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NUCLEATION OF WAVES IN EXCITABLE MEDIA BY NOISE*

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Abstract. We are interested in reaction-diffusion equations that model excitable media under the influence of an additive noise. In many models of this type, the homogeneous zero state is stable, and interesting dynamics are observed only for certain initial data. In the presence of noise, the excitable media is stimulated, and the noise may be sufficiently large to nucleate wave forms. In computations, we see for small noise that only target waves are nucleated when the time scales for excitation and inhibition are sufficiently separated. We provide a theorem that supports this observation.

Key words. reaction-diffusion equations, stochastic PDEs, excitable media

AMS subject classifications. 35K57, 60H15

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1. Introduction.

We are interested in reaction-diffusion equations that model excitable media under the influence of an additive noise. In many models of this type, the homogeneous zero state is stable, and interesting dynamics are observed only for certain initial data. In the presence of additive noise, the excitable media is stimulated, and the noise may be sufficiently large to nucleate wave forms. In this paper, we analyze the type of waves that are stimulated in the small noise limit. This question is of interest, for example, in models of cardiac muscle (see [2]), where noise and inhomogeneities are thought to be responsible for the creation of spiral waves and the onset of arrhythmias.

One example we have in mind is Barkley's model [1]. Consider an excitation field $u(t, \mathbf{x})$ and inhibitor field $v(t, \mathbf{x})$ such that

$$\begin{aligned} u_t &= \Delta u + f(u, v), \qquad u(0) = u_0, \\ v_t &= \epsilon\, g(u, v), \qquad v(0) = v_0, \end{aligned} \tag{1.1}$$

with time $t > 0$ on a smooth domain Ω in $\mathbf{R}^2$. The small parameter $\epsilon > 0$ separates the time scales of the excitation field and the inhibitor field. We have chosen to scale time and space so the inhibitor field is slow, which is important for our analysis. We have the reaction terms

$$f(u, v) = u(1-u)\Big(u - \frac{v+b}{a}\Big), \qquad g(u, v) = u - v \tag{1.2}$$

for parameters $a, b > 0$. Typical values are $a = 0.75$ and $b = 0.01$. This model is one of many reaction-diffusion equations that have been used to model excitable media; see [16] for a review.

We now add noise and specify boundary conditions. Let $L^2(\Omega)$ denote the Hilbert space of measurable real valued functions on Ω with inner product $\langle u, v\rangle = \int_\Omega u(\mathbf{x})v(\mathbf{x})\, d\mathbf{x}$. Let $W(t)$ be an $L^2(\Omega)$ valued Wiener process with correlation opera-

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tor Q (see [4] for further details). Consider the stochastic reaction-diffusion equation

$$\begin{aligned} du &= \Big[\Delta u + f(u,v)\Big]dt + \sigma\, dW(t), \qquad u(0) = u_0, \\ v_t &= \epsilon\, g(u,v), \qquad v(0) = v_0, \end{aligned} \tag{1.3}$$

with Dirichlet conditions $u(t,\mathbf{x}) = v(t,\mathbf{x}) = 0$ for all $t > 0$ and $\mathbf{x}$ on the boundary of Ω. Dirichlet conditions are important for our analysis, though it is more natural to consider Neumann conditions. To gain regularity of the solutions, we make assumptions on the correlation operator and the reaction terms.

Following the notation of [4], for nonsingular Q define the inner product and norm for $h, h_1, h_2 \in L^2(\Omega)$ by

$$\langle h_1, h_2\rangle_0 := \langle h_1, Q^{-1} h_2\rangle, \qquad \|h\|_0 := \langle h, h\rangle_0^{1/2}.$$

Let $\mathcal{W}_0 = Q^{1/2} L^2(\Omega) = \{u \in L^2(\Omega) \colon \|u\|_0 < \infty\}$. Denote by $H^k(\Omega)$ the Sobolev space of real valued functions on Ω with k weak derivatives in $L^2(\Omega)$ and by $H_0^1(\Omega)$ the $H^1(\Omega)$ functions equal to zero on the boundary of Ω.

Assumption 1.1. The correlation operator Q is diagonal in the following sense: for some $q_i \in \mathbf{R}$,

$$Q\mathbf{h} = \sum_{i=1}^{\infty} h_i\, q_i\, \mathbf{e}_i, \qquad \text{each } \mathbf{h} = \sum_{i=1}^{\infty} h_i \mathbf{e}_i, \quad h_i \in \mathbf{R}, \tag{1.4}$$

where $\mathbf{e}_i$ are the unit $L^2(\Omega)$ eigenfunctions of Δ with homogeneous Dirichlet boundary conditions. Further, we require that Q is nonsingular and that $\mathcal{W}_0$ is continuously embedded in $H^2(\Omega)$.

Assumption 1.2. The reaction term f is globally Lipschitz, $\mathbf{R}^2 \to \mathbf{R}$, and $g(u,v) = u - v$.

Under Assumptions 1.1–1.2, a unique weak solution of (1.3) exists on any time interval $[0,T]$ (see [11]) for initial data $u_0, v_0 \in L^2(\Omega)$. That is, an adapted $H_0^1(\Omega)^2$ valued process $(u(t), v(t))$ exists such that ($\langle\cdot,\cdot\rangle$ is the $L^2(\Omega)$ inner product)

$$\begin{aligned} \langle u(t), \chi_1\rangle &= \langle u_0, \chi_1\rangle + \int_0^t \langle \nabla u(s), \nabla\chi_1\rangle + \langle f(u(s), v(s)), \chi_1\rangle\, ds + \sigma\langle W(t), \chi_1\rangle, \\ \langle v(t), \chi_2\rangle &= \langle v_0, \chi_2\rangle + \epsilon \int_0^t \langle g(u(s), v(s)), \chi_2\rangle\, ds \end{aligned}$$

holds for all $\chi_1, \chi_2 \in H_0^1(\Omega)$ for $0 \le t \le T$. The assumption on Q is very strong, and solutions are available under weaker conditions, for example when Q has finite trace. The nonsingularity of Q is not necessary for the existence of solutions but will be important for considering large deviations. The embedding condition will be important later for proving regularity. The assumptions on f are also strong and not satisfied by Barkley's f in (1.2). This f is only locally Lipschitz. As all the important reaction kinetics takes place in the region $0 \le u, v \le 1$, we assume, for convenience of analysis, the global Lipschitz condition, which holds after modifying the definition in (1.2) outside a ball of radius R. By choosing $R > 1$, this does not change the underlying reaction kinetics exhibited by (1.3). The choice of g is not essential to our analysis, but it allows exact integration of the second equation.

The main theorem of this paper is now presented. The key features of the reaction equations (1.2) that we exploit are contained in the following.

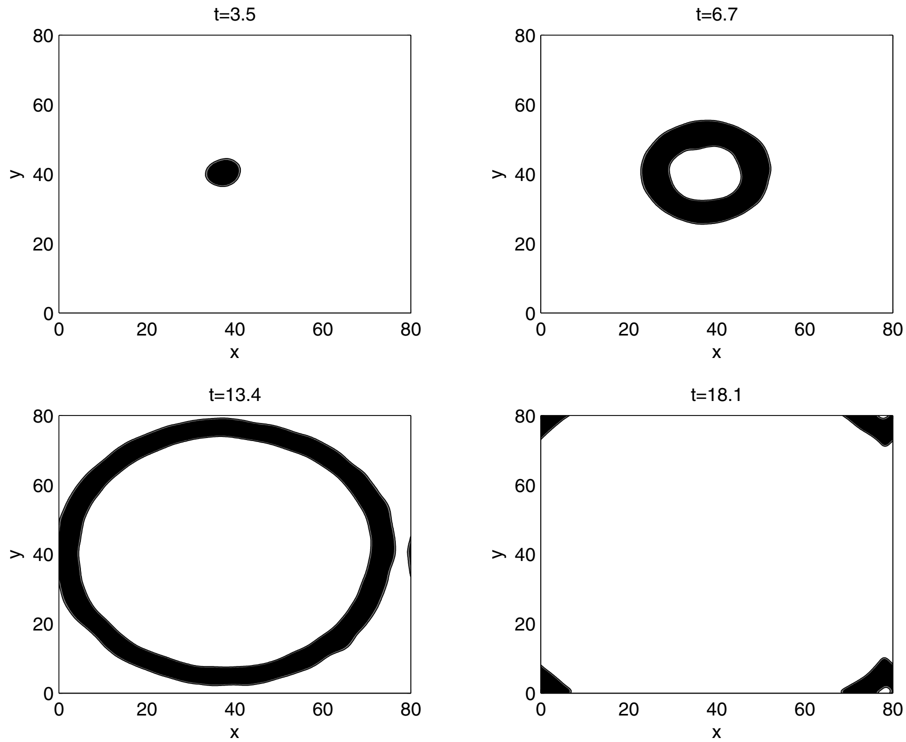

FIG. 1.1. *Nucleation of a target pattern for the Wiener process and PDE discussed in section* 5 *(black indicates* $u \approx 1$ *and white indicates* $u \approx 0$*).*

Assumption 1.3. For the reaction terms f, g,

1. $(0,0)$ is a stable equilibrium point of (1.1);
2. $f(u,0) > 0$ if $u < 0$.

Under (1), $(0,0)$ has a domain of attraction $\mathcal{D}_\epsilon$; that is, all solutions $(u(t), v(t))$ of (1.2) with $(u_0, v_0) \in \mathcal{D}_\epsilon$ approach $(0,0)$ as $t \to \infty$. When $\sigma > 0$ and the initial data $u_0 = v_0 = 0$, the noise causes a solution $(u(t), v(t))$ of (1.3) to wander around the set $\mathcal{D}_\epsilon$ until eventually it leaves this set. Our goal is to utilize large deviation techniques to determine the point at which $(u(t), v(t))$ leaves $\mathcal{D}_\epsilon$.

The behaviors can be observed numerically. For small noise intensities, the type of wave form nucleated depends on the size of ϵ. When ϵ is very small, only target patterns (radially symmetric waveforms) are observed, as in Figure 1.1. When ϵ is larger, spiral waves may be nucleated, as in Figure 1.2. We provide a theorem that explains this behavior. In particular cases, we are able to show that $(u(t), v(t))$ exits $\mathcal{D}_\epsilon$ very close to a radially symmetric configuration with probability converging to one as $\sigma \to 0$ when ϵ is small. As spiral waves are not radially symmetric, the theorem indicates that only target waves are nucleated in the small ϵ, σ regime.

The outline of the argument is as follows: the time scale on which v varies is very slow compared to u, and hence on an $\mathcal{O}(1)$ time scale the behavior of u can be approximated by the equation

$$dU = \Big[\Delta U + f(U,0)\Big]dt + \sigma\, dW(t), \qquad U(0) = U_0,$$

subject to Dirichlet boundary conditions. This is a gradient PDE, and the small σ

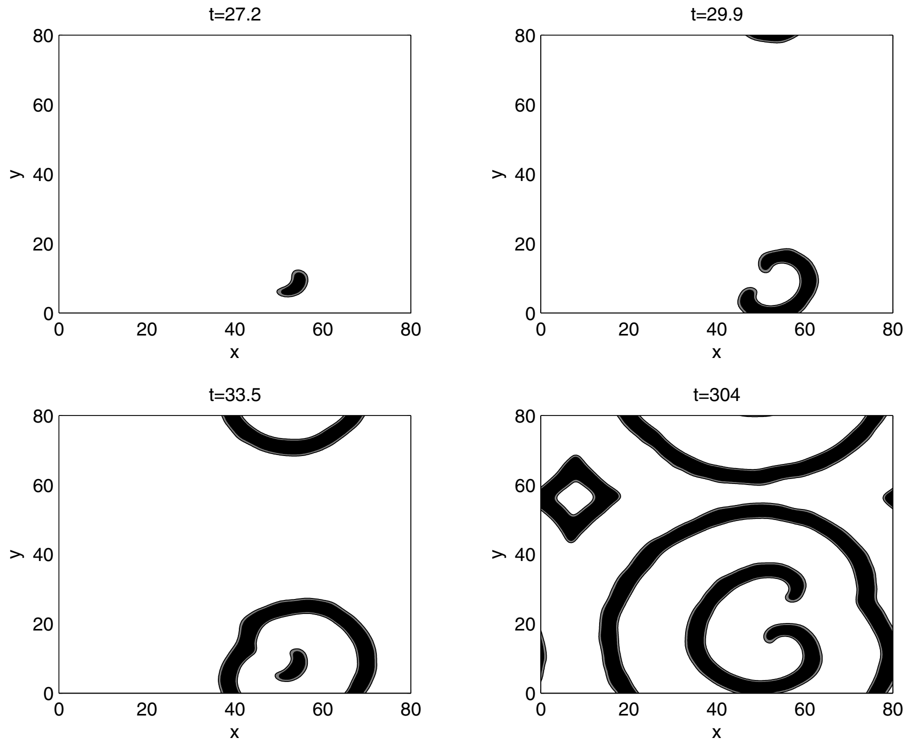

FIG. 1.2. *Nucleation of a spiral pattern for the second example described in section* 5.

behavior of this system is well understood. In particular, we know that $U(t)$ will exit the domain of attraction of 0 near a U^*, which satisfies the elliptic PDE

$$\Delta U + f(U, 0) = 0,$$

with appropriate boundary conditions. At this point, we make use of Assumption 1.3(2) and the maximum principle to show that U^* is positive. Further assuming that the domain Ω is a ball, we deduce that U^* is radially symmetric by applying a theorem of Gidas, Ni, and Nirenberg [8]. The next step is to relate the exit behavior of the gradient approximation back to the full system (1.3). Even though the approximation is good only on finite time intervals and the exit time grows like e^{K/σ^2}, some $K > 0$, this can be achieved. The key observation is that most of the time to exit is spent very close to $(0, 0)$, and the time to reach the boundary after leaving a neighborhood of $(0, 0)$ can be controlled.

We state the main theorem formally. Let $\Omega = B(\mathbf{0}, L)$, with the ball of radius L in $\mathbf{R}^2$ with center $\mathbf{0}$. The theorem concerns exit from a set $\mathcal{D}_{\epsilon,T}$. This set is uniformly attracted to the origin $(u, v) = (0, 0)$ and is defined precisely in section 4. As $T \to \infty$, the set $\mathcal{D}_{\epsilon,T}$ converges to the full domain of attraction. We have not been able to prove the result for exit from the full domain of attraction, because the attraction to the origin is not uniform.

Writing $u \in L^2(\Omega)$ in polar coordinates $u(r, \theta)$, we define the projection

$$\Theta u = u - \frac{1}{2\pi} \int_0^{2\pi} u(r, \theta)\, d\theta$$

to the nonradial component of u.

THEOREM 1.4. *Let* $(u(t), v(t))$ *be the solution of* (1.3) *with initial condition* $(u_0, v_0) = (0, 0)$. *Let* τ *be the exit time of* $(u(t), v(t))$ *from* $\mathcal{D}_{\epsilon,T}$. *Fix* $\delta > 0$. *For* T *sufficiently large and* ϵ *sufficiently small,*

$$\mathbf{P}(\|\Theta u(\tau)\|_{L^2(\Omega)} \geq \delta) \to 0 \qquad \text{as } \sigma \to 0.$$

The proof is given over the next three sections by developing Theorem 4.5, a more detailed version of Theorem 1.4. We develop the large deviation principle for (1.3) in section 2. The gradient approximation and its exit behavior are described in section 3. The argument is completed in section 4, where we make the link between the exit behavior of the gradient system and that of the spiral system. We complete the paper by providing some numerical simulations in section 5.

The theorem explains behavior seen in numerical simulations reasonably well, though the assumptions used are strong. The theorem requires Dirichlet boundary conditions on a ball, but simulations performed with periodic conditions on a domain $\Omega = [0, L]^2$ show the same type of behavior: targets, rather than spirals patterns, are nucleated for small ϵ, σ. The simulations indicate that the center of the nucleation has no preferred location for periodic boundary conditions. However, for Dirichlet conditions on $\Omega = B(\mathbf{0}, L)$, careful reading of the proof indicates that the target should be nucleated at the center of Ω. This is a reflection of the small gradient of the potential in the direction that controls that center of the nucleation. For Neumann conditions the situation is different. The energy is lower at the boundary, and consequently wave forms tend to form at the boundary, but spirals are still not nucleated for small ϵ, σ.

Further work should address the stability of the radial pattern that is nucleated. It may be possible for the noise to drive a radially symmetric field into a spiral wave after leaving the domain of attraction. It would be beneficial to understand the stability of the symmetry and show that this event is rare. It would also be interesting to examine the problem on an unbounded domain, where the methods used in this paper break down. We expect the same phenomenon to arise, though multiple target patterns may emerge in different parts of the spatial domain.

2. Large deviations for (1.3).

We introduce large deviations theory for (1.3). Consider $u_0, v_0 \in L^2(\Omega)$, $\phi(t)\colon [0,T] \to L^2(\Omega)$, and

$$\psi(t) := v_0 + \epsilon \int_0^t e^{-\epsilon(t-s)} \phi(s)\, ds. \tag{2.1}$$

If $\phi(t) \in H^2(\Omega) \cap H_0^1(\Omega)$ and $Z(t) := \phi_t(t) - (\Delta\phi(t) + f(\phi(t), \psi(t))) \in \mathcal{W}_0$ for almost all $t \in [0, T]$, define

$$S^{\epsilon}_{[0,T]}(\phi, \psi) = \frac{1}{2} \int_0^T \|Z(t)\|_0^2\, dt. \tag{2.2}$$

In all other cases, let $S^{\epsilon}_{[0,T]}(\phi, \psi) = \infty$. This defines the action functional for (1.3) in the following sense: for a normed vector space H, let $C([0,T], H)$ denote the set of continuous functions from $[0, T]$ to H.

DEFINITION 2.1. $S_{[0,T]}$ *is the action functional for a process* $X\colon [0,T] \to H$ *with* $X(0) = X_0$ *if we have the following:*

1. *For any* $\phi \in C([0,T],H)$ *with* $\phi(0) = X_0$ *and any* $h, \delta > 0$, *there exists* $\sigma_0 > 0$ *such that for* $\sigma < \sigma_0$

$$\mathbf{P}\Big(\sup_{0\le t\le T} \|X(t) - \phi(t)\|_H < \delta\Big) \ge \exp\Big[-\sigma^{-2}\Big(S_{[0,T]}(\phi) + h\Big)\Big].$$

2. *For any* $h, \delta > 0$ *and* $K < \infty$, *there exists* $\sigma_0 > 0$ *such that for* $\sigma < \sigma_0$

$$\mathbf{P}\Big(\inf_{\phi\in\Phi_K} \sup_{0\le t\le T} \|X(t) - \phi(t)\|_H \ge \delta\Big) \le \exp\Big[-\sigma^{-2}(K - h)\Big],$$

where

$$\Phi_K = \{\phi \in C([0,T],H)\colon S_{[0,T]}(\phi) \le K, \quad \phi(0) = X_0\}.$$

3. *The functional* $S_{[0,T]}(\phi)$ *is lower semicontinuous in* $C([0,T],H)$.
4. *The set* Φ_K *is compact in* $C([0,T],H)$.

PROPOSITION 2.2. *Let* $(u(t), v(t))$ *be the* $H = L^2(\Omega)^2$ *valued process defined by* (1.3) *with initial data* $(u_0, v_0) \in H$. *The action functional for* $(u(t), v(t))$ *on* H *is* $S^{\epsilon}_{[0,T]}$.

To prove this result, we make use of the following lemma concerning the action functional of a linear stochastic heat equation.

LEMMA 2.3. *Define* $S^{lin}_{[0,T]}(\phi)$ *for* $\phi\colon [0,T] \to L^2(\Omega)$ *as follows: if* $\phi(t)$ *belongs to* $H^2(\Omega) \cap H^1_0(\Omega)$ *and* $\phi_t(t) - \Delta\phi(t)$ *belongs to* $\mathcal{W}_0$ *for almost all* $t \in [0,T]$, *let*

$$S^{lin}_{[0,T]}(\phi) := \frac{1}{2}\int_0^T \|\phi_t(s) - \Delta\phi(s)\|_0^2\, ds.$$

Otherwise, set $S^{lin}_{[0,T]}(\phi) = \infty$. *Then* $S^{lin}_{[0,T]}$ *is the action functional for the process* $X(t)$ *on* $L_2(\Omega)$ *defined by*

$$dX = \Delta X\, dt + \sigma dW(t), \qquad X(0) = 0, \tag{2.3}$$

with homogeneous Dirichlet boundary conditions on Ω.

Proof. See [4]. □

Proof of Proposition 2.2. The method of proof follows [6]. Let $X(t)$ solve the linear system (2.3). Introduce the system

$$\frac{dw}{dt} = \Delta w + f(w + X, \psi), \qquad w(0) = u_0,$$
$$\frac{d\psi}{dt} = \epsilon(w + X - \psi), \qquad \psi(0) = v_0,$$

subject to homogeneous Dirichlet boundary conditions on Ω. Now $(\phi, \psi) = (w+X, \psi)$ is a solution of (1.3). We can define a transformation that gives the action functional for (1.3) in terms of $S^{lin}_{[0,T]}$. For fixed initial data (u_0, v_0), define $\Lambda(X) = (w+X, \psi) = (\phi, \psi)$. We also make use of the inverse of Λ, which is well defined when (ϕ, ψ) are related by (2.1) and defined by $\Lambda^{-1}(\phi, \psi) = \phi - w$.

Our claim is that the action functional for (1.3) is

$$S^{\epsilon}_{[0,T]}(\phi, \psi) = S^{lin}_{[0,T]}(\Lambda^{-1}(\phi, \psi)).$$

First note that this agrees with the definition of $S^{\epsilon}_{[0,T]}$ provided in (2.2). In the case when (2.1) holds, substitute the definition of Λ^{-1}:

$$\begin{aligned} S^{lin}_{[0,T]}(\Lambda^{-1}(\phi,\psi)) &= \frac{1}{2}\int_0^T \|(\phi_t - \Delta\phi) - (w_t - \Delta w)\|_0^2 \, ds \\ &= \frac{1}{2}\int_0^T \|(\phi_t - \Delta\phi) - f(w + X, \psi)\|_0^2 \, ds \\ &= \frac{1}{2}\int_0^T \|(\phi_t - \Delta\phi) - f(\phi, \psi)\|_0^2 \, ds, \end{aligned}$$

as required. If (2.1) does not hold for (ϕ,ψ), set $S^{\epsilon}_{[0,T]}(\phi,\psi) = \infty$ to indicate the event has zero probability.

The topological properties of the action functional are a consequence of the regularity of $S^{lin}_{[0,T]}$, Λ, and Λ^{-1}. Lower semicontinuity of $S_{[0,T]}$ follows from the lower semicontinuity of $S^{lin}_{[0,T]}$ and the continuity of Λ^{-1}. To show that Λ^{-1} is continuous, consider $\phi_i, \psi_i \in C([0,T], L^2(\Omega))$ for $i = 1, 2$ related by (2.1). Let w_i be the solution of

$$\frac{dw_i}{dt} = \Delta w_i + f(\phi_i, \psi_i), \qquad w_i(0) = u_0,$$

with Dirichlet boundary conditions on Ω. Given regularity of f, w_i depends continuously on (ϕ_i, ψ_i) and

$$\begin{aligned} \|\Lambda^{-1}(\phi_1,\psi_1) - \Gamma^{-1}(\phi_2,\psi_2)\|_{L^2(\Omega)} &= \|(\phi_1 - w_1) - (\phi_2 - w_2)\|_{L^2(\Omega)} \\ &\le \|\phi_1 - \phi_2\|_{L^2(\Omega)} + \|w_1 - w_2\|_{L^2(\Omega)}. \end{aligned}$$

Hence Λ^{-1} is continuous.

For given ψ, the set $\{\phi\colon S^{\epsilon}_{[0,T]}(\phi,\psi) \le K\} = \{\Lambda(X)\colon S^{lin}_{[0,T]}(X) \le K\}$, which is compact because Λ is continuous. This implies that the set $\{(\phi,\psi)\colon S^{\epsilon}_{[0,T]}(\phi,\psi) \le K\}$ is compact, because ψ is determined by (2.1) when $S^{\epsilon}_{[0,T]}$ is finite. This completes the proof. ∎

2.1. An estimate for the action functional. When $(u(t), v(t))$ belongs to a set that is uniformly attracted to the origin, there is a limit on the amount of time $(u(t), v(t))$ can spend outside a neighborhood of the origin even in the presence of noise. We quantify this in Lemma 2.5 by finding a lower bound on the action functional and further making sure the bound is uniform in ϵ.

First, we give some notation. Denote by $B_{L^2(\Omega)}(u,\delta)$ the open ball of radius δ with center u in $L^2(\Omega)$. Denote by $B((u_0,v_0),\delta)$ the set of $\{(u,v) \in L^2(\Omega)^2\colon \max\{\|u - u_0\|_{L^2(\Omega)}, \|v - v_0\|_{L^2(\Omega)}\} < \delta\}$. Similarly for a set $\mathcal{D} \subset L^2(\Omega)^2$ let $B(\mathcal{D},\delta)$ equal the union of $B((u_0,v_0),\delta)$ over $(u_0,v_0) \in \mathcal{D}$.

To develop this lemma, consider a neighborhood $\mathcal{O}$ of $(0,0)$. Let

$$\mathcal{O}_\epsilon := \bigcup_{(u_0,v_0)\in\mathcal{O}} B((u_0,v_0),\epsilon^{1/2})$$

and $\mathcal{D}_{\epsilon,T}$ equal the set of (u_0,v_0) such that the solution $(u(t),v(t))$ of (1.1) enters $\mathcal{O}_\epsilon$ in time less than or equal to T.

LEMMA 2.4. *Fix $R > 0$ and $T_1 > 0$. Consider $\epsilon_n \to 0$ as $n \to \infty$. Suppose that $\phi_n, \psi_n \in C([0,T_1], H_0^1(\Omega))$ such that $\|(\phi_n(0), \psi_n(0))\|_{L^2(\Omega)^2} \le R$ and $(\phi_n(t), \psi_n(t)) \in \mathcal{D}_{\epsilon_n,T}$ for $0 \le t \le T_1$.*

If $S^{\epsilon_n}_{[0,T_1]}(\phi_n, \psi_n) \to 0$, then there exists a limit $(\phi, \psi) \in C([0,T_1], L^2(\Omega)^2)$ such that $S^0_{[0,T_1]}(\phi, \psi) = 0$ and $(\phi(t), \psi(t)) \in \mathcal{D}_{0,T}$ for $0 \le t \le T_1$.

Let $L^2([0,T], H)$ (resp., $L^\infty([0,T], H)$) denote the functions $\phi\colon [0,T] \to H$ with norm $(\int_0^T \|\phi(t)\|_H^2 dt)^{1/2}$ (resp., with norm $\sup_{0\le t\le T} \|\phi(t)\|_H$).

Proof. We can write for some $\delta_n \to 0$, strongly in $L^2([0,T_1], L^2(\Omega))$,

$$\frac{d}{dt}\phi_n = \Delta\phi_n + f(\phi_n, \psi_n) + \delta_n, \qquad \frac{d}{dt}\psi_n = \epsilon(\phi_n - \psi_n).$$

Because of the boundedness on the initial data and Assumption 1.2, ϕ_n is bounded uniformly in n in $L^\infty([0,T_1], L^2(\Omega))$. Further, taking inner product with ϕ_n and using boundary conditions,

$$\frac{d}{dt}\|\phi_n\|^2_{L^2(\Omega)} + \|\nabla\phi_n\|^2_{L^2(\Omega)} = \langle f(\phi_n, \psi_n) + \delta_n, \phi_n\rangle,$$

which implies that ϕ_n are uniformly bounded in $L^2([0,T_1]H_0^1(\Omega))$. By weak compactness [15], there exists a weak (in $L^\infty([0,T_1], L^2(\Omega)) \cap L^2([0,T_1], H_0^1(\Omega))$) limit point

$$\phi \in C([0,T_1], H_0^1(\Omega)).$$

Define ψ from ϕ as in (2.1). For a test function $\chi \in H_0^1(\Omega)$,

$$\frac{d}{dt}\langle\chi, \phi_n\rangle = \langle\Delta\phi_n, \chi\rangle + \langle f(\phi_n, \psi_n), \chi\rangle + \langle\delta_n, \chi\rangle, \qquad 0 \le t \le T_1.$$

Taking limits in $n \to \infty$,

$$\frac{d}{dt}\langle\chi, \phi\rangle = \langle\Delta\phi, \chi\rangle + \langle f(\phi, \psi), \chi\rangle.$$

The convergence of $f(\phi_n, \psi_n)$ to $f(\phi, \psi)$ follows from the compactness argument of Lions. As this holds for all χ, we conclude that the limit $\phi(t)$ is a solution of (1.1) with $\sigma = \epsilon = 0$, and hence $S^0_{[0,T_1]}(\phi, \psi) = 0$.

The solution of (1.1) with initial data $(\phi(t), \psi(t))$, for $0 \le t \le T_1$, will enter any neighborhood of $\mathcal{O}$ in time T. This implies that $(\phi(t), \psi(t)) \in \mathcal{D}_{0,T}$. ∎

LEMMA 2.5. *Fix $T > 0$. Let $\mathcal{A}_\epsilon$ equal $\mathcal{D}_{\epsilon,T} - \mathcal{O}_\epsilon$. Then there exists $T_1 > 0$ and $B > 0$ such that for all ϵ sufficiently small $S^\epsilon_{[0,T_2]}(\phi, \psi) \ge BT_2$ if $T_2 > T_1$ and $(\phi, \psi) \in C([0,T_2], \mathcal{A}_\epsilon)$ and $\|(\phi(0), \psi(0))\|_{L^2(\Omega)^2} \le R$.*

Proof. Note that $\mathcal{O}_\epsilon$ is open and nonempty for all $\epsilon > 0$ and that a solution of (1.1) with initial data $(u_0, v_0) \in \mathcal{A}_\epsilon$ enters $\mathcal{O}_\epsilon$ in time less than or equal to T. Let $T_1 > T$. Denote by $\mathcal{Q}_\epsilon$ the set of functions from $C([0,T_1], L^2(\Omega)^2)$ assuming values in $\mathcal{A}_\epsilon$. Then $\mathcal{Q}_\epsilon$ is closed in $C([0,T_1], L^2(\Omega)^2)$ and nonempty. Further, $S^\epsilon_{[0,T_1]}(\phi, \psi)$ is lower semicontinuous: if $(\phi_n, \psi_n) \to (\phi^*, \psi^*)$, then $S^\epsilon_{[0,T_1]}(\phi^*, \psi^*) \le \liminf S^\epsilon_{[0,T_1]}(\phi_n, \psi_n)$. Thus, by using Definition 2.1(4), the functional $S^\epsilon_{[0,T_1]}$ attains its infimum $(\phi, \psi) \in \mathcal{Q}_\epsilon$. This infimum is different from zero, since otherwise some trajectory starting in $\mathcal{A}_\epsilon$ would enter $\mathcal{O}_\epsilon$ in time T_1. We conclude that $S^\epsilon_{[0,T_1]}(\phi, \psi) \ge B$ for $(\phi, \psi)\colon [0,T_1] \to \mathcal{A}_\epsilon$.

To check that the lower bound B can be chosen uniformly over ϵ small, apply Lemma 2.4: if B cannot be chosen uniformly, we can find (ϕ,ψ) such that $S^0_{[0,T_1]}(\phi,\psi) = 0$ and $(\phi(t),\psi(t)) \in \mathcal{A}_0$ for $0 \le t \le T_1$. This cannot be, due to the definition of $\mathcal{A}_0$ and because $T_1 > T$.

This is easily extended to $(\phi,\psi)\colon [0,T_2] \to \mathcal{A}_\epsilon$ for $T_2 \ge T_1$ by applying the additive properties of the action functional: $S^\epsilon_{[0,T_2]}(\phi,\psi) \ge S^\epsilon_{[0,T_1]}(\phi,\psi)(T_2/T_1)$ for $T_2 > T_1$. This concludes the proof. □

3. Approximation by a gradient system. We develop an approximation to (1.3) by a gradient system, exploiting the slow time scale in the dynamics of v. It is much easier to understand large deviations for a gradient system. Let $F(u,v)\colon \mathbf{R}^2 \to \mathbf{R}$ be such that $\nabla_u F(u,v) = -f(u,v)$. For $u,v \in H^1_0(\Omega)$, define the Lyapunov function

$$\mathcal{L}(u,v) := \int_\Omega \|\nabla u(\mathbf{x})\|^2_{\mathbf{R}^d} + F(u(\mathbf{x}), v(\mathbf{x}))\, d\mathbf{x}.$$

The spiral system (1.3) is written as

$$\begin{aligned} du &= -\nabla_u \mathcal{L}(u,v)\, dt + \sigma\, dW(t),\\ \frac{\partial v}{\partial t} &= \epsilon\, g(u,v). \end{aligned} \tag{3.1}$$

This leads to an approximate gradient system as follows.

Proposition 3.1. *Consider the following two initial value problems on $[0,T]$ for $U_0, V_0, u_0, v_0 \in L^2(\Omega)$: let $U(t)$ satisfy the gradient system*

$$dU = -\nabla_u \mathcal{L}(U, V_0)\, dt + \sigma\, dW(t), \qquad U(0) = U_0, \tag{3.2}$$

and let $(u(t), v(t))$ solve

$$\begin{aligned} du &= -\nabla_u \mathcal{L}(u,v)\, dt + \sigma\, dW(t), \qquad u(0) = u_0,\\ v_t &= \epsilon(u - v), \qquad v(0) = v_0. \end{aligned}$$

Under the condition $\|u(s)\|_{L^2(\Omega)} \le R$ for $0 \le s \le T$, we can find $K > 0$ such that, subject to $\|v_0 - V_0\|_{L^2(\Omega)} + \|u_0 - U_0\|_{L^2(\Omega)} \le \epsilon^{1/2}$,

$$\|v(s) - V_0\|_{L^2(\Omega)}, \|u(s) - U(s)\|_{L^2(\Omega)} \le K\epsilon^{1/2}, \qquad 0 \le s \le T.$$

Under the condition $\|u(s)\|_{L^2(\Omega)} \le R$ for $0 \le s \le T$, we have that for $(u_0, v_0) = (U_0, V_0)$ and all ϵ sufficiently small

$$\|v(s) - V_0\|_{L^2(\Omega)}, \|u(s) - U(s)\|_{L^2(\Omega)} \le \epsilon^{1/2}, \qquad 0 \le s \le T.$$

Proof. We prove only the first of the two statements, the second one being similar. Let $\delta(t) = u(t) - U(t)$. Then

$$\begin{aligned} \delta_t &= \Delta\delta + \Big(f(u,v) - f(U,V_0)\Big)\\ &= \Delta\delta + \Big(f(u,v) - f(u,V_0)\Big) + \Big(f(u,V_0) - f(U,V_0)\Big),\\ v(t) &= v_0 + \epsilon \int_0^t \exp(-\epsilon(t-s)) u(s)\, ds. \end{aligned}$$

Now

$$\|v(t)-v_0\|_{L^2(\Omega)} \le \epsilon\, t \max_{0\le s\le t} \|u(s)\|_{L^2(\Omega)}.$$

Under the condition $\|u(s)\|_{L^2(\Omega)} \le R$ for $0 \le s \le T$,

$$\|v(t)-V_0\|_{L^2(\Omega)} \le R\,\epsilon\, t + \|v_0 - V_0\|_{L^2(\Omega)}.$$

Using the Lipschitz condition on f and applying standard techniques, we can find $K>0$ such that for $0\le s\le T$

$$\begin{aligned}\|\delta(t)\|_{L^2(\Omega)} &\le \|\delta(0)\|_{L^2(\Omega)} + K\int_0^t \|v(s)-V_0\|_{L^2(\Omega)} + \|\delta(s)\|_{L^2(\Omega)}\,ds\\ &\le \|\delta(0)\|_{L^2(\Omega)} + K\int_0^t (R\,\epsilon\, s + \|v_0-V_0\|_{L^2(\Omega)}) + \|\delta(s)\|_{L^2(\Omega)}\,ds.\end{aligned}$$

We conclude that (for a possibly larger K)

$$\|\delta(t)\|_{L^2(\Omega)} \le K e^{Kt}\Big(\|u_0 - U_0\|_{L^2(\Omega)} + \epsilon t^2 + t\|v_0 - V_0\|_{L^2(\Omega)}\Big). \qquad \square$$

3.1. Large deviations for the gradient approximation (3.2). Our reason for using the gradient approximation is the convenience with which the action functional can be studied. If $\phi(t) \in H^2(\Omega)\cap H_0^1(\Omega)$ and $Z(t) := \phi_t(t) - (\Delta\phi(t) + f(\phi(t),V_0)) \in \mathcal{W}_0$ for almost all $t\in[0,T]$, the action functional on $L^2(\Omega)$ for (3.2) is simply

$$S_{[0,T]}(\phi;V_0) = \frac{1}{2}\int_0^T \|Z(t)\|_0^2\,dt = \frac{1}{2}\int_0^T \|\phi_t(s) + \nabla_u \mathcal{L}(\phi(s),V_0)\|_0^2\,ds.$$

The second component indicates the dependence of (3.2) on V_0. This can be developed further:

$$S_{[0,T]}(\phi;V_0) = \frac{1}{2}\int_0^T \|\phi_t(s) - \nabla_u \mathcal{L}(\phi(s),V_0)\|_0^2\,ds + \frac{1}{2}2\int_0^T \langle \phi_t(s), \nabla_u\mathcal{L}(\phi(s),V_0)\rangle_0 ds.$$

Using the structural assumptions on Q in (1.4) and the boundary conditions on ϕ,

$$\begin{aligned}\int_0^T \langle \phi_t(s), \nabla_u\mathcal{L}(\phi(s),V_0)\rangle_0\,ds &= -\int_0^T \langle \phi_t(s), \Delta\phi(s) + f(\phi(s),V_0)\rangle_0\,ds\\ &= -\int_0^T \Big\langle \phi_t(s), Q^{-1}\Delta\phi(s) + Q^{-1}f(\phi(s),V_0)\Big\rangle\,ds\\ &= \int_0^T \frac{d}{ds}\frac{1}{2}\Big\langle \nabla Q^{-1/2}\phi(s), \nabla Q^{-1/2}\phi(s)\Big\rangle\\ &\quad - \Big\langle Q^{-1} f(\phi(s),V_0)\phi_t(s), 1\Big\rangle\,ds\\ &= \int_0^T \frac{d}{ds}\frac{1}{2}\|\nabla\phi(s)\|_0^2 + \frac{d}{ds}\langle F(\phi(s),V_0),1\rangle_0\,ds.\end{aligned}$$

Hence,

(3.3)

$$\begin{aligned}S_{[0,T]}(\phi;V_0) &= \frac{1}{2}\int_0^T \|\phi_t(s) + \nabla_u\mathcal{L}(\phi(s),V_0)\|_0^2\,ds\\ &= \frac{1}{2}\int_0^T \|\phi_t(s) - \nabla_u\mathcal{L}(\phi(s),V_0)\|_0^2\,ds + \bar{\mathcal{L}}(\phi(T),V_0) - \bar{\mathcal{L}}(\phi(0),V_0),\end{aligned}$$

where we have the modified Lyapunov function

$$\bar{\mathcal{L}}(U,V_0) = \frac{1}{2}\|\nabla U\|_0^2 + \langle F(U,V_0),1\rangle_0.$$

We define the domain of attraction of 0 for (3.2) (with $\sigma = 0$) as a subset of $L^2(\Omega)^2$. It is useful to work in $L^2(\Omega)^2$, rather than the phase space of (3.2), to facilitate comparison to (1.3). Let $\mathcal{D}^g$ be the set of (U_0,V_0) such that the solution $U(t)$ of

$$\frac{\partial U}{\partial t} = -\nabla_u \mathcal{L}(U,V_0), \qquad U(0) = U_0, \tag{3.4}$$

converges to 0 as $t \to \infty$. The quasi potential for initial data in $\mathcal{D}^g$ may be represented using the modified Lyapunov functional in the following way: for $(U_0,V_0) \in \mathcal{D}^g$,

$$\mathcal{V}^g(0,U_1;V_0) = \min_{\phi(0)=0,\ \phi(T)=U_1,\ T>0} S_{[0,T]}(\phi;V_0) = \bar{\mathcal{L}}(U_1,V_0). \tag{3.5}$$

This follows from (3.3) exactly as in [6].

3.2. Symmetry. Let $\mathcal{L}^* = \min\{\mathcal{V}^g(0,U;0)\colon (U,0) \in \partial\mathcal{D}^g\}$. For certain domains Ω, any critical point U^* of $\mathcal{L}(U,0)$ is radially symmetric. We will exploit the following result.

THEOREM 3.2 (Gidas, Ni, and Nirenberg). *Let $\Omega = B(\mathbf{0},L)$, and let u be a twice differentiable positive solution of*

$$\Delta u + a(u) = 0 \text{ in } \Omega, \qquad u = 0 \text{ on the boundary of } \Omega.$$

If a is Lipschitz, then u is radial (i.e., $u(\mathbf{x}) = u(\|\mathbf{x}\|_{\mathbf{R}^2})$), and $u_r(\mathbf{x}) < 0$ for $0 < \|\mathbf{x}\|_{\mathbf{R}^2} \le L$.

Proof. See [8, 3]. □

PROPOSITION 3.3. *Suppose that $\Omega = B(\mathbf{0},L)$. Any minimizer U^* of $\mathcal{L}$ over $(U,0) \in \partial\mathcal{D}^g$ is radial. Further, there exists a radial function $U\colon \mathbf{R} \to L^2(\Omega)$ such that we have the following:*

1. (3.2) *holds with $V_0 = 0$ and $\sigma = 0$.*
2. *U approaches 0 (resp., U^*) as $t \to \infty$ (resp., $-\infty$).*
3. *For any $T_1 < T_2$, $S_{[T_1,T_2]}(U;0) \le \mathcal{L}^* = \mathcal{L}(U^*,0)$.*

Proof. Any minimum U^* of $\mathcal{V}^g(0,U;0)$ over $(U,0) \in \partial\mathcal{D}^g$ is a saddle point of $\bar{\mathcal{L}}(U,0)$. Further, $\|\nabla U^*\|_0 < \infty$. By Assumption 1.1 and the Sobolev embedding theorem, U^* is twice differentiable, and

$$\nabla\bar{\mathcal{L}}(U^*,0) = \Delta U^* + f(U^*,0) = 0.$$

As f is Lipschitz, U^* satisfies an equation covered by Theorem 3.2. We now show that U^* is positive by an application of the maximum principle. Consider a minimum $(x,y) \in \Omega$ of U^*. At the minimum, $\Delta U^* \ge 0$, and hence $f(U^*,0) \le 0$. By assumption on f, we have that $f(u,0) > 0$ for $u < 0$. This implies that any negative minimum must be $u = 0$. The boundary conditions imply that u is positive. From Theorem 3.2, we conclude that U^* is radially symmetric if $\Omega = B(\mathbf{0},L)$.

For the final part, let $U(t)$ denote the heteroclinic connection from U^* to 0. All the properties follow. □

4. Proof of the main result. We start our proof by providing lower bounds on the potential $\mathcal{V}^g(U_0, U_1; V_0)$ for an initial point (U_0, V_0) close to the origin and an exit point U_1 close to U^*.

LEMMA 4.1. *If* $(U_1^n, V^n) \to (\bar{U}_1, \bar{V})$ *in* $L_2(\Omega)^2$*, then*

$$\liminf_{n\to\infty} \bar{\mathcal{L}}(U_1^n; V^n) \ge \bar{\mathcal{L}}(\bar{U}_1; \bar{V}).$$

Proof. Let $\mathcal{W}_1$ denote the Banach space of $U_1 \in \mathcal{W}_0$ with $\|U_1\|_{\mathcal{W}_1} := \|\nabla U_1\|_0 < \infty$. Because the dual of $\mathcal{W}_1$ can be continuously embedded in $L^2(\Omega)$, we have $U_1^n \to \bar{U}_1$ weakly in $\mathcal{W}_1$, and hence

$$\|\nabla \bar{U}_1\|_0 \ge \liminf_n \|\nabla U_1^n\|_0.$$

It suffices to consider $\bar{\mathcal{L}}(U_1^n; V^n)$ and $\bar{\mathcal{L}}(\bar{U}_1, \bar{V})$ uniformly bounded and hence that $F^{1/2}(\bar{U}_1^n, \bar{V}^n)$ and $F^{1/2}(\bar{U}_1, \bar{V})$ are uniformly bounded in $L^2(\Omega)$. As $F^{1/2}(U, V)$ is locally Lipschitz, we can show $F^{1/2}(U_1^n, V^n) \to F^{1/2}(\bar{U}_1, \bar{V})$ in $L^2(\Omega)$ and hence converges weakly in $\mathcal{W}_0$. This implies that

$$\|F^{1/2}(\bar{U}_1, \bar{V})\|_0 \ge \liminf_n \|F^{1/2}(U_1^n, V^n)\|_0.$$

Because $\bar{\mathcal{L}}(U, V) = \frac{1}{2}\|\nabla U\|_0^2 + \|F^{1/2}(U, V)\|_0^2$, we have completed the proof. □

LEMMA 4.2. *For* $d, \delta_0 > 0$*, define* $\mathcal{D}_T^g$ *as the set of points* (U_0, V_0) *that take less (as evolved by the gradient system* (3.4)*) than time* T *to enter* $\mathcal{O}$*, where*

$$\mathcal{O} := \{(U, V) \in L^2(\Omega)^2 \colon\ \mathcal{V}^g(U, V) \le d \text{ and } \|V\|_{L^2(\Omega)} < \delta_0\}.$$

Fix $\delta_1 > 0$*. Let*

$$Y_T = \{(U_1, 0) \in \partial\mathcal{D}_T^g \colon \|\Theta U_1\|_{L^2(\Omega)} \ge \delta_1\}, \quad Y = \{(U_1, 0) \in \partial\mathcal{D}^g \colon \|\Theta U_1\|_{L^2(\Omega)} \ge \delta_1\}.$$

For all $d > 0$*, there exists* $T, \delta_0 > 0$ *such that for* $(U_1, V) \in B(Y_T, 3\delta_0) \cap \mathcal{D}_T^g$

$$\mathcal{V}^g(0, U_1; V) \ge \inf_{(U_1', V') \in Y} \mathcal{V}^g(0, U_1'; V') - d.$$

Proof. Consider $R \ge \inf_{(U_1', V') \in Y} \mathcal{V}^g(0, U_1'; V')$. It is enough to consider $(U_1, V) \in Y_{T,\delta_0,R} := B(Y_T, 3\delta_0) \cap \mathcal{D}_T^g \cap \{\mathcal{V}^g(0, U; V) \le R\}$. Suppose the above were not true. Then, for any sequences $\delta_0^n \to 0$ and $T^n \to \infty$, we could find a sequence $(U_1^n, V^n) \in Y_{T^n,\delta_0^n,R}$ such that

$$\mathcal{V}^g(0, U_1^n; V^n) < \inf_{(U_1', V') \in Y} \mathcal{V}^g(0, U_1'; V') - d. \tag{4.1}$$

As $(U_1^n, V^n) \in Y_{T^n,\delta_0^n,R}$, we have $\frac{1}{2}\|\nabla U_1^n\|_0^2 \le R$, and hence U_1^n must have an $L^2(\Omega)$ limit point $\bar{U}_1$. We may choose T^n and $\delta_0^n > 0$ such that $\mathcal{D}_{T^n}^g \subset \mathcal{D}_{T^{n'}}^g$ for $n < n'$. In this case, we can easily show $(\bar{U}_1, 0) \in \partial\mathcal{D}^g$. Clearly $V^n \to \bar{V} = 0$ in $L^2(\Omega)$. For n sufficiently large, by Lemma 4.2 and (3.5)

$$\mathcal{V}^g(0, U_1^n; V^n) > \mathcal{V}^g(0, \bar{U}_1; \bar{V}) - d. \tag{4.2}$$

But $(\bar{U}_1, \bar{V}) \in Y$. Consequently, (4.1) and (4.2) are in contradiction. □

LEMMA 4.3. *Suppose that* $\Omega = B(\mathbf{0}, L)$. *Fix* $\delta_1 > 0$. *There exists* $T, d, \delta_0 > 0$ *such that*

$$\mathcal{V}^g(U_0, U_1; V) \geq \mathcal{L}^* + 2d$$

if $(U_0, V) \in \mathcal{O}$ *and* $(U_1, V) \in B(Y_T, 3\delta_0) \cap \mathcal{D}^g_T$.

Proof. Under Proposition 3.3, every minimizer of $\mathcal{V}^g(0, U; 0)$ over $(U, 0) \in \partial\mathcal{D}^g$ is radial. Hence, for any $d > 0$ sufficiently small,

$$\mathcal{V}^g(0, U_1; 0) \geq \mathcal{L}^* + 4d$$

for $(U_1, 0) \in \partial\mathcal{D}^g$ with $\|\Theta U_1\|_{L^2(\Omega)} \geq \delta_1$. By Lemma 4.2, we can find T, δ_0 such that

$$\mathcal{V}^g(0, U_1; V) \geq \mathcal{L}^* + 3d$$

if $(U_1, V) \in B(Y_T, 3\delta_0) \cap \mathcal{D}^g_T$. By definition of $\mathcal{O}$, $\mathcal{V}^g(0, U_0; V) \leq d$ and

$$\mathcal{V}^g(U_0, U_1; V) \geq \mathcal{V}^g(0, U_1; V) - \mathcal{V}^g(0, U_0; V) \geq \mathcal{L}^* + 3d - d = \mathcal{L}^* + 2d.$$

This completes the proof. ∎

LEMMA 4.4. *Suppose that* $\Omega = B(\mathbf{0}, L)$. *Fix* $\delta_1, d > 0$. *There exists* $T > 0$ *and a radial* ϕ *with* $\phi(0) = 0$ *and* $\phi(T) \in B_{L^2(\Omega)}(U^*, \delta_1)$ *such that* $S_{[0,T]}(\phi; 0) \leq \mathcal{L}^* + d$.

Proof. Let $U(t)$ be the trajectory from Proposition 3.3. Choose T_2 so that $U(T_2) \in B_{L^2(\Omega)}(U^*, \delta_1)$. Let $U_1 = U(T_2)$. Then $\mathcal{V}^g(0, U_1; 0) \leq \mathcal{L}^*$. Hence,

$$\inf_{\phi(0)=0,\ \phi(T)=U_1,\ T>0} S_{[0,T]}(\phi; 0) \leq \mathcal{L}^*.$$

Because the optimal ϕ is radial, we can take the infimum over all radial ϕ only. In particular, we can find a radial ϕ and T to complete by proof by taking T large such that

$$S_{[0,T]}(\phi; 0) \leq \mathcal{L}^* + d. \qquad ∎$$

THEOREM 4.5. *Fix* $\delta > 0$. *Let* $(u(t), v(t))$ *be the solution of* (1.3) *with initial data* $(u_0, v_0) = (0, 0)$. *There exists* $d, T > 0$ *such that for* ϵ *sufficiently small we have the following:*

1. *For the path* $\hat{\phi}$ *described in Lemma* 4.4, *we have as* $\sigma \to 0$

$$\mathbf{P}\Big(\sup_{0 \leq t \leq T} \|u(t) - \hat{\phi}(t)\|_{L^2(\Omega)} \leq 2\delta\Big) \geq \exp\Big(-\sigma^{-2}(\mathcal{L}^* + d/2)\Big).$$

2. *Let* τ *denote the exit time of* $(u(t), v(t))$ *from* $\mathcal{D}_{\epsilon,T}$. *Then, as* $\sigma \to 0$,

$$\mathbf{P}(\|\Theta u(\tau)\|_{L^2(\Omega)} \geq 2\delta) \leq \exp\Big(-\sigma^{-2}(\mathcal{L}^* + 3d/2)\Big).$$

Proof. We first set up a number of estimates and constants that will enable us to complete the proof. Let $\delta_1 = \delta$. Choose $d, \delta_0 < \delta, T$ such that Lemmas 4.4 and 4.3 hold. Let $\mathcal{O}$ be as in Lemma 4.2, so that for $(U, V) \in \mathcal{O}$ we have $\mathcal{V}^g(U, V) \leq d$ and $\|V\|_{L^2(\Omega)} < \delta_0$. Recall that

$$\mathcal{O}_\epsilon = \bigcup_{(U,V)\in\mathcal{O}} B((U, V), \epsilon^{1/2}).$$

Denote by $\mathcal{V}^\epsilon((u_0, v_0), (u_1, v_1))$ the quasi potential for (1.3), that is, the minimum of $S^\epsilon_{[0,T]}(\phi, \psi)$ over all $T > 0$ and all paths (ϕ, ψ) that connect (u_0, v_0) to (u_1, v_1) on the interval $[0, T]$. Choose $K_3 \gg \mathcal{L}^*$. Further, choose R sufficiently large such that

$$\inf_{\|(u,v)\|_{L^2(\Omega)^2} > R} \mathcal{V}((0, 0); (u, v)) \geq K_3. \tag{4.3}$$

Any solution of the gradient system (3.4) with initial data in $\mathcal{D}^g_T$ satisfying $\|u\|_{L^2(\Omega)} \leq R$ enters $\mathcal{O}$ in time T by definition of $\mathcal{D}^g_T$. Hence, by Proposition 3.1 with ϵ sufficiently small, any solution of (1.3) with $\sigma = 0$ with the same initial data must enter $\mathcal{O}_\epsilon$ in time T. We conclude that

$$\mathcal{D}^g_T \cap \{\|u\|_{L^2(\Omega)} \leq R\} \subset \mathcal{D}_{\epsilon,T}. \tag{4.4}$$

By applying Lemma 2.5, choose $T_1 > T$ to get

$$S^\epsilon_{[0,T_1]}(\phi, \psi) \geq K_3 \quad \text{if } (\phi(t), \psi(t)) \in \mathcal{A}_\epsilon = \mathcal{D}_{\epsilon,T} - \mathcal{O}_\epsilon, \quad 0 \leq t \leq T_1. \tag{4.5}$$

By Proposition 3.1, we can choose ϵ sufficiently small such that if $\|u(s)\|_{L^2(\Omega)} \leq R$ for $0 \leq s \leq T_1$,

$$\|v(t) - V_0\|_{L^2(\Omega)}, \|u(t) - U(t)\|_{L^2(\Omega)} \leq \delta_0, \qquad 0 \leq t \leq T_1, \tag{4.6}$$

subject to $\|(u_0, v_0) - (U_0, V_0)\|_{L^2(\Omega)^2} \leq \epsilon^{1/2}$.

1. Consider the path $\hat{\phi}$ constructed in Lemma 4.4. Then, for $K_1 = \mathcal{L}^* + d/2$, by Definition 2.1(1),

$$\mathbf{P}\Big(\sup_{0 \leq t \leq T} \|\hat{\phi}(t) - U(t)\|_{L^2(\Omega)} < \delta\Big) \geq e^{-K_1/\sigma^2}$$

for all σ suitably small. By (4.6), this implies the following for the full system (1.3):

$$\mathbf{P}\Big(\sup_{0 \leq t \leq T} \|u(t) - \hat{\phi}(t)\|_{L^2(\Omega)} \leq 2\delta\Big) \geq e^{-K_1/\sigma^2},$$

where $(u(0), v(0)) = (0, 0)$.

2. Let $K_2 = \mathcal{L}^* + d$. Let $U(t)$ be the solution of (3.2) with $(U_0, V_0) \in \mathcal{O}$. From Lemma 4.3 and Definition 2.1(2), for σ sufficiently small

$$\mathbf{P}(\text{entry of } (U(t), V_0) \text{ into } B(Y_T, 3\delta_0) \text{ in } [0, T_1]) \leq e^{-K_2/\sigma^2}. \tag{4.7}$$

Consider $(u_0, v_0) \in \mathcal{O}_\epsilon$. For some $(U_0, V_0) \in \mathcal{O}$, the distance $\|(U_0, V_0) - (u_0, v_0)\|_{L^2(\Omega)^2} < \epsilon^{1/2}$. Using (4.3), the probability that $(u(t), v(t))$ leaves the ball $B((0, 0), R)$ in $[0, T_1]$ is bounded above by e^{-K_3/σ^2}. Any $(u(t), v(t))$ that does remain in the ball $B((0, 0), R)$ in $[0, T_1]$ satisfies $\|v(t) - V_0\|_{L^2(\Omega)}$, $\|u(t) - U(t)\|_{L^2(\Omega)} \leq \delta_0$ by (4.6). Taking these two observations together with (4.7),

$$\mathbf{P}(\text{entry of } (u(t), v(t)) \text{ into } B(Y_T, 2\delta_0) \text{ in } [0, T_1]) \leq e^{-K_2/\sigma^2} + e^{-K_3/\sigma^2}. \tag{4.8}$$

Let $Z := \partial\mathcal{D}_T^g \cap \{\|\Theta U\|_{L^2(\Omega)} \geq \delta_1\}$. We would like this inequality to imply a bound on the entry of $(u(t), v(t))$ into Z in time T_1. Consider a solution $(u(t), v(t))$ of (1.3) that enters Z in time T_1. We need only consider the case where $(u(t), v(t))$ remains in $B((0,0), R)$ on the time interval $[0, T_1]$. We have from (4.6) that $\|v(t) - V_0\|_{L^2(\Omega)} \leq \delta_0$ and hence that $\|v(t)\|_{L^2(\Omega)} \leq \|V_0\|_{L^2(\Omega)} + \delta_0 < 2\delta_0$ for $0 \leq t \leq T_1$. Hence, as $Y_T = \{(U_1, 0) \in \partial\mathcal{D}_T^g \colon \|\Theta U_1\|_{L^2(\Omega)} \geq \delta_1\}$, $(u(t), v(t))$ must enter $B(Y_T, 2\delta_0)$ before it enters Z. Thus,

$$\mathbf{P}(\text{entry of } (u(t), v(t)) \text{ into } Z \text{ in } [0, T_1]) \leq e^{-K_2/\sigma^2} + e^{-K_3/\sigma^2}. \tag{4.9}$$

Let p equal the probability that $(u(t), v(t))$ exits $\mathcal{D}_{\epsilon,T}$ in time T_1 and satisfies $\|\Theta u(t)\|_{L^2(\Omega)} \geq \delta_1$ at the time of exit t. By (4.4), any $(u(t), v(t))$ that exits $\mathcal{D}_{\epsilon,T}$ in time T_1 must exit $\mathcal{D}_T^g \cap \{\|u\|_{L^2(\Omega)} < R\}$ in time T_1. With (4.9),

$$p \leq \mathbf{P}((u(t), v(t)) \text{ enters } Z \text{ in time } T_1) \leq e^{-K_2/\sigma^2} + e^{-K_3/\sigma^2}.$$

Let τ be the time of exit from $\mathcal{D}_{\epsilon,T}$. Consider a random τ_1 such that $(u(\tau_1), v(\tau_1))$ is the last exit from $\mathcal{O}_\epsilon$ before exiting $\mathcal{D}_{\epsilon,T}$. Then, for $\tau_1 < t < \tau$, $(u(t,0), v(t,0)) \in \mathcal{A}_\epsilon$, and by (4.5), $P(\tau - \tau_1 \geq T_1) \leq e^{-K_3/\sigma^2}$.
Finally, for $(u_0, v_0) \in \mathcal{O}_\epsilon$,

$$\mathbf{P}(\|\Theta u(\tau)\|_{L^2(\Omega)} \geq \delta_1) \leq p + \mathbf{P}(\tau - \tau_1 \geq T_1) \leq e^{-K_2/\sigma^2} + 2e^{-K_3/\sigma^2}.$$

This completes the proof. $\square$

5. Example. Numerical simulations illustrate the conclusion of Theorem 1.4 very well: for ϵ below some critical value, small noise nucleates only target waves. We make a number of changes to the setting of Theorem 1.4 for convenience of computations and comparison with [1]. Consider a domain $\Omega = [0, L]^2$ and the rescaled PDE ($x \mapsto x/\epsilon^{1/2}$ and $t \mapsto t/\epsilon$)

$$\begin{aligned} du &= \left[\Delta u + \tilde{f}(u,v)/\epsilon\right] dt + \sigma\, dW(t), \qquad u(0) = u_0, \\ v_t &= \tilde{g}(u,v), \qquad v(0) = v_0, \end{aligned} \tag{5.1}$$

with periodic boundary conditions. It is convenient to work with the following reaction terms to avoid instabilities [13] in (1.2) when (u, v) are large:

$$\tilde{f}(u,v) = \begin{cases} f(u,v), & u \leq 1, \\ -|f(u,v)|, & u \geq 1, \end{cases} \qquad \tilde{g}(u,v) = \begin{cases} g(u,v), & v \geq 0, \\ |g(u,v)|, & v < 0, \end{cases}$$

where f, g are defined as in (1.2). The Wiener process $W(t)$ has correlation length ξ and is defined by

$$W(t) = \sum_{i,j \geq 0} \alpha_{ij} \mathbf{e}_{ij} \beta_{ij}(t), \tag{5.2}$$

where $\beta_{ij}(t)$ are independent standard Brownian motions,

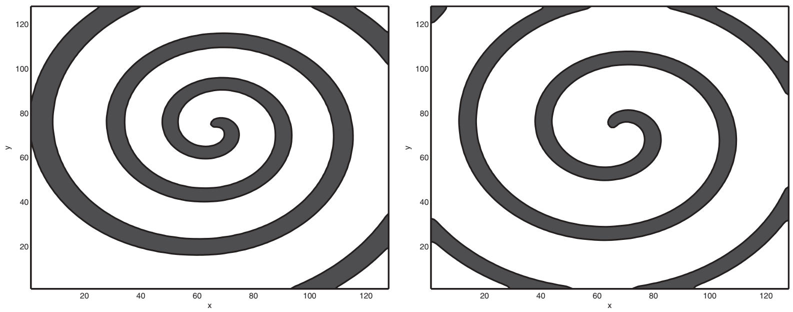

FIG. 5.1. *A spiral wave for the examples:* $\epsilon = 0.03$ *(left) and* $\epsilon = 0.05$ *(right).*

$$\alpha_{ij}^2 = \exp\Big(\frac{-\lambda_{ij}\xi^2}{\pi}\Big),$$

and $\mathbf{e}_{ij}$ are orthonormal eigenfunctions of Δ with periodic boundary conditions and corresponding eigenvalues $\lambda_{ij} = (2\pi/L)^2(i^2 + j^2)$. It may be shown [13, 7] that such an expansion leads to correlations

$$\mathbf{E}W(t,\mathbf{x})W(t',\mathbf{x}') \approx C(\mathbf{x}-\mathbf{x}')\min\{t,t'\}, \qquad C(\mathbf{x}) = \frac{1}{4\xi^2}\exp\Big(-\frac{\pi}{4}\frac{\|\mathbf{x}\|^2}{\xi^2}\Big).$$

The approximation is good when ξ is small and boundary effects are not important.

The correlation operator of $W(t)$ satisfies Assumption 1.1, as the coefficients α_{ij} are nonzero and decay exponentially.

The numerical simulations presented in the figures were generated by the splitting method described in [13]. The diffusion and noise terms in the excitation equation are approximated over one time step by working in Fourier space, using the above spectral representation of the noise. The reaction terms in both fields are approximated by a forward Euler step. Performing these steps successively provides an approximation to (1.3). We are unaware of any formal convergence analysis of this particular method. Convergence analysis of similar methods for stochastic reaction-diffusion equations are available in [10, 12, 9, 5]. For the smooth type of noise above, this type of analysis can yield convergence in the root mean square sense of an L^2 error of order $\Delta t^{1/2}$, where the time step Δt is fixed to some constant multiple of the grid spacing squared. See also [14] for a discussion of convergence in the weak sense.

We present simulations for the following values: the reaction terms' parameters $a = 0.75$, $b = 0.01$; the domain length $L = 80$; and the correlation length $\xi = 2$. We take homogeneous initial data and vary ϵ and σ to understand their effect on the nucleation of waves. A spatial grid of 256^2 grid points is used with time step 0.02. Two cases are considered: $\epsilon = 0.03$ with $\sigma = 0.09$ and $\epsilon = 0.05$ with $\sigma = 0.125$. It is important to understand that for both values of ϵ the PDE (1.1) supports spiral waves, as indicated in Figure 5.1. Figure 1.1 shows the case where Theorem 1.4 is biting; i.e., ϵ is small enough so that only target waves are nucleated. The figure shows the nucleation of the wave and its development until it destroys itself. Out of 30 nucleation events observed with noise level $\sigma = 0.09$, $\epsilon = 0.03$, a target wave was nucleated each time. Figure 1.2 shows the nucleation and development of a spiral wave. This experiment features a larger value $\epsilon = 0.05$, which yields spiral waves. Spirals are self-sustaining, and the wave does not die out. This is observed in the figure.

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