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PERIODIC ORBITS AND UNSTABLE MANIFOLDS FOR ORDINARY DIFFERENTIAL EQUATIONS

Tony Shardlow

Scientific Computing & Computational Mathematics, Gates 2B—MC 9025, Stanford University, CA 94305, USA
shardlow@sccm.stanford.edu

Abstract

Consider the unstable manifold of a hyperbolic periodic orbit of an ordinary differential equation under C^1 perturbations of the vector field and under approximation by a one-step numerical method, which is at least first order. Trajectories bounded backwards in time near the periodic orbit perturb Hausdorff continuously. This result as applied to numerical perturbations improves on Alouges-Debussche [1], who give only continuity of the unstable manifold, and on Beyn [3], who gives continuity of trajectories only when the periodic orbit is unstable. As a corollary, we find that attractors perturb Hausdorff continuously *when* the attractor equals a union of locally continuous unstable manifolds of invariant sets.

1 Introduction and Derivation of Equations

We consider two perturbations of the ordinary differential equation

$$\frac{dx}{dt} = f(x), \qquad x \in \mathbb{R}^d, \qquad f \in C^1(\mathbb{R}^d, \mathbb{R}^d). \tag{1.1}$$

Perturbing the vector field. We study

$$\frac{dx}{dt} = f(x) + F(x,\epsilon), \qquad x \in \mathbb{R}^d, \tag{1.2}$$

where the perturbation, F, belongs to $C^1(\mathbb{R}^d \times \mathbf{R}; \mathbb{R}^d)$ and vanishes when $\epsilon = 0$.

Perturbing numerically. We approximate (1.1) with a pth order, $p > 0$, one step numerical method. To be precise, let $S(\cdot)$ denote the semigroup for (1.1), so that $S(t)x$ solves (1.1) at time t for initial condition x; and let $S_{\Delta t}(\cdot)$ denote the numerical semigroup applied with time step Δt, so that $S_{\Delta t}(n)x$ is the nth iterate of the method, the approximation to $S(n\Delta t)x$. Define the truncation error T by

$$T(x,\Delta t) := S_{\Delta t}(1)x - S(\Delta t)x.$$

For every bounded set B, suppose that $L > 0$ exists such that, for $x, \tilde{x} \in B$,

$$\begin{aligned} ||T(x,\Delta t)|| &\le L\Delta t^{p+1}, \\ ||T(x,\Delta t) - T(\tilde{x},\Delta t)|| &\le L\Delta t^{p+1}||x - \tilde{x}||. \end{aligned} \tag{1.3}$$

Humphries–Stuart [9] derives these estimates for Runge-Kutta methods and shows how to apply such estimates to linear multi-step methods.

Suppose that (1.1) admits a C^2 hyperbolic periodic solution and denote this orbit by Γ_0. We recall the definition of hyperbolicity of periodic orbits.

Definition 1.1 Consider a cross section M of the flow around Γ_0. The *Poincaré mapping* P takes elements of M to the point of first return to M. In other words, with $S(t)$ denoting the solution operator for (1.1) and $x \in \mathcal{M}$, the Poincaré mapping $P(x) := S(T)x$ where $T := \min\{t > 0 \colon S(t)x \in M\}$. At $x_0 \in M \cap \Gamma_0$, the Taylor expansion is

$$P(x_0 + \Delta x) = x_0 + A\,\Delta x + \text{ nonlinear terms in } \Delta x, \qquad A := \left.\frac{dP}{dx}\right|_{x=x_0}.$$

The periodic orbit Γ_0 is *hyperbolic* if no eigenvalue of A has unit magnitude.

Theorem 1.2 *For every $\delta > 0$, there exists $\epsilon^* > 0$ such that, for every $\epsilon \in (-\epsilon^*, \epsilon^*)$, the perturbed equation (1.2) has a periodic solution, Γ_ϵ, within the δ neighbourhood of Γ_0.* □

This result is proved in Hale [7]. The analogous result for numerical methods was proved by Beyn [3].

Theorem 1.3 *Suppose that $f \in C^3(\mathbb{R}^d, \mathbb{R}^d)$. Consider a one step method whose truncation error satisfies (1.3). In addition, suppose that $T(x, \Delta t)$ is differentiable and that $S_{\Delta t}(1)x$ is uniformly in Δt Lipschitz for x in a neighbourhood of Γ_0. Then, for each $\delta > 0$, there exists $\Delta t^* > 0$ such that this method, applied with $\Delta t \leq \Delta t^*$, has an invariant curve, $\Gamma_{\Delta t}$, that converges to Γ_0 with order Δt^p.* □

We suppose that a perturbed periodic orbit Γ_ϵ (respectively, invariant curve $\Gamma_{\Delta t}$) exists for the perturbed systems that tend to Γ_0 as $\epsilon \to 0$ (resp., $\Delta t \to 0$). Possibly, the numerically perturbed curve, $\Gamma_{\Delta t}$, may exist for a dynamical system that fails the hypothesis of Beyn's theorem. Such a curve still yields to the analysis of this paper.

Definition 1.4 Let W_0, W_ϵ, and $W_{\Delta t}$ be the unstable manifolds of the invariant sets Γ_0, Γ_ϵ, and $\Gamma_{\Delta t}$. This means W_0 (respectively, W_ϵ, $W_{\Delta t}$) is the set of initial conditions for systems (1.1) (the respective perturbed systems) whose trajectories tend to the invariant set Γ_0 (resp., Γ_ϵ, $\Gamma_{\Delta t}$) as time t (resp., t, n) goes to negative infinity.

The Hausdorff distance between two sets A, B is defined by

$$d_H(A, B) := \sup_{a \in A} \inf_{b \in B} \|a - b\| + \sup_{b \in B} \inf_{a \in A} \|b - a\|.$$

The main work of this paper gives the existence and Hausdorff continuity of a trajectory bounded backwards in time for every point on the linearised unstable manifold; each point on the linearised manifold gives a bounded trajectory of the perturbed equation by perturbing the initial condition in the stable direction. Existence and continuity for continuous perturbations is Theorem 2.3; existence for numerical perturbations is Theorem 3.7; and continuity for numerical pertubations is Theorem 4.2. The continuity of trajectories with respect to the Hausdorff distance is the strongest available; trajectories bounded near Γ_0 become out of phase for large time, because the dynamics on Γ_0 exhibit no contraction. Consequently, the analysis can give no pointwise estimates, even though a uniform pointwise estimate does exist for trajectories bounded near a hyperbolic equilibria [4, 12].

As corollaries of these results, W_ϵ perturbs Hausdorff continuously in a neighbourhood of Γ_0 (Corollary 2.4); and analogously, $W_{\Delta t}$ perturbs Hausdorff continuously in a neighbourhood of Γ_0 (Corollary 4.4).

For the numerical perturbations, we gain an estimate between the radial variables of order Δt^p (Corollary 4.4). But, as Beyn [3] points out, this only gives order Δt convergence of the trajectories (Corollary 4.3), because the distance between neighbouring points of the orbit on the invariant curve is order Δt. (When the rotation number is irrational, the invariant curve will fill in and the Hausdorff distance, as in the continuous case, is controlled only by the radial estimate.) This result improves on Beyn [3] (only continuity of trajectories of unstable periodic orbits is proved) and on Alouges–Debussche [1] (only continuity of unstable manifolds is proved).

The results of this paper apply to the lower semi-continuity of attractors. An *attractor* is a invariant set that uniformly attracts every point in a neighbourhood of the attractor. Many attractors are unstable, even to the smoothest perturbations [12]. It is known, however, that attractors are upper semi-continuous [8]. Furthermore, thanks to Humphries [10], attractors made from the unstable manifolds of hyperbolic equilibria (i.e., equal to the closure of the union of the unstable manifolds) are lower semi-continuous. In the appendix, this result is extended to give lower semi-continuity of attractors made from the unstable manifolds of invariant sets that are locally lower semi-continuous. Thus, with the results of sections 2–4, attractors made from the unstable manifolds of hyperbolic equilibria and of hyperbolic periodic orbits perturb continuously with respect to C^1 perturbation of the vector field and to approximation by pth order, $p > 0$, one step methods. This applies, for example, to Duffing's equation

$$\frac{d^2x}{dt^2} - \lambda\frac{dx}{dt} - x + x^3 = \gamma\cos t, \qquad \text{parameters } \lambda,\ \gamma.$$

For small γ, this equation admits a chaotic attractor, which is equal to closure of the unstable manifold of the periodic solution (numerical evidence suggests this holds for large γ) [6]. Thus, we guarantee the convergence of the attractor appearing in numerical approximations of Duffing's equation to the true attractor. (See also Benedicks & Carleson [2], who relate the Hénon attractor to the unstable manifolds of equilibria.)

Equations in polar coordinates

We develop (1.2) by introducing polar coordinates. The next lemma comes from Hale [7].

Lemma 1.5 *There exist coordinates (θ, ρ) on a neighbourhood of Γ_0 defined for $\|\rho\| \le \delta$, some $\delta > 0$, such that*

(i) The coordinates are C^1;

(ii) $\Gamma_0 = \{(\theta, 0) \colon 0 \le \theta \le 2\pi\}$;

(iii) Written in (θ, ρ) coordinates, for $\|\rho\| \le \delta$, system (1.2) becomes

$$\frac{d\theta}{dt} = 1 + g_1(\theta, \rho, \epsilon) \tag{1.4}$$

$$\frac{d\rho}{dt} = A(\theta)\rho + g_2(\theta, \rho, \epsilon), \tag{1.5}$$

where

$$g_1 \in C^1(\mathbb{R} \times \mathbb{R}^{d-1}, \mathbb{R}), \qquad g_2 \in C^1(\mathbb{R} \times \mathbb{R}^{d-1}, \mathbb{R}^{d-1}),$$

$$A \in C^1(\mathbb{R}, \mathbb{R}^{(d-1)\times(d-1)}),$$

and g_1, g_2, A are 2π periodic in θ. Further, g_1 and g_2 are small:

$$|g_1(\theta, \rho, 0)| = O(\|\rho\|) \qquad \text{and} \qquad \|g_2(\theta, \rho, 0)\| = O(\|\rho\|^2).$$

Following Hale [7], we reduce (1.4–1.5) to (1.7), an equivalent system of one less dimension that has no dependence on θ in the linear term. First, by using the smallness and continuity of g_1, we choose δ and ϵ^* such that

$$|g_1(\theta, \rho, \epsilon)| \le \frac{1}{2}, \qquad |\epsilon| \le \epsilon^*, \quad \|\rho\| \le \delta.$$

We may now eliminate t from (1.4–1.5) by dividing, to find the non-autonomous equation

$$\frac{d\rho}{d\theta} = A(\theta)\rho + g_3(\theta, \rho, \epsilon), \qquad |\epsilon| \le \epsilon^*, \quad \|\rho\| \le \delta; \tag{1.6}$$

$$g_3(\theta, \rho, \epsilon) := -\frac{A(\theta)\rho g_1(\theta, \rho, \epsilon)}{1 + g_1(\theta, \rho, \epsilon)} + \frac{g_2(\theta, \rho, \epsilon)}{1 + g_1(\theta, \rho, \epsilon)}.$$

This equation simplifies by using Floquet's theorem [7].

Floquet's Theorem. *For any continuous real valued $n \times n$ matrix $A(t)$ with period T, the solution to*

$$\frac{dX}{dt} = A(t)X, \quad X(0) = I$$

may be written

$$X(t) = P(t)\exp(Bt)$$

where B is a constant $n \times n$ real matrix and $P(t)$ is a $n \times n$ real matrix valued function with period $2T$. □

Let B and $P(\theta)$ be the matrices arising from $d\rho/d\theta = A(\theta)\rho$ by use of this theorem. Substituting $P(\theta)r$ for ρ in (1.6), we find

$$\frac{dr}{d\theta} = Br + g(\theta, r, \epsilon) \tag{1.7}$$

where

$$g(\theta, r, \epsilon) := P(\theta)^{-1} g_3(\theta, P(\theta)r, \epsilon).$$

Using (1.7), the Poincaré mapping may be written

$$r \mapsto \exp(4\pi B)r + \text{ nonlinear terms},$$

and hence, by definition of hyperbolic, the unstable and stable eigenspaces of B together span $\mathbb{R}^{d-1}$. Denote by $\pi_\pm$ the projections to the stable (π_-) and unstable (π_+) eigenspaces of B. We state three properties of $\pi_\pm$ that are crucial to all later proofs [7].

Lemma 1.6 *There exists $K, \alpha > 0$ such that*

(i) $\|\exp(B\theta)r_\pm\| \leq Ke^{\pm\alpha\theta}\|r_\pm\|$ *for all* $r_\pm \in \pi_\pm\mathbb{R}^{d-1}$ *and all* $\theta \in \mathbb{R}$;

(ii) $\|\exp(B\theta)r\| \leq K\|r\|e^{\alpha|\theta|}$ *for all* $r \in \mathbb{R}^{d-1}$ *and all* $\theta \in \mathbb{R}$;

(iii) $\|\pi_\pm r\| \leq K\|r\|$ *for all* $r \in \mathbb{R}^{d-1}$. □

The existence and continuity for pertubations of the vector field derives from a Lyapunov-Perron approach to (1.7); that is, we integrate (1.7) forward along the stable directions of B and backwards along the unstable directions, to gain an integral equation that bounded solutions solve. We can then apply standard contraction arguments to prove existence and continuity. This approach follows Hale [7]; this equation does not directly fall to his theory as it is non-autonomous. The continuity of numerical perturbations also comes from (1.7) by a Lyapunov-Perron argument. For the existence of bounded numerical solutions, a different approach is needed to avoid the difficulty of applying the numerical method to an equation without time.

The θ derivative in (1.7) makes the analysis of the numerical method hard. To analyse the numerical perturbations, we use an equation with only time derivatives. Multiplying (1.7) and (1.4) gives an ordinary differential equation with time derivatives whose dominant terms are still constant:

$$\frac{d\theta}{dt} = 1 + h_1(\theta, r), \qquad \frac{dr}{dt} = Br + h_2(\theta, r), \tag{1.8}$$

where

$$\begin{aligned} h_1(\theta, r) &:= g_1(\theta, P(\theta)r, 0), \\ h_2(\theta, r) &:= g_1(\theta, P(\theta)r, 0) \cdot (Br + g(\theta, r, 0)) + g(\theta, r, 0). \end{aligned} \tag{1.9}$$

Working with (1.8) is harder than working with (1.7), as the expression for $d\theta/dt$ involves neither expansion nor contraction. Yet another change of coordinates would resolve this problem: ideally, we would introduce an angular variable ϕ so that the angular part of the numerical method

$$\theta_{n+1} = \theta_n + \Delta t + \text{small terms} \qquad \text{is replaced by} \qquad \phi_{n+1} = \phi_n + \widetilde{\Delta t}.$$

Then, the angular dynamics is simply a rotation and the Lyapunov–Perron method applies as before. Finding such coordinates seems difficult; even, for such coordinates to exists on the invariant curve, more regularity is necessary (see [11]).

Instead, this paper proves bounded solutions exist by introducing a metric that weights the positions at later times increasingly strongly. Because solutions on the unstable manifold approach a solution on the invariant curve exponentially fast, the weighted distance between a bounded solution and the invariant curve is finite. Further, there exists a mapping that fixes bounded solutions and that contracts with respect to the weighted metric. This suggests employing the Contraction Mapping Theorem. In fact, we prefer to work directly with sequences—this avoids checking that the contraction maps into a particular space.

The contraction argument for the perturbed ODE finds a single fixed point for each $\theta \in [0, 2\pi]$ and $r_+ \in \pi_+\mathbb{R}^{d-1}$. When applying contractions to (1.7), it is impossible to fix θ and, to find a solution for each initial angle, we must introduce a space of closed curves and find curves where each point on the curve belongs to a bounded numerical solution. This approach is similar to Alogues-Debussche [1], who gave a proof of continuity of the unstable manifold with respect to numerical approximation in time for a general class of PDEs by considering a space of curves $\{(\theta, r(\theta)) \colon r(\theta) \text{ is } C^1 \text{ and small}\}$. We take a larger space of closed curves, $\{(\theta(\zeta), r(\zeta)) \colon \theta(\zeta), r(\zeta) \text{ are } C^1 \text{ and small}\}$. This space makes for a simpler argument, but one depending on the Floquet theorem of finite dimensions.

2 Perturbing the Vector Field

The perturbed differential equation (1.7) is considered. We require two lemmas for the main theorem. The first establishes a Lipschitz property for the nonlinear term g; the second is a non-autonomous version of a Theorem in Hale [7] that characterizes bounded solutions using an integral relation. Let $\{r \in \mathbb{R}^{d-1} \colon \|r\| \leq \delta\}$ be the neighbourhood for which (1.7) is equivalent to (1.2) and suppose that Γ_ϵ belongs to this neighbourhood for $|\epsilon| \leq \epsilon^*$.

Lemma 2.1 *If $r, \tilde{r} \in \mathbb{R}^{d-1}$ and $\|r\|, \|\tilde{r}\| \leq \delta$, then*

$$\|g(\theta, r, \epsilon) - g(\theta, \tilde{r}, \epsilon)\| \leq \eta(\delta, \epsilon)\|r - \tilde{r}\|$$

where the Lipschitz constant $\eta(\delta, \epsilon) \to 0$ as $\max\{\delta, \epsilon\} \to 0$.

Proof. Let $\eta_1(\delta, \epsilon)$, $\eta_2(\delta, \epsilon)$ be the Lipschitz constant of $g_1(\theta, P(\theta)r, \epsilon)$, $g_2(\theta, P(\theta)r, \epsilon)$ with respect to r for $\|r\| \leq \delta$. Since $g_2(\theta, \rho, \epsilon)$ is order $\|\rho\|^2$ at $\epsilon = 0$, the Lipschitz constant $\eta_2(\delta, \epsilon)$ goes to zero as $\max\{\delta, \epsilon\} \to 0$. Calculate the Lipschitz constant of $g(\theta, r, \epsilon)$ with respect to r:

$$\begin{aligned} &\|g(\theta, r, \epsilon) - g(\theta, \tilde{r}, \epsilon)\| \\ \leq& 2\|P(\theta)^{-1}\| \cdot \| - A(\theta)P(\theta)(r\, g_1(\theta, P(\theta)r, \epsilon) - \tilde{r}\, g_1(\theta, P(\theta)\tilde{r}, \epsilon)) \end{aligned}$$

$$+ (g_2(\theta, P(\theta)r, \epsilon) - g_2(\theta, P(\theta)\tilde{r}, \epsilon))\|$$
$$\leq 2\|P(\theta)^{-1}\|\Big\{\|A(\theta)\| \cdot \|P(\theta)\| \cdot \Big[\|r - \tilde{r}\| \cdot |g_1(\theta, P(\theta)r, \epsilon)|$$
$$+ \|r\| \cdot |g_1(\theta, P(\theta)r, \epsilon) - g_1(\theta, P(\theta)\tilde{r}, \epsilon)|\Big] + \eta_2(\delta, \epsilon) \cdot \|r - \tilde{r}\|\Big\}$$
$$\leq 2\|P(\theta)^{-1}\|\Big\{\|A(\theta)\| \cdot \|P(\theta)\|\Big[|g_1(\theta, P(\theta)r, \epsilon)| + \delta\, \eta_1(\delta, \epsilon)\Big] +$$
$$\eta_2(\delta, \epsilon)\Big\}\|r - \tilde{r}\|.$$

Thus, g is Lipschitz with respect to r and the Lipschitz constant has the required property. □

Lemma 2.2 *For every solution of (1.7) that is bounded for $\theta \leq \sigma$, there exists $r_+ \in \pi_+\mathbb{R}^{d-1}$ such that, for $\theta \leq \sigma$,*

$$\begin{aligned} r(\theta) = \exp B(\theta - \sigma)r_+ &+ \int_\sigma^\theta \exp B(\theta - s)\pi_+ g(s, r(s), \epsilon)ds \\ &+ \int_{-\infty}^0 \exp(-Bs)\pi_- g(\theta + s, r(\theta + s), \epsilon)ds. \end{aligned} \tag{2.1}$$

When $2K^2\eta(\delta, \epsilon) < \alpha$, any $r(\theta)$ that satisfies (2.1) and is bounded tends to the invariant curve Γ_ϵ.

Proof. We establish that any bounded solution satisfies (2.1) by developing expressions for $\pi_\pm r(\theta)$. As $\pi_\pm$ commute with B, the Variation of Constants formula applied to (1.7) gives

$$\pi_\pm r(\theta) = \exp B(\theta - \sigma)\pi_\pm r(\sigma) + \int_\sigma^\theta \exp B(\theta - s)\pi_\pm g(s, r(s), \epsilon)ds.$$

For the unstable projection π_-, the first term vanishes as σ gets large and negative:

$$\|\exp B(\theta - \sigma)\pi_- r(\sigma)\| \leq K e^{-\alpha(\theta - \sigma)}\|\pi_- r(\sigma)\| \leq \text{constant } e^{-\alpha(\theta - \sigma)}.$$

Also, the second term is bounded as $\sigma \to -\infty$:

$$\begin{aligned} \Big\|\int_\sigma^\theta \exp B(\theta - s)\pi_- g(s, r(s), \epsilon)ds\Big\| &\leq K \int_\sigma^\theta e^{-\alpha(\theta - s)}\|\pi_- g(s, r(s), \epsilon)\|ds \\ &\leq \text{constant } \times |e^{-\alpha(\theta - \sigma)} - 1|/\alpha. \end{aligned}$$

Consequently, the expression for $\pi_- r(\theta)$ simplifies by putting $\sigma = -\infty$ and by translating the variable of integration:

$$\pi_- r(\theta) = \int_{-\infty}^{0} \exp(-Bs)\pi_- g(s+\theta, r(s+\theta), \epsilon) ds.$$

As $r(\theta) = \pi_+ r(\theta) + \pi_- r(\theta)$, we have (2.1).

To show that any bounded orbit satisfying (2.1) tends to Γ_ϵ, let $r(\theta)$ be a general orbit satisfying (2.1) and let $\tilde{r}(\theta)$ denote a trajectory on Γ_ϵ. Certainly, $\tilde{r}$ is a bounded trajectory and must satisfy (2.1). Let $L := \limsup_{\theta \le \sigma} \|r(\theta) - \tilde{r}(\theta)\|$ and choose $\beta > 1$ such that $2K^2\eta(\delta,\epsilon)\beta/\alpha < 1$. Then, if $L > 0$, there exists θ^* such that $\|r(\theta) - \tilde{r}(\theta)\| < \beta L$ for $\theta \le \theta^*$. Now, when $\theta \le \theta^*$,

$$\begin{aligned} &\|r(\theta) - \tilde{r}(\theta)\| \\ &\le K e^{\alpha(\theta-\sigma)} \|\pi_+ r(\sigma) - \pi_+ \tilde{r}(\sigma)\| + K^2 \eta(\delta,\epsilon) \int_\sigma^{\theta^*} e^{\alpha(\theta - s)} \|r(s) - \tilde{r}(s)\| ds \\ &\quad + \beta L K^2 \eta(\delta,\epsilon) \Big(\int_{-\infty}^{0} e^{\alpha s} ds + \int_{\theta^*}^{\theta} e^{\alpha(\theta-s)} ds \Big) \\ &\le K L \beta e^{\alpha(\theta-\sigma)} + K^2\eta(\delta,\epsilon) \int_\sigma^{\theta^*} e^{\alpha(\theta-s)} \|r(s) - \tilde{r}(s)\| ds + K^2 \eta(\delta,\epsilon) L\beta 2/\alpha. \end{aligned}$$

Taking the limsup as $\theta \to -\infty$, we have

$$L \le \frac{2K^2\eta(\delta,\epsilon)\beta}{\alpha} L.$$

As $2K^2\eta(\delta,\epsilon)\beta < \alpha$, this implies that $\limsup \|r(\theta) - \tilde{r}(\theta)\|$ is zero. Hence, as θ goes to negative infinity, $r(\theta)$ tends to $\tilde{r}(\theta)$ and therefore to Γ_ϵ. □

The next theorem proves that bounded solutions exist and perturb continuously. We need two bounds for this theorem: reduce $\max\{\delta, \epsilon^*\}$ until

$$\eta(\delta,\epsilon) \le \frac{\alpha}{6K^2} \qquad \text{and} \qquad \|g(r,0,\epsilon)\| \le \frac{\alpha\delta}{6K^2}, \qquad |\epsilon| \le \epsilon^*, \quad \|r\| \le \delta.$$

Theorem 2.3 *Suppose that $|\epsilon| < \epsilon^*$, that $r_+ \in \pi_+\mathbb{R}^{d-1}$ where $\|r_+\| \le \delta/3K$, and that $\sigma \in \mathbb{R}$. Then, there exists a unique solution, $r(\theta;\sigma,r_+,\epsilon)$, of (1.7) that satisfies*

$$\pi_+ r(\sigma;\sigma,r_+,\epsilon) = r_+$$

and tends to Γ_ϵ as $\theta \to -\infty$. Further, this solution depends continuously on ϵ: for a constant L independent of σ and r_+,

$$\sup_\theta \|r(\theta;\sigma,r_+,\epsilon) - r(\theta;\sigma,r_+,0)\| \le L\epsilon.$$

Proof. This proof uses the Lyapunov–Perron method [7]. Let $Y(\sigma,\delta)$ denote the set of continuous functions from $(-\infty,\sigma]$ to $\mathbb{R}^{d-1}$ that satisfy

$$\sup_\theta \|r(\theta)\| \le \delta \qquad \text{and} \qquad \|\pi_+ r(\sigma)\| \le \delta/3K.$$

We consider this set as a Banach space with the supremum norm $\|\cdot\|_\infty$. For $r \in Y(\sigma,\delta)$, set

$$(\mathcal{F}_\epsilon r)(\theta) := \exp(B(\theta-\sigma))r_+ + \int_\sigma^\theta \exp(B(\theta-s))\pi_+ g(s,r(s),\epsilon)ds$$
$$+ \int_{-\infty}^0 \exp(-Bs)\pi_- g(\theta+s, r(\theta+s),\epsilon)ds.$$

We want to apply the Contraction Mapping Theorem to $\mathcal{F}_\epsilon$ on $Y(\sigma,\delta)$. We show that $\mathcal{F}_\epsilon$ maps into $Y(\sigma,\delta)$ and that $\mathcal{F}_\epsilon$ contracts on $Y(\sigma,\delta)$. Consider $r \in Y(\sigma,\delta)$; by lemma 1.6,

$$\|(\mathcal{F}_\epsilon r)(\theta)\| \le Ke^{\alpha(\theta-\sigma)}\|r_+\| + K^2\int_\sigma^\theta e^{\alpha(\theta-s)}\|g(s,r(s),\epsilon)\|ds$$
$$+ K^2\int_{-\infty}^0 e^{\alpha s}\|g(s+\theta, r(s+\theta),\epsilon)\|ds.$$

Using the Lipschitz constant $\eta(\delta,\epsilon)$, we have

$$\|(\mathcal{F}_\epsilon r)(\theta)\|$$
$$\le Ke^{\alpha(\theta-\sigma)}\|r_+\| + K^2\int_\sigma^\theta e^{\alpha(\theta-s)}\Big(\|g(s,0,\epsilon)\| + \eta(\delta,\epsilon)\|r(s)\|\Big)ds$$
$$+ K^2\int_{-\infty}^0 e^{\alpha s}\Big(\|g(s+\theta,0,\epsilon)\| + \eta(\delta,\epsilon)\|r(s+\theta)\|\Big)ds$$
$$\le Ke^{\alpha(\theta-\sigma)}\|r_+\| + K^2\Big(\alpha\delta/6K^2 + \eta(\delta,\epsilon)\delta/3K\Big)(2-e^{\alpha(\theta-\sigma)})/\alpha.$$

The bounds on δ and ϵ imply that $\|(\mathcal{F}_\epsilon r)(\theta)\|$ is less than δ for $\theta \le \sigma$, and hence $\mathcal{F}_\epsilon$ maps into Y. Let $r, \tilde{r} \in Y(\sigma,\delta)$, and consider the contractivity of $\mathcal{F}_\epsilon$:

$$\|(\mathcal{F}_\epsilon r)(\theta) - (\mathcal{F}_\epsilon \tilde{r})(\theta)\| \le \int_\sigma^\theta Ke^{\alpha(\theta-s)}K\eta(\delta,\epsilon)\|r(s)-\tilde{r}(s)\|ds$$
$$+ \int_{-\infty}^0 Ke^{\alpha s}\eta(\delta,\epsilon)K\|r(\theta+s)-\tilde{r}(\theta+s)\|ds$$
$$\le 2\frac{K^2\eta(\delta,\epsilon)}{\alpha}\|r-\tilde{r}\|_\infty.$$

With the bounds on $\eta(\delta,\epsilon)$, we see that $\mathcal{F}_\epsilon$ contracts by a factor $\frac{1}{2}$ on Y.

We now apply the Contraction Mapping Theorem: let the unique solution to $\mathcal{F}_\epsilon r = r$ in $Y(\sigma,\delta)$ be denoted by $r(\theta;\sigma,r_+,\epsilon)$. By lemma 2.2,

$r(\theta;\sigma,r_+,\epsilon)$ is the unique bounded solution to (1.2) subject to $\pi_+ r(\sigma) = r_+$. And, again by lemma 2.2, this solution tends to Γ_ϵ as θ goes to negative infinity.

Finally, we prove continuity of the fixed points with respect to ϵ by applying the Uniform Contraction Principle. Let r be a fixed point of $\mathcal{F}_0$ and let r_ϵ be a fixed point of $\mathcal{F}_\epsilon$; then

$$\begin{aligned}\|r - r_\epsilon\|_\infty &= \|\mathcal{F}_0 r - \mathcal{F}_\epsilon r_\epsilon\|_\infty \\ &\le \|\mathcal{F}_0 r - \mathcal{F}_0 r_\epsilon\|_\infty + \|\mathcal{F}_0 r_\epsilon - \mathcal{F}_\epsilon r_\epsilon\|_\infty \\ &\le \frac{1}{2}\|r - r_\epsilon\|_\infty + \|\mathcal{F}_0 r_\epsilon - \mathcal{F}_\epsilon r_\epsilon\|_\infty,\end{aligned}$$

and we have

$$\|r - r_\epsilon\|_\infty \le 2\|\mathcal{F}_0 r_\epsilon - \mathcal{F}_\epsilon r_\epsilon\|_\infty.$$

Continuity follows from estimating $\|\mathcal{F}_0 r - \mathcal{F}_\epsilon r\|$. Let J denote the Lipschitz constant of f with respect to ϵ; then

$$\begin{aligned}\|(\mathcal{F}_0 r_\epsilon)(\theta) - (\mathcal{F}_\epsilon r_\epsilon)(\theta)\| &\le K^2\Big(\int_\sigma^\theta e^{\alpha(\theta-s)} J\epsilon ds + \int_{-\infty}^0 e^{\alpha s} J\epsilon ds\Big) \\ &\le K^2 J\epsilon(2 - e^{\alpha(\theta-\sigma)})/\alpha,\end{aligned}$$

and, because $\theta \le \sigma$, we conclude

$$\|r - r_\epsilon\|_\infty \le (2K^2 J/\alpha)\epsilon. \qquad \Box$$

Corollary 2.4 *The unstable manfifold, W_ϵ, of Γ_ϵ perturbs Hausdorff continuously in a neighbourhood of Γ_0. In particular, the sets*

$$W_{\epsilon,\delta} := \{(\theta, r) \in W_\epsilon : \|\pi_+ r\| \le \delta/3K\}$$

satisfy, for a constant L,

$$d_H(W_{\epsilon,\delta}, W_{0,\delta}) \le L\epsilon.$$

Proof. By the preceding theorem, any point in $W_{\epsilon,\delta}$ may be denoted $(\theta, r(\theta;\theta,r_+,\epsilon))$. Further, whenever $0 \le \theta \le 2\pi$ and $r_+ \in \pi_+\mathbb{R}^{d-1}$ with $\|r_+\| \le \delta/3K$, the set $W_{\epsilon,\delta}$ contains $(\theta, r(\theta;\theta,r_+,\epsilon))$. Therefore

$$W_{\epsilon,\delta} = \{(\theta, r(\theta;\theta,r_+,\epsilon)) : r_+ \in \pi_+\mathbb{R}^{d-1} \text{ with } \|r_+\| \le \delta/3K,\ 0 \le \theta \le 2\pi\}.$$

Because of the uniform estimate in Theorem 2.3, we have the Lipschitz continuity of $W_{\epsilon,\delta}$. $\Box$

3 Numerical Perturbations: existence

To understand the numerical perturbations, we need to work with (1.8), viz.

$$\frac{d\theta}{dt} = 1 + h_1(\theta, r), \qquad \frac{dr}{dt} = Br + h_2(\theta, r).$$

Let (Θ, R) be the solution operator for (1.8). That is, $(\Theta, R)(s; \theta, r)$ solves equation (1.8) at time s for the initial condition (θ, r). Then the Variation of Constants formulae applied to (1.8) says

$$\begin{aligned}\theta((n+1)\Delta t) &= \theta(n\Delta t) + \Delta t + \int_0^{\Delta t} h_1\Big((\Theta, R)(s; \theta(n\Delta t), r(n\Delta t))\Big)ds,\\ r((n+1)\Delta t) &= \exp(B\Delta t) r(n\Delta t)\\ &\quad + \int_0^{\Delta t} \exp(B(\Delta t - s)) h_2\Big((\Theta, R)(s; \theta(n\Delta t), r(n\Delta t))\Big)ds.\end{aligned}$$

Let (T_1, T_2) denote the truncation error T written in polar coordinates (θ, r). We have an expression for a general one step method approximating (1.8) by adding in (T_1, T_2):

$$\begin{aligned}\theta_{n+1} &= \theta_n + \Delta t + H_1(\theta_n, r_n)\\ r_{n+1} &= \exp(B\Delta t) r_n + H_2(\theta_n, r_n)\end{aligned} \tag{3.1}$$

where

$$H_1(\theta, r) := \int_0^{\Delta t} h_1\Big((\Theta, R)(s; \theta, r)\Big)ds + T_1(\theta, r)$$

$$H_2(\theta, r) := \int_0^{\Delta t} \exp(B(\Delta t - s)) h_2\Big((\Theta, R)(s; \theta, r)\Big)ds + T_2(\theta, r).$$

As the coordinate changes used in §1 are C^1, assumption (1.3) gives the following regularity of (T_1, T_2) and then of (H_1, H_2). We use the norm $\|\cdot\|$ on $\mathbb{R} \times \mathbb{R}^{d-1}$ defined by

$$\|(\theta, r)\| := \max\{|\theta|, \|r\|\}.$$

Proposition 3.1 *There exists $L_1 > 0$ such that, for all $r, \tilde{r} \in \mathbb{R}^{d-1}$ satisfying $\|r\|, \|\tilde{r}\| \leq \delta$, we have*

$$|T_1(\theta, r)| \leq L_1 \Delta t^{p+1}, \qquad |T_1(\theta, r) - T_1(\tilde{\theta}, \tilde{r})| \leq L_1 \Delta t^{p+1} \|(\theta, r) - (\tilde{\theta}, \tilde{r})\|,$$

$$\|T_2(\theta, r)\| \leq L_1 \Delta t^{p+1}, \qquad \|T_2(\theta, r) - T_2(\tilde{\theta}, \tilde{r})\| \leq L_1 \Delta t^{p+1} \|(\theta, r) - (\tilde{\theta}, \tilde{r})\|.$$

□

Lemma 3.2 *Let*

$$\kappa_1 := \frac{1}{\Delta t}\int_0^{\Delta t} e^{s\lambda} ds, \qquad \kappa_2 := e^{\alpha \Delta t}\frac{1}{\Delta t}\int_0^{\Delta t} e^{s(\lambda-\alpha)} ds$$

There exists $L_2 > 0$ such that, for all $r, \tilde{r} \in \mathbb{R}^{d-1}$ with $\|r\|, \|\tilde{r}\| \le \delta$,

$$|H_1(\theta, r)| \le \Delta t(L_2\delta + L_1\Delta t^p),$$
$$\|H_2(\theta, r)\| \le \Delta t(L_2 e^{\alpha\Delta t}\delta^2 + L_1\Delta t^p),$$
$$|H_1(\theta, r) - H_1(\tilde{\theta}, r)| \le \Delta t(L_2\kappa_1\delta + L_1\Delta t^p)|\theta - \tilde{\theta}|,$$
$$|H_1(\theta, r) - H_1(\theta, \tilde{r})| \le \Delta t(L_2\kappa_1 + L_1\Delta t^p)\|r - \tilde{r}\|,$$
$$\|H_2(\theta, r) - H_2(\tilde{\theta}, \tilde{r})\| \le \Delta t(L_2\kappa_2\delta + L_1\Delta t^p)\|(\theta, r) - (\tilde{\theta}, \tilde{r})\|.$$

Proof. Throughout the proof we consider $\|r\|$ and $\|\tilde{r}\|$ less than δ. By lemma 1.5 and equation (1.9), there exists $L_2 > 0$ such that

$$|h_1(\theta, r) - h_1(\theta, \tilde{r})| \le L_2\|r - \tilde{r}\|, \qquad |h_1(\theta, r) - h_1(\tilde{\theta}, r)| \le L_2\delta\|\theta - \tilde{\theta}\|,$$

$$\|h_2(\theta, r) - h_2(\theta, \tilde{r})\| \le L_2\delta\|r - \tilde{r}\|, \qquad \|h_2(\theta, r) - h_2(\tilde{\theta}, r)\| \le L_2\delta^2\|\theta - \tilde{\theta}\|.$$

Under these conditions, we can choose $\lambda \ge \alpha$ such that

$$\|(\Theta, R)(s;\theta, r) - (\Theta, R)(s;\tilde{\theta}, \tilde{r})\| \le e^{\lambda s}\|(\theta, r) - (\tilde{\theta}, \tilde{r})\|, \qquad s \ge 0. \tag{3.2}$$

We find the Lipschitz constant for H_2 as a function of r only; the other cases are similar.

$$\begin{aligned}
&\|H_2(\theta, r) - H_2(\tilde{\theta}, \tilde{r})\| \\
&\le \int_0^{\Delta t} \|\exp B(\Delta t - s)\| \cdot \|h_2\Big((\Theta, R)(s;\theta, r)\Big) - h_2\Big((\Theta, R)(s;\tilde{\theta}, \tilde{r})\Big)\| ds \\
&\qquad + \|T_2(\theta, r) - T_2(\tilde{\theta}, \tilde{r})\| \\
&\le \int_0^{\Delta t} K e^{\alpha(\Delta t - s)} L_2(\delta^2 + \delta)\|(\Theta, R)(s;\theta, r) - (\Theta, R)(s;\tilde{\theta}, \tilde{r})\| ds \\
&\qquad + L_1\Delta t^{p+1}\|(\theta, r) - (\tilde{\theta}, \tilde{r})\|.
\end{aligned}$$

Applying (3.2) and increasing L_2, we find

$$\begin{aligned}
&\|H_2(\theta, r) - H_2(\tilde{\theta}, \tilde{r})\| \\
&\le \Big(K e^{\alpha\Delta t} L_2\delta \int_0^{\Delta t} e^{(\lambda-\alpha)s} ds + L_1\Delta t^{p+1}\Big)\|(\theta, r) - (\tilde{\theta}, \tilde{r})\|.
\end{aligned}$$

Then

$$\|H_2(\theta,r) - H_2(\tilde\theta,\tilde r)\| \le \Delta t(L_2\kappa_2\delta + L_1\Delta t^p)\|(\theta,r) - (\tilde\theta,\tilde r)\|. \qquad \square$$

The existence argument

We must develop a space of closed curves, to gain existence of a bounded solution for each initial angle.

(i) Let S^1 denote the unit circle; for $\theta, \tilde\theta \in S^1$, let $|\theta - \tilde\theta|$ denote the angular distance between θ and $\tilde\theta$.

(ii) Define

$$Z^\theta := \{\theta \in C(S^1, S^1) : \theta \text{ is surjective }\}$$
$$Z^r := \{r \in C(S^1, \mathbb{R}^{d-1}) : \|r(\zeta)\| \le \delta \text{ for } \zeta \in S^1\}.$$

Thus, if $(\theta, r) \in Z^\theta \times Z^r$, then $\{(\theta(\zeta), r(\zeta)) : \zeta \in S^1\}$ is a closed curve in $\mathbb{R}^d$ near Γ_0. Let Z denote the sequences (θ_n, r_n), indexed by $n \in \{0, -1, \ldots\}$, where

$$\theta_n \in Z^\theta, \qquad r_n \in Z^r, \qquad \text{and for all } \zeta \in S^1 \quad \|\pi_+ r_0(\zeta)\| \le \delta/3K.$$

(iii) Consider $z = (\theta_n, r_n) \in Z$; then define

$$\mathcal{A}_n(z;\zeta) := \theta_{n-1}(\zeta) + \Delta t + H_1(\theta_{n-1}(\zeta), r_{n-1}(\zeta))$$
$$\mathcal{B}_n(z;\zeta) := \exp(Bn\Delta t)r_+ + \sum_{i=0}^{n-1} \exp(B(n-i)\Delta t)\pi_+ H_2(\theta_i(\zeta), r_i(\zeta))$$
$$+ \sum_{i=-\infty}^{0} \exp(-Bi\Delta t)\pi_- H_2(\theta_{i+n}(\zeta), r_{i+n}(\zeta)).$$

Let $(\mathcal{A}, \mathcal{B})z$ be the sequence $(\mathcal{A}_n(z;\zeta), \mathcal{B}_n(z;\zeta))$.

Lemmas 3.3–3.4 show that $(\mathcal{A}, \mathcal{B})$ is a well defined mapping from Z into Z. Bounded numerical solutions arise as fixed points of $(\mathcal{A}, \mathcal{B})$—if $(\theta_n, r_n) \in Z$ fixes $(\mathcal{A}, \mathcal{B})$ then $(\theta_n(\zeta), r_n(\zeta))$, $n \le 0$, is a bounded numerical solution for each $\zeta \in S^1$. This is the content of lemma 3.5.

The final ingredient is the metric d_β.

(iv) Consider $z = (\theta_n, r_n)$ and $\tilde z = (\tilde\theta_n, \tilde r_n)$, elements of Z. For $\beta \ge 0$, define

$$d_\beta(z, \tilde z) := \sup_{n \le 0} e^{-\beta n \Delta t} \max\left\{\|r_n(\cdot) - \tilde r_n(\cdot)\|_\infty, |\theta_n(\cdot) - \tilde\theta_n(\cdot)|_\infty\right\}.$$

The metric d_0 is the standard supremum on Z and hence (Z, d_0) is a complete metrics space. The weighted metrics ($\beta > 0$) are not bounded on Z and hence (Z, d_β) is not a metric space. The weighted metrics are used in lemma 3.6 to understand how $(\mathcal{A}, \mathcal{B})$ contracts. For $\beta = 0$, there is no contraction. In fact, a contraction for $\beta = 0$ would allow use of the Contraction Mapping Theorem and give unique solutions in Z. There is no such uniqueness because $(\theta_n(\zeta), r_n(\zeta))$ fixing $(\mathcal{A}, \mathcal{B})$ implies that $(\theta_n(\zeta + \tau), r_n(\zeta + \tau))$ fixes $(\mathcal{A}, \mathcal{B})$ for any τ—rotating gives more fixed points.

The fixed points arise as the limit of $(\mathcal{A}, \mathcal{B})^n z_0$ when $d_\beta(z_0, (\mathcal{A}, \mathcal{B})z_0)$ is finite. The proof of theorem 3.7 describes a z_0 with this property.

Lemma 3.3 *Suppose $L_2\delta\Delta t + L_1\Delta t^{p+1} \le \pi/4$, then $\mathcal{A}_n(z; \cdot)$ maps S^1 onto S^1 for any $z \in Z$.*

Proof. We have defined

$$\mathcal{A}_n(z; \zeta) := \theta_{n-1}(\zeta) + \Delta t + H_1(\theta_{n-1}(\zeta), r_{n-1}(\zeta))$$

where $\theta_{n-1}(\zeta)$ is onto and we have

$$|H_1(\theta, r)| \le \Delta t(L_2\delta + L_1\Delta t^p) \le \pi/4, \qquad \|r\| \le \delta, \quad \theta \in S^1.$$

Fix $\theta \in S^1$. Let U denote a segment whose image under

$$\zeta \mapsto \theta_{n-1}(\zeta) + \Delta t \tag{3.3}$$

equals the segment running from $\theta - \pi/2$ to θ to $\theta + \pi/2$. As $\mathcal{A}_n(z; \zeta)$ differs from function (3.3) by $\pi/4$ at most, the image of U under $\mathcal{A}_n(z; \zeta)$ must include θ. As θ was arbitrary, we have proved $\mathcal{A}_n(z; \zeta)$ maps onto S^1. □

Lemma 3.4 *Suppose that*

$$\|r_+\| \le \delta/3K \qquad \text{and} \qquad K(L_2 e^{\alpha\Delta t}\delta^2 + L_1\Delta t^p) < \alpha\delta/3K.$$

Then, $\|\mathcal{B}_n(z, \zeta)\|$ is less that δ for any $\zeta \in S^1$ and any $z \in Z$.

Proof.

$$\begin{aligned}\|\mathcal{B}_n(z; \zeta)\| \le & K e^{\alpha n\Delta t}\|r_+\| + \sum_{i=0}^{n-1} e^{\alpha(n-i)\Delta t} K\Delta t(L_2 e^{\alpha\Delta t}\delta^2 + L_1\Delta t^p) \\ & + \sum_{i=-\infty}^{0} e^{\alpha i\Delta t} K\Delta t(L_2 e^{\alpha\Delta t}\delta^2 + L_1\Delta t^p) \\ \le & K\|r_+\| + 2K(L_2 e^{\alpha\Delta t}\delta^2 + L_1\Delta t^p)/\alpha.\end{aligned}$$

Our hypothesis now show that $\|\mathcal{B}_n(z;\zeta)\| \le \delta$. □

Lemma 3.5 *Any solution of numerical method (3.1) bounded as $n \to -\infty$ may be written, for some $r_+ \in \pi_+\mathbb{R}^{d-1}$, as follows.*

$$\begin{aligned}\theta_n &= \theta_{n-1} + \Delta t + H_1(\theta_{n-1}, r_{n-1}),\\ r_n &= \exp(Bn\Delta t)r_+ + \sum_{i=0}^{n-1} \exp[B(n-i)\Delta t]\pi_+ H_2(\theta_i, r_i)\\ &\quad + \sum_{i=-\infty}^{0} \exp[-Bi\Delta t]\pi_- H_2(\theta_{i+n}, r_{i+n}).\end{aligned}$$

Conversely, any sequence satisfying this equation solves (3.1).

Proof. This is similar to the proof of lemma 2.2 and is omitted. □

Lemma 3.6 *If*

$$\Delta t L_2 \kappa_1 \le 1, \qquad L_2\kappa_1\delta + L_1\Delta t^p \le \alpha/4,$$

and

$$K(L_2\kappa_2\delta + L_1\Delta t^p) \le \alpha/8,$$

then there exists $\mu < 1$ such that

$$d_{\alpha/2}((\mathcal{A},\mathcal{B})z, (\mathcal{A},\mathcal{B})\tilde{z}) \le \mu\, d_{\alpha/2}(z,\tilde{z}), \qquad z,\tilde{z} \in Z.$$

Proof. The proof proceeds by calculating $e^{-\alpha n\Delta t/2}|\mathcal{A}_n(z;\zeta) - \mathcal{A}_n(\tilde{z};\zeta)|$ and $e^{-\alpha n\Delta t/2}\|\mathcal{B}_n(z;\zeta) - \mathcal{B}_n(\tilde{z};\zeta)\|$. When these quantities are bounded by $\mu\, d_{\alpha/2}(z,\tilde{z})$ for some $\mu < 1$, we are done. Let $z, \tilde{z}$ be sequences in Z with components (θ_n, r_n) and $(\tilde{\theta}_n, \tilde{r}_n)$. First, consider $\mathcal{A}$:

$$\begin{aligned}&e^{-\alpha n\Delta t/2}|\mathcal{A}_n(z;\zeta) - \mathcal{A}_n(\tilde{z};\zeta)|\\ \le &e^{-\alpha n\Delta t/2}\Big(|\theta_{n-1} - \tilde{\theta}_{n-1}| + \Delta t L_2\kappa_1\delta|\theta_{n-1} - \tilde{\theta}_{n-1}|\\ &+ \Delta t L_2\kappa_1\|r_{n-1} - r_{n-1}\| + L_1\Delta t^{p+1}\|(\theta_{n-1}, r_{n-1}) - (\tilde{\theta}_{n-1}, \tilde{r}_{n-1})\|\Big)\end{aligned}$$

By hypothesis, we have

$$1 + \Delta t(L_2\kappa_1\delta + L_1\Delta t^p) \le 1 + \alpha\Delta t/4 \le e^{\alpha\Delta t/4}$$

and

$$\Delta t L_2 \kappa_1 + L_1 \Delta t^p \le 1 + \alpha\Delta t/4 \le e^{\alpha\Delta t/4},$$

and therefore

$$e^{-\alpha n\Delta t/2}|\mathcal{A}_n(z;\zeta) - \mathcal{A}_n(\tilde{z};\zeta)| \le e^{-\alpha\Delta t/4} d_{\alpha/2}(z,\tilde{z}).$$

Now, consider $\mathcal{B}$:

$$\begin{aligned}
&e^{-\alpha n\Delta t/2}\|\mathcal{B}_n(z;\zeta) - \mathcal{B}_n(\tilde{z};\zeta)\| \\
\le&\Delta t(L_2\kappa_2\delta + L_1\Delta t^p)\Big(\sum_{i=0}^{n-1} K e^{\alpha(n-i)\Delta t}\|(\theta_i, r_i) - (\tilde{\theta}_i, \tilde{r}_i)\| e^{-\alpha n\Delta t/2} \\
&+ \sum_{i=-\infty}^{0} K e^{\alpha i\Delta t} e^{-\alpha n\Delta t/2}\Delta t\|(\theta_{i+n}, r_{i+n}) - (\tilde{\theta}_{i+n}, \tilde{r}_{i+n})\|\Big) \\
\le&d_{\alpha/2}(z,\tilde{z}) K (L_2\kappa_2\delta + L_1\Delta t^p) \\
&\times \Big(\Delta t \sum_{i=0}^{n-1} e^{\Delta t(\alpha-\alpha/2)(n-i)} + \Delta t \sum_{i=-\infty}^{0} e^{(\alpha+\alpha/2)i\Delta t}\Big) \\
\le&d_{\alpha/2}(z,\tilde{z}) 4K(L_2\kappa_2\delta + L_1\Delta t^p)/\alpha.
\end{aligned}$$

Using the bounds of the hypothesis, we see

$$e^{-\alpha n\Delta t/2}\|\mathcal{B}_n(z;\zeta) - \mathcal{B}_n(\tilde{z};\zeta)\| \le \tfrac{1}{2} d_{\alpha/2}(z,\tilde{z}).$$

Thus, we have the result with $\mu := \max\{\frac{1}{2}, e^{-\alpha\Delta t/4}\}$. □

The next theorem states that bounded numerical solutions exist. Its proof depends on the preceding results; thus we choose δ and Δt^* such that the hypothesis of lemmas 3.3–3.6 hold for $\Delta t \le \Delta t^*$. Furthermore, we suppose that $\Gamma_{\Delta t}$ is within the δ neighbourhood of Γ_0 for $\Delta t \le \Delta t^*$.

Theorem 3.7 *For each $r_+ \in \pi_+\mathbb{R}^{d-1}$ with $\|r_+\| \le \delta/3K$ and each $\sigma \in S^1$, numerical method (3.1), applied with step size $\Delta t \le \Delta t^*$, has a unique solution that satisfies $\pi_+ r_0 = r_+$ and $\theta_0 = \sigma$ and tends to $\Gamma_{\Delta t}$ as $n \to -\infty$.*

Proof. Choose a $(\theta^0, r^0) \in Z^\theta \times Z^r$ that parameterises $\Gamma_{\Delta t}$. Denote the iterates of (θ^0, r^0) under numerical method (3.1) by (θ_n^0, r_n^0). As $\Gamma_{\Delta t}$ is invariant, each (θ_n^0, r_n^0) also parameterises $\Gamma_{\Delta t}$ and thus defines an element of Z; denote this element by z_0. We now prove that the $(\mathcal{A}, \mathcal{B})$ iterates of

z_0 converge to a fixed point of $(\mathcal{A}, \mathcal{B})$. By lemma 3.5,

$$\begin{aligned}\theta_n^0 &= \theta_{n-1}^0 + \Delta t + H_1(\theta_{n-1}^0, r_{n-1}^0),\\ r_n^0 &= \exp(Bn\Delta t)\pi_+ r_0^0 + \sum_{i=0}^{n-1} \exp(B(n-i)\Delta t)\pi_+ H_2(\theta_i^0, r_i^0)\\ &\quad + \sum_{i=-\infty}^{0} \exp(-Bi\Delta t)\pi_- H_2(\theta_{i+n}^0, r_{i+n}^0).\end{aligned}$$

In other words,

$$\theta_n^0(\zeta) = \mathcal{A}_n(z^0;\zeta), \qquad r_n^0(\zeta) = \mathcal{B}_n(z^0;\zeta) + \exp(Bn\Delta t)\pi_+(r_0^0(\zeta) - r_+).$$

Define $z^i := (\mathcal{A}, \mathcal{B})^i z^0$; then

$$\begin{aligned}d_{\alpha/2}(z^0, z^1) &= \sup_{n\le 0} e^{-\alpha n\Delta t/2} \max\left\{|\theta_n^0(\cdot) - \mathcal{A}_n(z^0;\cdot)|_\infty, \|r_n^0(\cdot) - \mathcal{B}_n(z^0;\cdot)\|_\infty\right\}\\ &= \sup_{n\le 0} e^{-\alpha n\Delta t/2} \|\exp(Bn\Delta t)\pi_+(r_0^0(\cdot) - r_+)\|_\infty\\ &< \infty.\end{aligned}$$

With lemma 3.6, this shows that $d_{\alpha/2}(z^i, z^{i+1})$ is a convergent geometric series, and hence $\{z^i\}$ is a Cauchy sequence with respect to $d_{\alpha/2}$. As

$$d_0(z,\tilde{z}) \le d_\beta(z,\tilde{z}), \qquad z,\tilde{z} \in Z, \quad \beta > 0,$$

the sequence is also Cauchy with respect to d_0 and will converge to a fixed point in Z; denote the limit point by $(\theta_n(\zeta), r_n(\zeta))$.

The $d_{\alpha/2}$ distance between z_0 and the limit point $(\theta_n(\zeta), r_n(\zeta))$ is finite. This implies that

$$\max\{|\theta_n(\zeta) - \theta_n^0(\zeta)|, \|r_n(\zeta) - r_n^0(\zeta)\|\} \to 0 \qquad \text{as} \qquad n \to -\infty.$$

Hence, for fixed ζ, $(\theta_n(\zeta), r_n(\zeta))$ represents a solution to the numerical method which tends to the invariant curve $\Gamma_{\Delta t}$. Because θ_0 is onto, we can choose ζ to find a bounded solution that tends to $\Gamma_{\Delta t}$ for any problem subject to $\pi_+ r_0 = r_+$ and $\theta_0 = \sigma$.

We discuss uniqueness: the argument above gives uniqueness of bounded solutions that tend to the invariant curve exponentially fast. We prove there exists only one solution converging to $\Gamma_{\Delta t}$ for each small $r_+ \in \mathbb{R}^{d-1}$ and each $\theta \in [0, 2\pi]$. To see this, consider any solution, z, that converges to $\Gamma_{\Delta t}$ and prove that z converges exponentially fast to $\Gamma_{\Delta t}$. Repeating the above argument gives a solution converging to z exponentially fast for each (σ, r_+). It is easily seen that these solutions vary continuously with r_+, and hence we can find such a solution on the stable manifold. As that solution is bounded, it must be the periodic orbit; i.e., z converges exponentially fast to the periodic orbit. □

4 Numerical Perturbations: continuity

The numerical solutions found in section 3 approximate the true bounded solutions uniformly. To see this, consider equation (1.7) with $\epsilon = 0$; denote $g(\theta, r, 0)$ by $g(\theta, r)$, then (1.7) becomes

$$\frac{dr}{d\theta} = Br + g(\theta, r). \tag{4.1}$$

Let (θ_n, r_n) be a bounded solution to (3.1), where we interpret θ_n as a monotonically increasing sequence in $\mathbb{R}$ rather than a sequence in S^1. From §3, we recall the solution operator $R(s; \theta, r)$, the radial position at time s given initial conditions (θ, r). Using this notation, the Variation of Constants formula applied to (4.1) states that

$$r(\theta_{n+1}) = \exp(B(\theta_{n+1} - \theta_n))r(\theta_n) + G(n, r(\theta_n)), \tag{4.2}$$

where

$$G(n, r) := \int_{\theta_n}^{\theta_{n+1}} \exp B(\theta_{n+1} - s) g(s, R(s; \theta_n, r)) ds.$$

To understand how the radial part of the numerical solution differs from the true solution, write r_n as a perturbation of the above expression. Let Δt_n be the time taken to reach θ_{n+1} given initial conditions $(\theta_n, r(\theta_n))$; then

$$r(\theta_{n+1}) = R(\Delta t_n; \theta_n, r(\theta_n)).$$

Also,

$$r_{n+1} = R(\Delta t; \theta_n, r_n) + T_2(\theta_n, r_n).$$

Hence, the numerical solution satisfies

$$r_{n+1} = \exp(B(\theta_{n+1} - \theta_n))r_n + G(n, r_n) + T^*(n, r_n) \tag{4.3}$$

where the perturbation

$$T^*(n,r) := R(\Delta t; \theta_n, r) - R(\Delta t_n; \theta_n, r) + T_2(\theta_n, r).$$

We prove this perturbation is small.

Lemma 4.1 *For some constant $L_3 > 0$,*

$$\|T^*(n,r)\| \leq L_3 \Delta t^{p+1}, \qquad \|r\| \leq \delta.$$

Proof. We introduce $t(\theta; \theta_0, r_0)$, the time taken to reach angle θ given initial conditions (θ_0, r_0). Previously, we restricted δ so that

$$1/2 \leq \theta_t \leq 3/2;$$

therefore we have the Lipschitz property

$$|t(\theta; \theta_0, r_0) - t(\tilde{\theta}; \theta_0, r_0)| \leq 2|\theta - \tilde{\theta}|, \qquad \|r\| \leq \delta.$$

Using Θ to denote the angular solution operator, we see

$$\begin{aligned}\|\Delta t_n - \Delta t\| &= \|t(\theta_{n+1}; \theta_n, r(\theta_n)) - t(\Theta(\Delta t; \theta_n, r(\theta_n)); \theta_n, r(\theta_n))\| \\ &\leq 2\|\theta_{n+1} - \Theta(\Delta t; \theta_n, r(\theta_n))\| \\ &= 2\|T_1(\theta_n, r(\theta_n))\|.\end{aligned}$$

As R, the radial solution operator, is Lipschitz with respect to time, we have the following for some constant J:

$$\begin{aligned}\|T^*(n,r)\| &\leq \|R(\Delta t_n; \theta_n, r(\theta_n)) - R(\Delta t; \theta_n, r(\theta_n))\| + \|T_2(\theta_n, r(\theta_n))\| \\ &\leq J\|\Delta t_n - \Delta t\| + \|T_2(\theta_n, r(\theta_n))\| \\ &\leq 2J\|T_1(\theta_n, r(\theta_n))\| + \|T_2(\theta_n, r(\theta_n))\|.\end{aligned}$$

With proposition 3.1, this gives the lemma. □

We are now in a position to estimate the radial distance between the true and numerical solutions. We need an extra bound on Δt for this argument to go through: reduce Δt^*, δ until

$$L_2 \kappa_1 \delta + L_1 \Delta t^p \leq 1/4, \qquad \Delta t \leq \Delta t^*. \tag{4.4}$$

Theorem 4.2 *Consider $r_+ \in \pi_+ \mathbb{R}^{d-1}$ with $\|r_+\| \leq \delta/3K$ and let $\sigma \in \mathbb{R}$.*

Let $(\theta(t), r(t))$ and (r_n, θ_n) be the true and numerical solutions associated to (σ, r_+). The following uniform estimate holds between the radial part of the solutions: if $4K^3\eta(\delta)\alpha < 1/2$ then

$$\|r(\theta_n) - r_n\| \le \frac{16K^2L_3}{3\alpha}\Delta t^p, \qquad n \le 0.$$

Proof. The estimate comes by considering r_n and $r(\theta_n)$ as fixed points of mappings to which we can apply the Uniform Contraction Principle. Let $\boldsymbol{s}$ denote a sequence with components s_n; define mappings $\mathcal{F}^{\Delta t}$ for $0 \le \Delta t \le \Delta t^*$ by

$$\mathcal{F}_n^{\Delta t}(\boldsymbol{s}) := \exp(B(\theta_n - \theta_0))r_+ + \sum_{i=0}^{n-1} \exp B(\theta_n - \theta_i)\pi_+\Big(G(i, s_i) + T^*(i, s_i)\Big)$$
$$+ \sum_{i=0}^{-\infty} \exp(-B\theta_i)\pi_-\Big(G(i+n, s_{i+n}) + T^*(i+n, s_{i+n})\Big).$$

(here, $T^* := 0$ at $\Delta t = 0$.) Denote the sequences $\{r(\theta_n)\}$, $\{r_n\}$ by $\boldsymbol{r}^0$, $\boldsymbol{r}^{\Delta t}$. By arguing with equations (4.2, 4.3) as in lemma 2.2, we see that $\boldsymbol{r}^0$ fixes $\mathcal{F}^0$ and that $\boldsymbol{r}^{\Delta t}$ fixes $\mathcal{F}^{\Delta t}$. The Uniform Contraction Principle gives continuity if $\mathcal{F}^{\Delta t}$ contracts and $\mathcal{F}^{\Delta t}\boldsymbol{r} - \mathcal{F}^0\boldsymbol{r}$ is small.

We show $\mathcal{F}^0$ contracts. First, derive a Lipschitz property for $G(n, \cdot)$.

$$\|G(n,r) - G(n,\tilde{r})\|$$
$$\le \int_{\theta_n}^{\theta_{n+1}} \|\exp(B(\theta_{n+1} - s))\| \cdot \|g(s, R(s;\theta_n, r)) - g(s, R(s;\theta_n, \tilde{r}))\| ds$$
$$\le \int_{\theta_n}^{\theta_{n+1}} e^{\alpha(\theta_{n+1}-s)} K\eta(\delta)\|R(s;\theta_n, r) - R(s;\theta_n,\tilde{r})\| ds.$$

Using (3.2), we find

$$\|G(n,r) - G(n,\tilde{r})\| \le \int_{\theta_n}^{\theta_{n+1}} e^{\alpha(\theta_{n+1}-s)} K\eta(\delta) e^{\lambda(s-\theta_{n+1})}\|r - \tilde{r}\| ds$$
$$\le (\theta_{n+1} - \theta_n)K\eta(\delta)\|r - \tilde{r}\|.$$

Let $\Delta\theta_{\max} := \max_n |\theta_n - \theta_{n-1}|$ and relate this Lipschitz property to $\mathcal{F}^0$.

$$\|\mathcal{F}_n^0(\boldsymbol{r}^0) - \mathcal{F}_n^0(\boldsymbol{r}^{\Delta t})\| \le \sum_{i=0}^{n-1} e^{\alpha(\theta_n - \theta_i)} 2K^3\eta(\delta)\Delta\theta_{\max}\|r(\theta_n) - r_n\|$$
$$+ \sum_{i=-\infty}^{0} e^{\alpha\theta_i} 2K^3\eta(\delta)\Delta\theta_{\max}\|r(\theta_{i+n}) - r_{i+n}\|.$$

Let $\Delta\theta_{\min} := \min_n |\theta_n - \theta_{n-1}|$; then by lemma 1.6

$$\|\mathcal{F}_n^0(\boldsymbol{r}^0) - \mathcal{F}_n^0(\boldsymbol{r}^{\Delta t})\| \leq \sup_{n\leq 0} \|r(\theta_n) - r_n\| 2K^3\eta(\delta)\Delta\theta_{\max}/\alpha\Delta\theta_{\min},$$

and, from lemma 3.2 and bound (4.4),

$$\frac{\Delta\theta_{\max}}{\Delta\theta_{\min}} \leq \frac{\Delta t + L_2\Delta t\delta + L_1\Delta t^{p+1}}{\Delta t - L_2\Delta t\delta - L_1\Delta t^{p+1}} \leq 2.$$

Consequently, $\mathcal{F}^0$ satisfies

$$\sup_{n\leq 0} \|\mathcal{F}_n^0(\boldsymbol{r}^0) - \mathcal{F}_n^0(\boldsymbol{r}^{\Delta t})\| \leq \sup_{n\leq 0} \|r(\theta_n) - r_n\| 4K^3\eta(\delta)/\alpha.$$

Therefore, with the above hypothesis, $\mathcal{F}^0$ contracts with factor 1/2.

The final step shows that $\mathcal{F}^0$ and $\mathcal{F}^{\Delta t}$ are close:

$$\begin{aligned}\|\mathcal{F}_n^0(\boldsymbol{r}) - \mathcal{F}_n^{\Delta t}(\boldsymbol{r})\| &\leq \sum_{i=0}^{n-1} K^2 e^{\alpha(\theta_n - \theta_i)} \|T^*(n, r_n)\| \\ &\quad + \sum_{i=-\infty}^{0} K^2 e^{\alpha\theta_i} \|T^*(i+n, r_{i+n})\| \\ &\leq 2K^2 L_3 \Delta t^{p+1}/\alpha\Delta\theta_{\min}.\end{aligned}$$

Now, the Uniform Contraction Principle gives

$$\|r_n - r(\theta_n)\| \leq 4K^2 L_3 \Delta t^{p+1}/\alpha\Delta\theta_{\min}, \qquad n \leq 0.$$

As $\Delta\theta_{\min} \geq 3\Delta t/4$, the proof is complete. □

Corollary 4.3 *Let $\theta \in \mathbb{R}$ and $r_+ \in \pi_+\mathbb{R}^{d-1}$ with $\|r_+\| \leq \delta/3K$. Denote the bounded true and numerical solutions associated to (θ, r_+) by $(\theta(t), r(t))$ and (θ_n, r_n). Then, the Hausdorff distance*

$$d\Big(\{(\theta(t), r(t)) : t \leq 0\}, \{(\theta_n, r_n) : n \leq 0\}\Big) = O(\Delta t).$$

Proof. Use the previous theorem with the fact that $\max_n |\theta_n - \theta_{n-1}|$ is order Δt. □

Corollary 4.4 *The unstable manifolds, W_ϵ, of Γ_ϵ perturb Hausdorff continuously in a neighbourhood of Γ_0. In particular, the sets*

$$W_{\epsilon,\delta} := \{(\boldsymbol{\theta}, r) \in W_\epsilon : \|\pi_+ r\| \leq \delta/3K\}$$

satisfy, for a constant L,

$$d_H(W_{\epsilon,\delta}, W_{0,\delta}) \leq L\Delta t^p.$$

Proof. Consider $(\theta, r) \in W_{0,\delta}$. By the preceding result, there exists a numerical solution that belongs to $W_{\Delta t,\delta}$ and satisfies $\pi_+ r_0 = \pi_+ r$ and $\theta_0 = \theta$. Further,

$$||(\theta, r) - (\theta_0, r_0)|| \leq 16K^2 L_3 \Delta t^p / 3\alpha.$$

Thus the Hausdorff semidistance

$$\sup_{y\in W_{0,\delta}} \inf_{y\in W_{\epsilon,\delta}} ||x - y|| = O(\Delta t^p).$$

The argument applies with the two manifolds interchanged, to find the same order of convergence for the second semidistance. Hence, the Hausdorff distance is order Δt^p. □

A Attractors

Definition A.1 Consider a Banach space $\mathcal{X}$ with norm $||\cdot||$. The *Hausdorff semi distance* between sets A and B

$$\text{dist}(A, B) = \sup_{a\in A} \inf_{b\in B} ||a - b||.$$

Consider a semigroup $S(\cdot)$ (either discrete or continuous time) acting on $\mathcal{X}$. A set A *attracts* B if

$$\text{dist}(S(t)B, A) \to 0 \quad \text{as } t \to \infty,$$

(t maybe discrete or continuous time.) An *attractor* for $S(\cdot)$ is a compact set A that attracts some neighbourhood of A. Sets $A_\lambda \subset X$ parameterised by $\lambda \in \mathbf{R}$ are *lower* (resp., *upper*) *semi-continuous* at $\lambda = 0$ if $\text{dist}(A, A_\lambda) \to 0$ (resp., $\text{dist}(A_\lambda, A) \to 0$) as $\lambda \to 0$.

Let Γ be an invariant set with unstable manifold W. Let N be a neighbourhood of Γ; then

$$\{x \in N \cap W : S(t)x \in N \text{ for all } t \leq 0\}$$

is called a *local unstable manifold* of Γ.

With these definitions, we now give sufficient conditions for an attractor to be lower semi-continuous (remember that upper semi-continuity is guaranteed). The theorem applies to both our perturbations: when peturbing the vector field, $S_\lambda(\cdot)$ is the solution operator of (1.2) with $\epsilon = \lambda$; Hale [7] gives estimates that verify the hypothesis of the theorem. When perturbing numerically, $S_\lambda(1)$ equals $S(\Delta t)$ at $\lambda = 0$ and equals $S_{\Delta t}(1)$ at $\lambda = \Delta t$; the hypothesis come from the bound on the truncation error and the Lipschitz property of $S(\cdot)x$ with respect to x.

Theorem A.2 *Consider a one parameter family of semigroups $S_\lambda(\cdot)$ (discrete or continuous) acting on a Banach space $\mathcal{X}$ such that for some increasing function $\zeta(t) : \mathbb{R}^+ \to \mathbb{R}^+$,*

$$\|S_\lambda(t)x - S_0(t)\tilde{x}\| \leq \zeta(t)\Big\{|\lambda| + \|x - \tilde{x}\|\Big\}, \qquad t \geq 0, \quad x, \tilde{x} \in \mathcal{X}.$$

Let $\mathcal{A}_0$ be an attractor with respect to $S_0(\cdot)$ that equals the closure of the union of the following sets: unstable manifolds, W_0, of invariant sets of $S_0(\cdot)$ that are locally lower semi continuous; that is, local unstable manifolds, $W_{\lambda,loc}$, exist for $S_\lambda(\cdot)$ subject to (i) $W_{0,loc} \subset W_0$ and (ii) $dist(W_{0,loc}, W_{\lambda,loc}) \to 0$ as $\lambda \to 0$.

Then, for λ in a neighbourhood of 0, attractors $\mathcal{A}_\lambda$ exist for $S_\lambda(\cdot)$ and the Hausdorff distance

$$d_H(\mathcal{A}_\lambda, \mathcal{A}_0) \to 0 \qquad \text{as} \qquad \lambda \to 0.$$

Proof. The existence and upper semi-continuity is given by Hale–Lin–Raugel [8]. The lower semi-continuity is an extension of Humphries work on unstable manifolds of hyperbolic equilibria [10]; the argument proves that the continuity of a local unstable manifold gives lower semi-continuity of the global manifold, when the manifold is compact. In general, upper semi-continuity does not extend beyond attractors [12] nor lower semi-continuity beyond unstable manifolds [5].

Fix $\epsilon > 0$. Let W_0 be one of the component unstable manifolds. Because this manifold is contained by an attractor, it has compact closure. Therefore, the closure of $W_0 - W_{0,loc}$ is compact and there exist $y_1, \ldots, y_N \in W_0 - W_{0,loc}$ whose ϵ balls cover $W_0 - W_{0,loc}$. Pick $T_i > 0$ for $i = 1, \ldots, N$ such that $S_0(-T_i)y_i \in W_{0,loc}$. For λ small, there exists $x_i \in W_{\lambda,loc}$ with $\|S_0(-T_i)y_i - x_i\| \leq \epsilon/\zeta(T_i)$. Now,

$$\begin{aligned}\|S_\lambda(T_i)x_i - y_i\| \le& \|S_\lambda(T_i)x_i - S_0(T_i)x_i\| \\ &+ \|S_0(T_i)x_i - S_0(T_i)S_0(-T_i)y_i\| \\ \le& \zeta(T_i)\lambda + \epsilon\end{aligned}$$

and hence, as $S_\lambda(T_i)x_i \in W_\lambda$,

$$\begin{aligned}\text{dist}(W_0 - W_{0,loc}, W_\lambda) &\le \epsilon + \sup_{i=1,\dots,N} \inf_{w_\lambda \in W_\lambda} \|y_i - w_\lambda\| \\ &\le \epsilon + \zeta(\max T_i)\lambda + \epsilon.\end{aligned}$$

Take $\lambda < \epsilon/\zeta(\max T_i)$, so that $\text{dist}(W_0 - W_{0,loc}, W_\lambda) < 3\epsilon$. As $W_{0,loc}$ is lower semi-continuous and as $W_\lambda \subset \mathcal{A}_\lambda$, we get

$$\text{dist}(W_0, \mathcal{A}_\lambda) \to 0 \qquad \text{as} \qquad \lambda \to 0.$$

The attractor $\mathcal{A}$ is made from the sets W_0; we conclude that $\mathcal{A}_\lambda$ is also lower semi-continuous at $\lambda = 0$. □

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