1,\theta^*\}-\varepsilon} + N^{-\kappa} + N_w^{-\kappa\theta^*}\right).$$

Proof. This is given in Section 4. □

To balance the error contributions of the different parts, we have to consider two cases: (i) $\theta^* \geq 1$. Here it is optimal to choose

$$N_w = \lceil c_w \cdot N^{1/\theta^*} \rceil$$

with $c_w > 0$, so we can use fewer modes to approximate the noise. Furthermore, balancing the Δt-terms gives

$$\Delta t = c_{\Delta t} \cdot N^{-\kappa}$$

with $c_{\Delta t} > 0$. So, for $\widehat{u}$ with such a choice of $\Delta t, N, N_w$ we have

$$\sup_{t\in[0,T]} \left(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\right)^{1/p} \leq \widetilde{K}_{\varepsilon,T,p} \cdot N^{-\kappa+\varepsilon}.$$

(ii) $\theta^* < 1$. Here we cannot save modes for the noise and have to choose

$$N_w = \lceil c_w \cdot N \rceil$$

with $c_w > 0$. Balancing the Δt-terms gives again

$$\Delta t = c_{\Delta t} \cdot N^{-\kappa}$$

with $c_{\Delta t} > 0$. So, here we obtain

$$\sup_{t\in[0,T]} \left(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\right)^{1/p} \leq \widetilde{K}_{\varepsilon,T,p} \cdot N^{-\kappa\theta^*+\varepsilon}.$$

Summarising, we have

$$\sup_{t\in[0,T]} \left(\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\right)^{1/p} \leq \widetilde{K}_{\varepsilon,T,p} \cdot N^{-\kappa\min\{1,\theta^*\}+\varepsilon} \tag{15}$$

with

$$N_w = \lceil c_w \cdot N^{\min\{1,1/\theta^*\}} \rceil, \qquad \Delta t = c_{\Delta t} \cdot N^{-\kappa}. \tag{16}$$

In the case where $-A$ is the one-dimensional Laplacian these error bounds coincide (up to the arbitrarily small $\varepsilon > 0$) with the results of Corollary 3.4 in [2].

2.2. Pathwise convergence rates

For the pathwise convergence rates, we need the following lemma, which is a straightforward consequence of the Borel–Cantelli lemma; see e.g. [4].

Lemma 1. *Let $\alpha > 0$ and $C_p \in [0, \infty)$ for $p \geq 1$. In addition, let Z_n, $n \in \mathbb{N}$, be a sequence of real-valued random variables such that*

$$(\mathbf{E}|Z_n|^p)^{1/p} \leq C_p \cdot n^{-\alpha}$$

for all $p \geq 1$ and all $n \in \mathbb{N}$. Then for all $\varepsilon > 0$ there exists a random variable η_ε such that

$$|Z_n| \leq \eta_\varepsilon \cdot n^{-\alpha+\varepsilon} \quad \mathbf{P}\text{-a.s.}$$

for all $n \in \mathbb{N}$. Moreover, $\mathbf{E}|\eta_\varepsilon|^p < \infty$ for all $p \geq 1$.

Applying this lemma we get the following result:

Corollary 1. *Let Assumptions 1–3 hold and let $\gamma + \kappa > d$ and $u_0 \in D(A)$. Moreover let N, N_w and Δt satisfy (16). Then for all $T > 0$ and $\varepsilon > 0$, there exists a random variable $\eta_{\varepsilon,T} > 0$ such that*

$$\|u(T,\omega) - \widehat{u}(T,\omega)\|_H \leq \eta_{\varepsilon,T}(\omega) \cdot N^{-\kappa\min\{1,\theta^*\}+\varepsilon}$$

for almost all $\omega \in \Omega$.

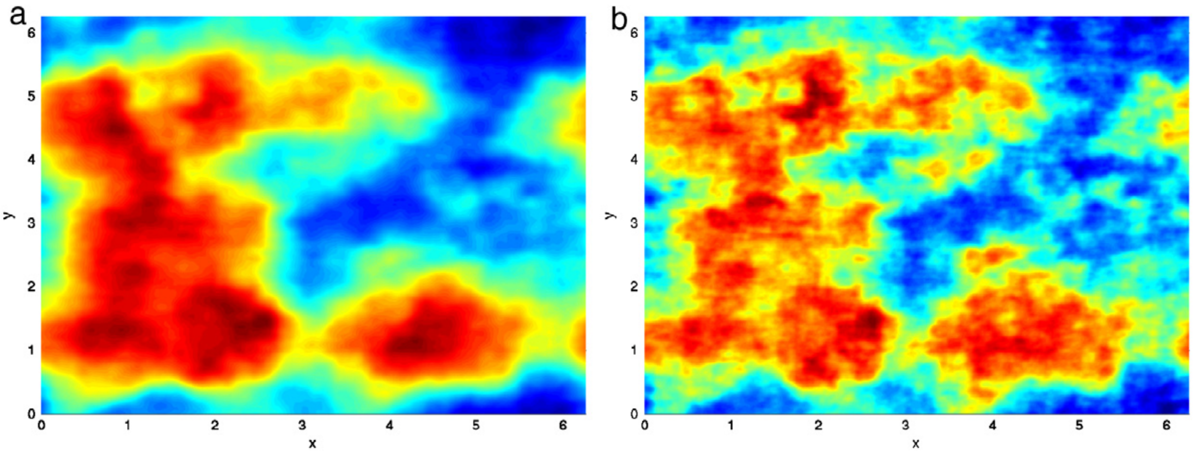

Fig. 1. Plot of two sample true solutions with 256×256 modes at time $T = 2$; (a) $\gamma = 4$ and (b) $\gamma = 3$. Note that the solution in (a) is smoother than the solution in (b) as the regularity of the noise decreases.

Since $\|h\|_H^2 = \int_{[a,b]^d} h(x)^2 \mathrm{d}x$ another application of the Borel–Cantelli lemma yields:

Corollary 2. *Let the same assumptions as in the previous corollary hold and assume additionally that $\gamma + \kappa > d + 2$ and $\kappa > 1$. Then we have*

$$\widehat{u}(T, x, \omega) \overset{N\to\infty}{\longrightarrow} u(T, x, \omega)$$

for almost all $\omega \in \Omega$ and almost all $x \in [a, b]^d$.

So, in the case of the d-dimensional Laplacian, i.e. $\kappa = 2$ and trace class noise, i.e. $\gamma > d$, the exponential integrator scheme converges for almost all $\omega \in \Omega$ and almost all $x \in [a, b]^d$.

3. A numerical illustration

Consider the Allen–Cahn equation in two dimensions

$$\mathrm{d}u(t) = [\nu\Delta u(t) + u(t) - u(t)^3]\mathrm{d}t + \mathrm{d}W(t)$$

with periodic boundary conditions on $[0, 2\pi] \times [0, 2\pi]$. Here we have the dimensional Laplacian operator, so $\kappa = 2$ in Assumption 3. We take noise that is white in time and vary the spatial regularity through the parameter γ in Assumption 2. With these values we see that $\theta^* = \gamma/4$ and we have a critical value of $\gamma = 4$. We integrate using (7) to a final time $T = 2$ with a time step of $\Delta t = 0.005$. For our numerical calculations, we take the diffusion coefficient $\nu = 0.004$. To test the numerics, "true" solutions were computed using 256×256 modes and two sample "true" solutions at $T = 2$ are plotted in Fig. 1. These solutions are computed with the same path and it is only the regularity of the noise that varies, in (a) $\gamma = 4$ and (b) $\gamma = 3$, and visually this is reflected in the regularity of the solution. In Fig. 2 we show that our results agree with the theoretical results and for $\gamma = 4$ we see convergence like N^{-2} (numerically we observe in the figure -2.05) both for a single realisation and for the mean over 100 realisations. For $\gamma = 3$ we have convergence like the predicted $N^{-3/2}$ (numerically we observe in the figure -1.53) again for a single realisation and for the mean over 100 realisations.

4. Proof of the convergence result

We prove Theorem 5 by estimating

$$\sup_{t\in[0,\tau]} \left[\mathbf{E}\|u(t) - \widehat{u}(t)\|_H^p\right]^{1/p}, \quad \tau \in [0, T],$$

for $p \geq 2$ and applying Gronwall's lemma. (*Note that the estimates for $1 \leq p < 2$ follow by Hölder's inequality*.)

4.1. Preliminaries

We first recall some basic facts of stochastic integration with respect to a Q-Wiener process. Let $(\Omega, \mathcal{A}, \mathcal{F}, \mathbf{P})$ be a filtered probability space and let $W = (W(t), t \in [0, T])$ be a Q-Wiener process on this space with respect to the filtration $\mathcal{F} = (\mathcal{F}_t, t \in [0, T])$. Denote by $L_2^0 := HS(Q^{1/2}(H), H)$ the space of Hilbert–Schmidt operators from $Q^{1/2}(H)$ to H and

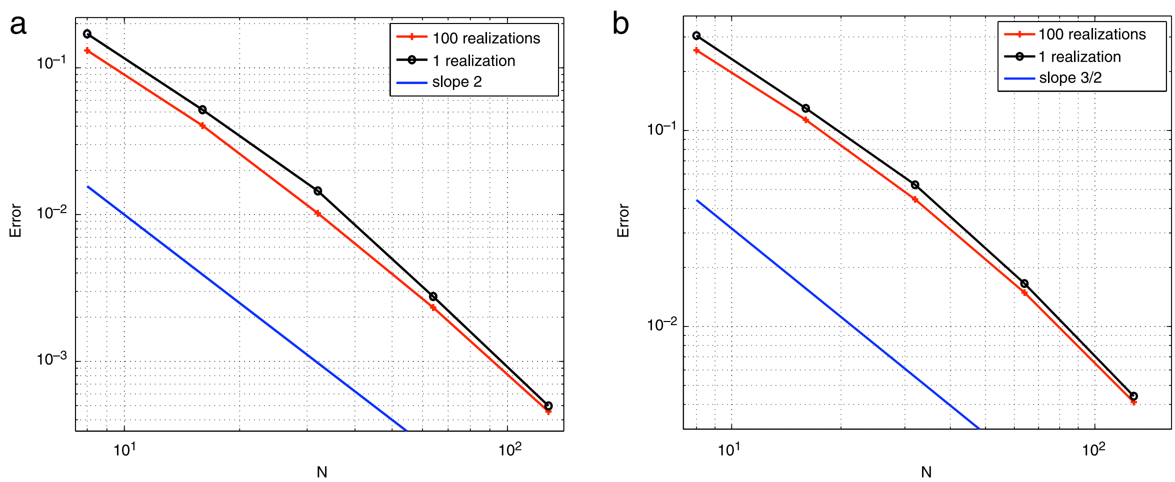

Fig. 2. Convergence in space for (a) $\gamma = 4$ and (b) $\gamma = 3$. The plot is a log log plot of the error on $[0, 2]$, i.e. of $\max_{t\in[0,2]} \|u(t,\omega) - \widehat{u}^N(t,\omega)\|_H$ for the single realisation and of $\max_{t\in[0,2]} \left(\frac{1}{100}\sum_{i=1}^{100} \|u(t,\omega_i) - \widehat{u}^N(t,\omega_i)\|_H^2\right)^{1/2}$ for the mean over 100 realisations, as the system size N is changed. Results are plotted for a single realisation and for the mean over 100 realisations. In (a) we see the predicted rate of N^{-2} and in (b) $N^{-3/2}$.

by $\|\cdot\|_{L_2^0}$ the corresponding norm given by

$$\|C\|_{L_2^0}^2 = \mathrm{Tr}(C^*QC) := \sum_{n\in\mathbb{N}^d} \langle C^*QC\varphi_n, \varphi_n\rangle,$$

where φ_n, $n \in \mathbb{N}^d$, is an arbitrary orthonormal basis of H. Moreover denote by $L_{\mathcal{F}}^2 := L_{\mathcal{F}}^2([0,T]; L_2^0)$ the space of all predictable stochastic processes $X = (X(t), t \in [0,T])$ with values in L_2^0 such that

$$\|X\|_{L_{\mathcal{F}}^2} := \left(\int_0^T \mathbf{E}\|X(t)\|_{L_2^0}^2 \mathrm{d}t\right)^{1/2} < \infty.$$

Then for $X \in L_{\mathcal{F}}^2$ the stochastic integral

$$\int_0^T X(t)\mathrm{d}W(t)$$

is well defined as an element of H and we have the following Itô isometry:

$$\mathbf{E}\left\|\int_0^T X(t)\mathrm{d}W(t)\right\|_H^2 = \int_0^T \mathbf{E}\|X(t)\|_{L_2^0}^2 \mathrm{d}t. \tag{17}$$

(A process X with values in L_2^0 is called predictable if $X : [0,T] \times \Omega \to L_2^0$ is a $\mathcal{P}_T - \mathcal{B}(L_2^0)$ measurable mapping, where $\mathcal{P}_T$ is the σ-field generated by the sets $(s,t] \times F$, with $s, t \in [0,T]$, $F \in \mathcal{F}_s$ and $\{0\} \times F$ with $F \in \mathcal{F}_0$.)

The Itô integral satisfies the following stability property (see e.g. Proposition 4.15 in [14]): Let $G : D(G) \to H$ be a closed operator, where $D(G)$ is a Borel subset of H and let moreover $X \in L_{\mathcal{F}}^2$ be such that $\mathbf{P}(X(t) \in D(G)$ for all $t \in [0,T]) = 1$ and $GX \in L_{\mathcal{F}}^2$. Then, we have

$$\mathbf{P}\left(\int_0^T X(s)\mathrm{d}W(s) \in D(G)\right) = 1$$

and

$$G\int_0^T X(s)\mathrm{d}W(s) = \int_0^T GX(s)\mathrm{d}W(s) \quad \mathbf{P}\text{-a.s.}$$

Moreover, one has the following version of the Burkholder–Davis–Gundy inequality (see e.g. Lemma 7.2 in [14]): For any $r \geq 1$ and any $X \in L_{\mathcal{F}}^2$ there exist constants $C_r > 0$ such that

$$\mathbf{E}\left\|\int_0^T X(s)\mathrm{d}W(s)\right\|^{2r} \leq C_r\mathbf{E}\left(\int_0^T \|X(s)\|_{L_2^0}^2 \mathrm{d}s\right)^r. \tag{18}$$

We need the following version of the stochastic Fubini theorem (see e.g. Theorem 4.18 in [14]): Let $Y : [0, T] \times \Omega \times [0, T] \to L_2^0$ be a $\mathcal{P}_T \times \mathcal{B}([0, T]) - \mathcal{B}(L_2^0)$-measurable mapping such that

$$\int_0^T \left(\mathbf{E} \int_0^T \|Y(t, s)\|_{L_2^0}^2 \mathrm{d}t \right)^{1/2} \mathrm{d}s < \infty.$$

Then we have **P**-a.s.

$$\int_0^T \int_0^T Y(t, s) \mathrm{d}W(t) \mathrm{d}s = \int_0^T \int_0^T Y(t, s) \mathrm{d}s \mathrm{d}W(t). \tag{19}$$

We also require the following properties of the operator A and the semigroup e^{-At}; see e.g. Theorem 6.13 in Chapter 2 in [13].

Lemma 2. *For arbitrary $\delta_1 \geq 0$, $0 \leq \delta_2 \leq 1$ there exist constants $C_5, C_6 > 0$ such that we have*

$$\|A^{\delta_1} \mathrm{e}^{-At}\|_{L(H;H)} \leq C_5 t^{-\delta_1} \tag{20}$$

and

$$\|A^{-\delta_2} (\mathrm{id} - \mathrm{e}^{-At})\|_{L(H;H)} \leq C_6 t^{\delta_2} \tag{21}$$

for any $t \in (0, T]$.

We denote by $\mathbb{P}_N : H \to H$ the orthogonal projection of H to the subspace generated by $\{\phi_n : |n| \leq N\}$, i.e.

$$\mathbb{P}_N u = \sum_{|n| \leq N} c_n \cdot \phi_n$$

for $u = \sum_{n \in \mathbb{N}^d} c_n \cdot \phi_n \in H$. Clearly, we have

$$\|\mathbb{P}_N u\|_H^2 = \sum_{|n| \leq N} |c_n|^2$$

and

$$\|(\mathrm{id} - \mathbb{P}_N) u\|_H^2 = \sum_{|n| > N} |c_n|^2,$$

for $u = \sum_{n \in \mathbb{N}^d} c_n \cdot \phi_n$, which we use several times in the following. We also have

$$\|(\mathrm{id} - \mathbb{P}_N) \mathrm{e}^{-At}\|_{L(H;H)} \leq \mathrm{e}^{-\min\{\alpha_n : |n| = N\} t} \tag{22}$$

for $t \in [0, T]$.

Finally, we require the following estimate, which can be obtained by straightforward calculations. Let $\delta > d$. Then, there exist constants $C_7, C_8 > 0$, which depend only on d and δ, such that

$$C_7 \cdot N^{-\delta + d} \leq \sum_{|n| > N} |n|^{-\delta} \leq C_8 \cdot N^{-\delta + d}. \tag{23}$$

After these preparations, we can now start with the error analysis. To estimate terms, we use a generic constant C which varies between instances but is independent of Δt, N, N_w and $t \in [0, T]$. Moreover, we write $\| \cdot \|$ instead of $\| \cdot \|_H$, $\| \cdot \|_{L(H;H)}$ respectively $\| \cdot \|_{L_2^0}$, if no misunderstanding is possible.

4.2. The initial value

For the error of the approximation of the initial value we have

$$\mathbf{INITIAL} = \sup_{t \in [0, \tau]} \|\mathrm{e}^{-At} (u(0) - \widehat{u}(0))\|.$$

Since

$$\sup_{t \in [0, \tau]} \|\mathrm{e}^{-At} (u(0) - \widehat{u}(0))\|^2 = \sup_{t \in [0, \tau]} \sum_{|n| > N} \mathrm{e}^{-2\alpha_n t} |u_n(0)|^2 = \sum_{|n| > N} |u_n(0)|^2$$

and $u(0) \in D(A)$ it follows that

$$\mathbf{INITIAL} = \left(\sum_{|n| > N} |u_n(0)|^2 \right)^{1/2} \leq \frac{1}{\alpha_N^*} \left(\sum_{|n| > N} |\alpha_n u_n(0)|^2 \right)^{1/2} \leq \frac{1}{\alpha_N^*} \|Au(0)\|,$$

where $\alpha_N^* = \min\{\alpha_n : |n| = N\}$. So, we obtain

$$\mathbf{INITIAL} \leq C \cdot N^{-\kappa} \tag{24}$$

by Assumption 3.

4.3. The noise terms

For estimating the noise terms recall that

$$W(t) = \sum_{n\in\mathbb{N}^d} \lambda_n^{1/2}\beta_n(t)\cdot\phi_n.$$

(i) Consider first the noise with modes $|n| \le N_w$. We have

$$\mathbf{NOISE}_1 = \sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\sum_{|n|\le N_w}\lambda_n^{1/2}\left(\int_0^t\left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)d\beta_n(s)\right)\cdot\phi_n\right\|^p\right]^{1/p}.$$

Since

$$\sum_{|n|\le N_w}\lambda_n^{1/2}\left(\int_0^t\left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)d\beta_n(s)\right)\cdot\phi_n = \int_0^t\varphi(t,s)dW(s)$$

with

$$\varphi(t,s) = \sum_{|n|\le N_w}\left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)\cdot\phi_n,$$

an application of the Burkholder–Davis–Gundy inequality (18) yields

$$\mathbf{NOISE}_1 \le C\sup_{t\in[0,\tau]}\left[\int_0^t\|\varphi(t,s)\|^2_{L_2^0}ds\right]^{1/2}.$$

However,

$$\|\varphi(t,s)\|^2_{L_2^0} = \sum_{|n|\le N_w}\lambda_n\left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2$$

and thus

$$\mathbf{NOISE}_1 \le C\sup_{t\in[0,\tau]}\left[\int_0^t\sum_{|n|\le N_w}\lambda_n\left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2 ds\right]^{1/2}.$$

Since for every $\theta\in[0,1]$ we have

$$|e^{-x} - e^{-y}| \le |x-y|^\theta, \quad x, y \ge 0,$$

we obtain

$$\int_0^t\left(e^{-(t-s)\alpha_n} - e^{-(t-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2 ds \le \int_0^t e^{-2(t-s)\alpha_n}\left(1 - e^{-(s-\lfloor s\rfloor_{\Delta t})\alpha_n}\right)^2 ds$$
$$\le \Delta t^{2\theta}\alpha_n^{2\theta}\int_0^t e^{-2(t-s)\alpha_n}ds \le C\Delta t^{2\theta}\alpha_n^{2\theta-1}$$

for $\theta\in(0,1)$. Hence we have

$$\mathbf{NOISE}_1 \le C\Delta t^\theta\left(\sum_{|n|\le N_w}\lambda_n\alpha_n^{2\theta-1}\right)^{1/2} \le C\Delta t^\theta\left(\sum_{|n|\le N_w}|n|^{-\gamma-\kappa+2\theta\kappa}\right)^{1/2},$$

since

$$0 \le \lambda_n\alpha_n^{2\theta-1} \le C\cdot|n|^{-\gamma-\kappa+2\theta\kappa}$$

by Assumptions 2 and 3. *Recall that* $\theta^* = \frac{\gamma+\kappa-d}{2\kappa}$. *So, for* $\theta < \theta^*$ *we have* $-\gamma-\kappa+2\theta\kappa < -d$ *and thus*

$$\sum_{n\in\mathbb{N}^d}|n|^{-\gamma-\kappa+2\theta\kappa} < \infty.$$

Hence we have obtained

$$\mathbf{NOISE}_1 \le C\cdot\Delta t^\theta \tag{25}$$

for $\theta < \min\{1, \theta^*\}$.

(ii) Now consider the noise with modes $|n| > N_w$, i.e.

$$\mathbf{NOISE}_2 = \sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \sum_{|n|>N_w} \int_0^t \lambda_n^{1/2} e^{-(t-s)\alpha_n} d\beta_n(s) \cdot \phi_n \right\|^p \right]^{1/p}$$

$$= \sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| (\mathrm{id} - \mathbb{P}_{N_w}) \int_0^t e^{-A(t-s)} dW(s) \right\|^p \right]^{1/p}.$$

Using the stability of the Itô integral and the Burkholder–Davis–Gundy inequality (see Section 4.1), we have

$$\mathbf{NOISE}_2 \le \sup_{t\in[0,\tau]} C \left(\sum_{|n|>N_w} \lambda_n \int_0^t e^{-2(t-s)\alpha_n} ds \right)^{1/2} \le C \left(\sum_{|n|>N_w} \frac{\lambda_n}{\alpha_n} \right)^{1/2}.$$

Assumptions 2 and 3 and the estimate (23) now give

$$\mathbf{NOISE}_2 \le C \cdot N_w^{(-\gamma-\kappa+d)/2}. \tag{26}$$

4.4. Nonlinear terms: modes $|n| > N$

Consider now the nonlinear terms of F not contributing to $\widehat{u}$: Using Jensen's inequality, estimate (22) and Assumption 1 we have

$$\mathbf{TAIL} = \sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \sum_{|n|>N} \int_0^t e^{-(t-s)\alpha_n} F_n(u(s)) ds \cdot \phi_n \right\|^p \right]^{1/p}$$

$$= \sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \int_0^t (\mathrm{id} - \mathbb{P}_N) e^{-A(t-s)} F(u(s)) ds \right\|^p \right]^{1/p}$$

$$\le \sup_{t\in[0,\tau]} \int_0^t \left[\mathbf{E} \left\| (\mathrm{id} - \mathbb{P}_N) e^{-A(t-s)} F(u(s)) \right\|^p \right]^{1/p} ds$$

$$\le C \sup_{t\in[0,\tau]} \int_0^t e^{-(t-s)\alpha_N^*} [\mathbf{E}(1 + \|u(s)\|)^p]^{1/p} ds,$$

where $\alpha_N^* = \min\{\alpha_n : |n| = N\}$. Since

$$\sup_{s\in[0,T]} \mathbf{E}(1 + \|u(s)\|)^p < \infty$$

by Theorem 4, we have

$$\sup_{t\in[0,\tau]} \int_0^t e^{-(t-s)\alpha_N^*} [\mathbf{E}(1 + \|u(s)\|)^p]^{1/p} ds \le C \frac{1}{\alpha_N^*}$$

and thus we obtain by Assumption 3 that

$$\mathbf{TAIL} \le C \cdot N^{-\kappa}. \tag{27}$$

4.5. Nonlinear terms: modes $|n| \le N$

We have

$$\sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})} \left(e^{-\alpha_n(\lfloor s\rfloor_{\Delta t}-s)} F_n(u(s)) - F_n(\widehat{u}(\lfloor s\rfloor_{\Delta t})) \right) ds \cdot \phi_n \right\|^p \right]^{1/p} \le C \cdot (\mathbf{NL}_1 + \mathbf{NL}_2 + \mathbf{NL}_3),$$

where

$$\mathbf{NL}_1 = \sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})} \left(F_n(u(s)) - F_n(u(\lfloor s\rfloor_{\Delta t})) \right) ds \cdot \phi_n \right\|^p \right]^{1/p},$$

$$\mathbf{NL}_2 = \sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})} \left(F_n(u(\lfloor s\rfloor_{\Delta t})) - F_n(\widehat{u}(\lfloor s\rfloor_{\Delta t})) \right) ds \cdot \phi_n \right\|^p \right]^{1/p},$$

$$\mathbf{NL}_3 = \sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \sum_{|n|\le N} \int_0^t e^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})} \left(e^{-\alpha_n(\lfloor s\rfloor_{\Delta t}-s)} - 1 \right) F_n(u(s)) ds \cdot \phi_n \right\|^p \right]^{1/p}.$$

(i) The first term. Note that

$$\sum_{|n|\le N}\int_0^t \mathrm{e}^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\left(F_n(u(s))-F_n(u(\lfloor s\rfloor_{\Delta t}))\right)\mathrm{d}s\cdot\phi_n=\mathbb{P}_N\left[\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\left(F(u(s))-F(u(\lfloor s\rfloor_{\Delta t}))\right)\mathrm{d}s\right].$$

Since F is twice Fréchet differentiable we can write

$$F(u(s))-F(u(\lfloor s\rfloor_{\Delta t}))=\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))(u(s)-u(\lfloor s\rfloor_{\Delta t}))+r_s$$

where

$$r_s=\frac{1}{2}\mathrm{d}^2F\left(\xi u(\lfloor s\rfloor_{\Delta t})+(1-\xi)u(s)\right)(u(s)-u(\lfloor s\rfloor_{\Delta t}),u(s)-u(\lfloor s\rfloor_{\Delta t}))$$

for some $\xi\in(0,1)$. Since d^2F is bounded by Assumption 1, we have

$$\|r_s\|\le C\|u(s)-u(\lfloor s\rfloor_{\Delta t})\|^2.$$

Moreover, we have

$$u(s)-u(\lfloor s\rfloor_{\Delta t})=\delta_s^i+\delta_s^d+\delta_s^w$$

with

$$\delta_s^i=(\mathrm{e}^{-A(s-\lfloor s\rfloor_{\Delta t})}-\mathrm{id})u(\lfloor s\rfloor_{\Delta t}),$$
$$\delta_s^d=\int_{\lfloor s\rfloor_{\Delta t}}^s \mathrm{e}^{-A(s-\tau)}F(u(\tau))\mathrm{d}\tau,$$
$$\delta_s^w=\int_{\lfloor s\rfloor_{\Delta t}}^s \mathrm{e}^{-A(s-\tau)}\mathrm{d}W(\tau),$$

so we can write

$$F(u(s))-F(u(\lfloor s\rfloor_{\Delta t}))=\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i+\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^d+\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^w+r_s.$$

Thus, we have

$$\begin{aligned}
&\left[\mathbf{E}\left\|\sum_{|n|\le N}\int_0^t \mathrm{e}^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\left(F_n(u(s))-F_n(u(\lfloor s\rfloor_{\Delta t}))\right)\mathrm{d}s\cdot\phi_n\right\|^p\right]^{1/p}\\
&=\left[\mathbf{E}\left\|\mathbb{P}_N\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\left(F(u(s))-F(u(\lfloor s\rfloor_{\Delta t}))\right)\mathrm{d}s\right\|^p\right]^{1/p}\\
&\le C\left[\mathbf{E}\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i\mathrm{d}s\right\|^p\right]^{1/p}+C\left[\mathbf{E}\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^d\mathrm{d}s\right\|^p\right]^{1/p}\\
&\quad+C\left[\mathbf{E}\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^w\mathrm{d}s\right\|^p\right]^{1/p}+C\left[\mathbf{E}\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}r_s\mathrm{d}s\right\|^p\right]^{1/p}.
\end{aligned}$$

For the first term note that $\mathbf{P}\left(u(\lfloor s\rfloor_{\Delta t})\in D(A^\theta)\text{ for all }s\in[0,T]\right)=1$ by Theorem 4 and thus $\mathbf{P}$-a.s.

$$\|(\mathrm{e}^{-A(s-\lfloor s\rfloor_{\Delta t})}-\mathrm{id})u(\lfloor s\rfloor_{\Delta t})\|\le\|A^{-\theta}(\mathrm{e}^{-A(s-\lfloor s\rfloor_{\Delta t})}-\mathrm{id})A^\theta u(\lfloor s\rfloor_{\Delta t})\|,$$

since A^θ and the semigroup e^{-At} commute. Now, Lemma 2 gives $\mathbf{P}$-a.s.

$$\|A^{-\theta}(\mathrm{e}^{-A(s-\lfloor s\rfloor_{\Delta t})}-\mathrm{id})A^\theta u(\lfloor s\rfloor_{\Delta t})\|\le C|s-\lfloor s\rfloor_{\Delta t}|^\theta\|A^\theta u(\lfloor s\rfloor_{\Delta t})\|.$$

So we obtain by the assumptions on the nonlinearity F and the boundedness of the semigroup generated by $-A$ that $\mathbf{P}$-a.s.

$$\begin{aligned}
\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i\mathrm{d}s\right\|&\le C\Delta t^\theta\int_0^t\|\mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\|\,\|A^\theta u(\lfloor s\rfloor_{\Delta t})\|\mathrm{d}s\\
&\le C\Delta t^\theta\int_0^t\|A^\theta u(\lfloor s\rfloor_{\Delta t})\|\mathrm{d}s
\end{aligned}$$

for $t\in[0,T]$. An application of Hölder's inequality and Theorem 4 yields that

$$\sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^i\mathrm{d}s\right\|^p\right]^{1/p}\le C\cdot\Delta t^\theta \tag{28}$$

for all $\theta<\min\{1,\theta^*\}$.

Now to the second term. Here we have by the assumptions on the nonlinearity F and the boundedness of the semigroup generated by $-A$ that

$$\|\delta_s^d\| \le C \int_{\lfloor s \rfloor_{\Delta t}}^{s} (1 + \|u(\tau)\|) \mathrm{d}\tau.$$

So we obtain

$$\left\| \int_0^t \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \delta_s^d \mathrm{d}s \right\| \le C \int_0^t \int_{\lfloor s \rfloor_{\Delta t}}^{\lfloor s \rfloor_{\Delta t}+\Delta t} (1 + \|u(\tau)\|) \mathrm{d}\tau \mathrm{d}s$$

and it again follows by Theorem 4 and an application of Hölder's inequality that

$$\sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \int_0^t \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \delta_s^d \mathrm{d}s \right\|^p \right]^{1/p} \le C \cdot \Delta t. \tag{29}$$

The third term: Since

$$\delta_s^w = \int_{\lfloor s \rfloor_{\Delta t}}^{s} \mathrm{e}^{-A(s-\tau)} \mathrm{d}W(\tau)$$

we have

$$\begin{aligned}
&\int_0^t \left[\mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \right] \int_{\lfloor s \rfloor_{\Delta t}}^{s} \mathrm{e}^{-A(s-\tau)} \mathrm{d}W(\tau) \mathrm{d}s \\
&\quad = \int_0^t \int_{\lfloor s \rfloor_{\Delta t}}^{s} \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \mathrm{d}W(\tau) \mathrm{d}s \\
&\quad = \int_0^T \int_0^T 1_{[\lfloor s \rfloor_{\Delta t}, s]}(\tau) 1_{[0,t]}(s) \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \mathrm{d}W(\tau) \mathrm{d}s
\end{aligned}$$

using the stability of the Itô integral; see Section 4.1. By the stochastic Fubini theorem (see again Section 4.1), it follows that $\mathbf{P}$-a.s.

$$\begin{aligned}
&\int_0^T \int_0^T 1_{[\lfloor s \rfloor_{\Delta t}, s]}(\tau) 1_{[0,t]}(s) \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \mathrm{d}W(\tau) \mathrm{d}s \\
&\quad = \int_0^T \int_0^T 1_{[\tau, \lceil \tau \rceil_{\Delta t}]}(s) 1_{[0,t]}(\tau) \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \mathrm{d}s \, \mathrm{d}W(\tau) \\
&\quad = \int_0^t \left[\int_\tau^{\lceil \tau \rceil_{\Delta t}} \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \mathrm{d}s \right] \mathrm{d}W(\tau),
\end{aligned}$$

where $\lceil \tau \rceil_{\Delta t} = \min_{k\in\mathbb{N}} \{t_k : t_k \ge \tau\}$. Since

$$\mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \phi_n = \mathrm{e}^{-\alpha_n(t-\lfloor s \rfloor_{\Delta t})} (\mathrm{d}F)_n (u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-\alpha_n(s-\tau)} \phi_n,$$

where $(\mathrm{d}F)_n$ denotes the n-th coefficient of $\mathrm{d}F$, we have

$$\begin{aligned}
\|\mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)}\|_{L_2^0}^2 &= \sum_{n\in\mathbb{N}^d} \lambda_n \mathrm{e}^{-2\alpha_n(t-\lfloor s \rfloor_{\Delta t}+s-\tau)} |(\mathrm{d}F)_n(u(\lfloor s \rfloor_{\Delta t}))|^2 \\
&\le \sum_{n\in\mathbb{N}^d} \lambda_n \mathrm{e}^{-2\alpha_n(t-\tau)} |(\mathrm{d}F)_n(u(\lfloor s \rfloor_{\Delta t}))|^2
\end{aligned}$$

and thus

$$\|\mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)}\|_{L_2^0}^2 \le C \sum_{n\in\mathbb{N}^d} \lambda_n \mathrm{e}^{-2\alpha_n(t-\tau)}$$

by the assumptions on F. Hence it follows that

$$\left\| \int_\tau^{\lceil \tau \rceil_{\Delta t}} \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \mathrm{d}s \right\|_{L_2^0}^2 \le C \Delta t^2 \sum_{n\in\mathbb{N}^d} \lambda_n \mathrm{e}^{-2\alpha_n(t-\tau)}.$$

Thus, we obtain by the Burkholder–Davis–Gundy inequality (18) that

$$\sup_{t\in[0,\tau]} \left[\mathbf{E} \left\| \int_0^t \int_\tau^{\lceil \tau \rceil_{\Delta t}} \mathrm{e}^{-A(t-\lfloor s \rfloor_{\Delta t})} \mathrm{d}F(u(\lfloor s \rfloor_{\Delta t})) \mathrm{e}^{-A(s-\tau)} \mathrm{d}s \mathrm{d}W(\tau) \right\|^p \right]^{1/p} \le C \Delta t \left(\sum_{n\in\mathbb{N}^d} \frac{\lambda_n}{\alpha_n} \right)^{1/2}.$$

Since $\gamma+\kappa>d$ we have $\sum_{n\in\mathbb{N}^d}\frac{\lambda_n}{\alpha_n}<\infty$ and hence

$$\sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\mathrm{d}F(u(\lfloor s\rfloor_{\Delta t}))\delta_s^w \mathrm{d}s\right\|^p\right]^{1/p} \le C\cdot\Delta t. \tag{30}$$

Finally, for the remainder term r_s we obtain by straightforward estimations and Theorem 4 that

$$\sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}r_s \mathrm{d}s\right\|^p\right]^{1/p} \le C\cdot\Delta t^\theta + C\cdot\Delta t \tag{31}$$

for all $\theta<1$. Thus combining the estimates (28)–(31) yields

$$\mathbf{NL}_1 \le C\cdot\Delta t^\theta \tag{32}$$

for all $\theta<\min\{1,\theta^*\}$.

(ii) The second term. Here we have

$$\mathbf{NL}_2=\sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\int_0^t\sum_{|n|\le N}\mathrm{e}^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\left(F_n(u(\lfloor s\rfloor_{\Delta t}))-F_n(\widehat{u}(\lfloor s\rfloor_{\Delta t}))\right)\mathrm{d}s\cdot\phi_n\right\|^p\right]^{1/p}.$$

Again, we can write

$$\mathbf{NL}_2=\sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\mathbb{P}_N\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\left(F(u(\lfloor s\rfloor_{\Delta t}))-F(\widehat{u}(\lfloor s\rfloor_{\Delta t}))\right)\mathrm{d}s\right\|^p\right]^{1/p}.$$

So we obtain by Jensen's inequality, the Lipschitz continuity of F and the boundedness of e^{-At} that

$$\mathbf{NL}_2\le C\int_0^\tau\sup_{t\in[0,s]}\left[\mathbf{E}\|u(t)-\widehat{u}(t)\|^p\right]^{1/p}\mathrm{d}s. \tag{33}$$

(iii) The third nonlinear term.

$$\mathbf{NL}_3=\sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\sum_{|n|\le N}\int_0^t \mathrm{e}^{-\alpha_n(t-\lfloor s\rfloor_{\Delta t})}\left(\mathrm{e}^{\alpha_n(s-\lfloor s\rfloor_{\Delta t})}-1\right)F_n(u(s))\mathrm{d}s\cdot\phi_n\right\|^p\right]^{1/p}.$$

Rewriting this expression using the projection operator and applying Jensen's inequality we have

$$\begin{aligned}
\mathbf{NL}_3 &= \sup_{t\in[0,\tau]}\left[\mathbf{E}\left\|\mathbb{P}_N\int_0^t \mathrm{e}^{-A(t-\lfloor s\rfloor_{\Delta t})}\left(\mathrm{e}^{A(s-\lfloor s\rfloor_{\Delta t})}-\mathrm{id}\right)F(u(s))\mathrm{d}s\right\|^p\right]^{1/p}\\
&\le \sup_{t\in[0,\tau]}\int_0^t\left[\mathbf{E}\left\|\mathrm{e}^{-A(t-s)}\left(\mathrm{id}-\mathrm{e}^{-A(s-\lfloor s\rfloor_{\Delta t})}\right)F(u(s))\right\|^p\right]^{1/p}\mathrm{d}s\\
&\le \sup_{t\in[0,\tau]}\int_0^t\left[\mathbf{E}\left\|A^\theta\mathrm{e}^{-A(t-s)}A^{-\theta}\left(\mathrm{id}-\mathrm{e}^{-A(s-\lfloor s\rfloor_{\Delta t})}\right)F(u(s))\right\|^p\right]^{1/p}\mathrm{d}s\\
&\le \sup_{t\in[0,\tau]}\int_0^t\|A^\theta\mathrm{e}^{-A(t-s)}\|\left\|A^{-\theta}\left(\mathrm{id}-\mathrm{e}^{-A(s-\lfloor s\rfloor_{\Delta t})}\right)\right\|\left[\mathbf{E}\|F(u(s))\|^p\right]^{1/p}\mathrm{d}s.
\end{aligned}$$

Now Theorem 4 and Lemma 2 give

$$\mathbf{NL}_3\le C\Delta t^\theta\sup_{t\in[0,\tau]}\int_0^t(t-s)^{-\theta}\left[\mathbf{E}\|F(u(s))\|^p\right]^{1/p}\mathrm{d}s$$

for all $\theta<1$. So using Assumption 1 and Theorem 4 we have

$$\mathbf{NL}_3\le C\cdot\Delta t^\theta \tag{34}$$

for all $\theta<1$.

(iv) Now, combining (32)–(34), we have

$$\mathbf{NL}\le C\int_0^\tau\sup_{t\in[0,s]}\left[\mathbf{E}\,\|u(t)-\widehat{u}(t)\|^p\right]^{1/p}\mathrm{d}s+C\cdot\Delta t^\theta \tag{35}$$

for all $\theta<\min\{1,\theta^*\}$.

4.6. Conclusion

Combining the estimates (24)–(27) and (35) we have achieved the following inequality:

$$\sup_{s\in[0,\tau]}\left[\mathbf{E}\|u(s)-\widehat{u}(s)\|^p\right]^{1/p}\le C\int_0^\tau \sup_{t\in[0,s]}\left[\mathbf{E}\|u(t)-\widehat{u}(t)\|^p\right]^{1/p}\mathrm{d}s + C\cdot N^{-\kappa}+C\cdot N_w^{(-\gamma-\kappa+d)/2}+C\cdot\Delta t^{\theta}$$

for all $\theta<\min\{1,\theta^*\}$. Gronwall's lemma provides now the assertion of Theorem 5.

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Appendix. Proof of Theorem 4

We first show the following lemma:

Lemma 3. *Let* $\kappa+\gamma>d, \theta<\theta^*=\frac{\gamma+\kappa-d}{2\kappa}$ *and* $\vartheta\in[0,1/2]$ *be such that* $\vartheta+\theta<\theta^*$. *Then there exist constants* $C_9, C_{10}, C_{11}>0$, *which are independent of* $s,t\in[0,T]$, *such that*

$$\int_0^t \|\mathrm{e}^{-Au}\|^2_{L_2^0}\mathrm{d}u\le C_9, \tag{A.1}$$

$$\int_s^t \|A^\theta \mathrm{e}^{-A(t-u)}\|^2_{L_2^0}\mathrm{d}u\le C_{10}\cdot|t-s|^{2\vartheta} \tag{A.2}$$

and

$$\int_0^s \|A^\theta(\mathrm{e}^{-A(t-u)}-\mathrm{e}^{-A(s-u)})\|^2_{L_2^0}\mathrm{d}u\le C_{11}\cdot|t-s|^{2\vartheta}. \tag{A.3}$$

Proof. Throughout this proof, we will denote constants which are independent of $s,t\in[0,T]$ by C regardless of their value.
(i) Recall that here L_2^0 denotes the space of Hilbert–Schmidt operators from $Q^{1/2}(H)$ to H and $\|\cdot\|_{L_2^0}$ is the corresponding norm given by $\|C\|^2_{L_2^0}=\mathrm{Tr}(C^*QC)$. Since e^{-Au} is self-adjoint with eigenvalues $\mathrm{e}^{-\alpha_j u}$ and eigenvectors ϕ_j and since moreover Q is self-adjoint with eigenvalues λ_j and eigenvectors ϕ_j, and $\phi_j, j\in\mathbb{N}^d$, is an orthonormal basis of H, we have

$$\mathrm{Tr}(\mathrm{e}^{-Au}Q\mathrm{e}^{-Au})=\sum_{j\in\mathbb{N}^d}\langle \mathrm{e}^{-Au}Q\mathrm{e}^{-Au}\phi_j,\phi_j\rangle=\sum_{j\in\mathbb{N}^d}\mathrm{e}^{-2\alpha_j u}\lambda_j.$$

Thus we obtain

$$\int_0^T \|\mathrm{e}^{-As}\|^2_{L_2^0}\mathrm{d}s=\sum_{j\in\mathbb{N}^d}\int_0^T \mathrm{e}^{-2\alpha_j s}\lambda_j\mathrm{d}s\le\sum_{j\in\mathbb{N}^d}\frac{\lambda_j}{\alpha_j}.$$

Since

$$0\le\frac{\lambda_j}{\alpha_j}\le C\cdot|j|^{-\gamma-\kappa}$$

by Assumptions 2 and 3, we obtain

$$\int_0^T \|\mathrm{e}^{-As}\|^2_{L_2^0}\mathrm{d}s\le C\sum_{j\in\mathbb{N}^d}|j|^{-\gamma-\kappa}<\infty$$

for $\kappa+\gamma>d$.
(ii) We have similarly that

$$\|A^\theta\mathrm{e}^{-A(t-u)}\|^2_{L_2^0}=\sum_{j\in\mathbb{N}^d}\mathrm{e}^{-2\alpha_j(t-u)}\lambda_j\alpha_j^{2\theta}$$

and hence

$$\int_s^t \|A^\theta\mathrm{e}^{-A(t-u)}\|^2_{L_2^0}\mathrm{d}u\le\sum_{j\in\mathbb{N}^d}\lambda_j\alpha_j^{2\theta-1}(1-\mathrm{e}^{-2\alpha_j(t-s)}).$$

Since for every $\theta \in [0, 1]$ we have

$$|e^{-x} - e^{-y}| \leq |x - y|^{\theta}, \quad x, y \geq 0,$$

it follows that

$$\int_s^t \|A^{\theta} e^{-A(t-u)}\|^2_{L_2^0} du \leq |t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} \lambda_j \alpha_j^{2(\theta+\vartheta)-1} \leq C|t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} |j|^{-\gamma-\kappa+2\kappa(\theta+\vartheta)}$$

for $\vartheta \in [0, 1/2]$ by Assumptions 2 and 3. Moreover, $\vartheta + \theta < \theta^*$ yields

$$2\kappa(\theta + \vartheta) < \gamma + \kappa - d$$

and thus

$$\sum_{j\in\mathbb{N}^d} |j|^{-\gamma-\kappa+2\kappa(\theta+\vartheta)} < \infty.$$

(iii) Like in (i) we obtain

$$\begin{aligned}\|A^{\theta}(e^{-A(t-u)} - e^{-A(s-u)})\|^2_{L_2^0} &= \|A^{\theta} e^{-A(s-u)}(e^{-A(t-s)} - \mathrm{id})\|^2_{L_2^0} \\ &= \sum_{j\in\mathbb{N}^d} \alpha_j^{2\theta} \lambda_j e^{-2\alpha_j(s-u)} (e^{-\alpha_j(t-s)} - 1)^2\end{aligned}$$

and thus

$$\|A^{\theta}(e^{-A(t-u)} - e^{-A(s-u)})\|^2_{L_2^0} \leq C|t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} \alpha_j^{2(\theta+\vartheta)} \lambda_j e^{-2\alpha_j(s-u)},$$

and also

$$\int_0^s \|A^{\theta}(e^{-A(t-u)} - e^{-A(s-u)})\|^2_{L_2^0} du \leq C|t-s|^{2\vartheta} \sum_{j\in\mathbb{N}^d} \lambda_j \alpha_j^{2(\theta+\vartheta)-1}.$$

Now we can proceed as in (ii). □

Proof of Theorem 4. We will again denote constants which are independent of $s, t \in [0, T]$ by C regardless of their value.

(i) Note first that the stochastic integrals

$$W_A(t) = \int_0^t e^{-A(t-s)} dW(s), \quad t \in [0, T],$$

are well defined if

$$\int_0^T \|e^{-As}\|_{L_2^0} ds < \infty$$

(see Theorem 5.2 in [14]). The latter is true for $\kappa + \gamma > d$ due to Lemma 3. The existence of a unique mild solution of Eq. (1) with $\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|^p < \infty$ for all $p \geq 1$ follows now from a straightforward generalisation of Theorem 7.6 in [14].

(ii) Now recall that $\theta^* = \frac{\gamma+\kappa-d}{2\kappa}$, let $\theta < \theta^*$ and consider

$$W_A(t) - W_A(s) = \int_s^t e^{-A(t-u)} dW(u) + \int_0^s \left(e^{-A(t-u)} - e^{-A(s-u)}\right) dW(u).$$

By Lemma 3 and the stability of the Itô integral (see Section 4.1), we have that $A^{\theta}(W_A(t) - W_A(s))$ is $\mathbf{P}$-a.s. well defined. Moreover, by the Burkholder–Davis–Gundy inequality (18) and the above lemma we obtain

$$\begin{aligned}&\left[\mathbf{E}\|A^{\theta}(W_A(t) - W_A(s))\|^p\right]^{1/p} \\ &\leq C\left[\mathbf{E}\left\|\int_s^t A^{\theta} e^{-A(t-u)} dW(u)\right\|^p\right]^{1/p} + C\left[\mathbf{E}\left\|\int_0^s A^{\theta}\left(e^{-A(t-u)} - e^{-A(s-u)}\right) dW(u)\right\|^p\right]^{1/p} \\ &\leq C\left[\int_s^t \|A^{\theta} e^{-A(t-u)}\|^2_{L_2^0} du\right]^{1/2} + C\left[\int_0^s \left\|A^{\theta}\left(e^{-A(t-u)} - e^{-A(s-u)}\right)\right\|^2_{L_2^0} du\right]^{1/2} \\ &\leq C|t-s|^{\vartheta}\end{aligned}$$

for all $\vartheta \in [0, 1/2]$ such that $\vartheta + \theta < \theta^*$. The Kolmogorov–Chentsov theorem now implies that there exists a modification $\widetilde{W}_A$ of W_A such that

$$\widetilde{W}_A(\cdot, \omega) \in \bigcap_{\theta < \theta^*} C\left([0, T]; D(A^\theta)\right)$$

for almost all $\omega \in \Omega$. Moreover, we have

$$\sup_{t\in[0,T]} \left[\mathbf{E}\|A^\theta \widetilde{W}_A(t)\|^p\right]^{1/p} < \infty, \qquad \left[\mathbf{E}\|\widetilde{W}_A(t) - \widetilde{W}_A(s)\|^p\right]^{1/p} \le C|t - s|^{\min\{1/2,\theta\}}$$

for all $s, t \in [0, T]$ and $\theta < \theta^*$.

(iii) Finally consider $A^\theta(u(t) - u(s))$, $s, t \in [0, T]$. We have $\mathbf{P}$-a.s.

$$A^\theta(u(t) - u(s)) = A^\theta(\mathrm{e}^{-At}u(0) - \mathrm{e}^{-As}u(0)) + A^\theta \int_s^t \mathrm{e}^{-A(t-\tau)}F(u(\tau))\mathrm{d}\tau + A^\theta \int_0^s \left(\mathrm{e}^{-A(t-\tau)} - \mathrm{e}^{-A(s-\tau)}\right) F(u(\tau))\mathrm{d}\tau + A^\theta(\widetilde{W}_A(t) - \widetilde{W}_A(s))$$

for all $s, t \in [0, T]$. So it follows that

$$\left[\mathbf{E}\|A^\theta(u(t) - u(s))\|^p\right]^{1/p} \le \mathbf{I_1} + \mathbf{I_2} + \mathbf{I_3} + \mathbf{I_4}$$

with

$$\mathbf{I_1} = \|A^\theta \mathrm{e}^{-As}(\mathrm{e}^{-A(t-s)} - \mathrm{id})u(0)\|,$$
$$\mathbf{I_2} = \left[\mathbf{E}\left\|A^\theta \int_s^t \mathrm{e}^{-A(t-\tau)}F(u(\tau))\mathrm{d}\tau\right\|^p\right]^{1/p},$$
$$\mathbf{I_3} = \left[\mathbf{E}\left\|A^\theta \int_0^s \mathrm{e}^{-A(s-\tau)}\left(\mathrm{e}^{-A(t-s)} - \mathrm{id}\right) F(u(\tau))\mathrm{d}\tau\right\|^p\right]^{1/p},$$
$$\mathbf{I_4} = \left[\mathbf{E}\|A^\theta(\widetilde{W}_A(t) - \widetilde{W}_A(s))\|^p\right]^{1/p}.$$

Since $u(0) \in D(A)$ we have by Lemma 2 that

$$\begin{aligned}\mathbf{I_1} &= \|A^\theta \mathrm{e}^{-As}(\mathrm{e}^{-A(t-s)} - \mathrm{id})u(0)\| \\ &\le \|\mathrm{e}^{-As}A^{\theta-1}(\mathrm{e}^{-A(t-s)} - \mathrm{id})Au(0)\| \\ &\le \|\mathrm{e}^{-As}\|\, \|A^{\theta-1}(\mathrm{e}^{-A(t-s)} - \mathrm{id})\|\, \|Au(0)\| \\ &\le C|t - s|^{1-\theta}\end{aligned}$$

for all $\theta < 1$. Moreover, by step (ii) we have

$$\mathbf{I_4} \le C|t - s|^\vartheta$$

for all $\vartheta \in [0, 1/2]$ such that $\vartheta + \theta < \theta^*$. For the second term we obtain by Jensen's inequality and the stability of the integral that

$$\begin{aligned}\mathbf{I_2} &= \left[\mathbf{E}\left\|\int_s^t A^\theta \mathrm{e}^{-A(t-\tau)}F(u(\tau))\mathrm{d}\tau\right\|^p\right]^{1/p} \\ &\le \int_s^t \|A^\theta \mathrm{e}^{-A(t-\tau)}\|[\mathbf{E}\|F(u(\tau))\|^p]^{1/p}\mathrm{d}\tau.\end{aligned}$$

Hence Assumption 1 and Lemma 2 give

$$\mathbf{I_2} \le C\int_s^t |t - \tau|^{-\theta}\left(1 + \sup_{t\in[0,T]} [\mathbf{E}\|u(t)\|^p]^{1/p}\right)\mathrm{d}\tau.$$

Since $\sup_{t\in[0,T]} \mathbf{E}\|u(t)\|^p < \infty$ by part (i) of the proof, it follows that

$$\mathbf{I_2} \le C|t - s|^{1-\theta}.$$

Finally, consider the third term. Here we have, proceeding as above,

$$\mathbf{I_3} = \left[\mathbf{E}\left\|\int_0^s A^\theta \mathrm{e}^{-A(s-\tau)}\left(\mathrm{e}^{-A(t-s)} - \mathrm{id}\right) F(u(\tau))\mathrm{d}\tau\right\|^p\right]^{1/p}$$

$$\leq C \int_0^s \|A^{\theta+\delta} e^{-A(s-\tau)}\| \left\|A^{-\delta}\left(e^{-A(t-s)} - \mathrm{id}\right)\right\| \left(1 + \sup_{t\in[0,T]} [\mathbf{E}\|u(t)\|^p]^{1/p}\right) d\tau$$
$$\leq C|t-s|^{\delta}$$

for $\delta < 1 - \theta$.

Combining the estimates for $\mathbf{I_1}$, $\mathbf{I_2}$, $\mathbf{I_3}$ and $\mathbf{I_4}$ we obtain

$$\left[\mathbf{E}\|A^{\theta}(u(t) - u(s))\|^p\right]^{1/p} \leq C|t-s|^{\vartheta} + C|t-s|^{\delta},$$

for all $\vartheta \in [0, 1/2]$ such that $\theta + \vartheta < \theta^*$ and $\delta \in [0, 1]$ such that $\delta < 1 - \theta$. Hence by the Kolmogorov–Chentsov theorem it follows that there exists a modification $\widetilde{u}$ of u such that

$$\widetilde{u}(\cdot, \omega) \in \bigcap_{\theta < \min\{1,\theta^*\}} C\left([0, T]; D(A^{\theta})\right)$$

for almost all $\omega \in \Omega$. Furthermore, the above estimates give

$$\sup_{t\in[0,T]} \left[\mathbf{E}\|A^{\theta}\widetilde{u}(t)\|^p\right]^{1/p} < \infty, \qquad \left[\mathbf{E}\|\widetilde{u}(t) - \widetilde{u}(s)\|^p\right]^{1/p} \leq C|t-s|^{\min\{1/2,\theta\}}$$

for all $s, t \in [0, T]$ and $\theta < \min\{1, \theta^*\}$. □

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