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WEAK CONVERGENCE OF A NUMERICAL METHOD FOR A STOCHASTIC HEAT EQUATION*

TONY SHARDLOW†

Department of Mathematics, Oxford Road, University of Manchester, Manchester M13 9PL, UK. email: `shardlow@ma.man.ac.uk`

Abstract.

Weak convergence with respect to a space of twice continuously differentiable test functions is established for a discretisation of a heat equation with homogeneous Dirichlet boundary conditions in one dimension, forced by a space-time Brownian motion. The discretisation is based on finite differences in space and time, incorporating a spectral approximation in space to the Brownian motion.

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1 Introduction.

Consider the following stochastic heat equation on $[0,1]$ with homogeneous Dirichlet boundary conditions:

$$du + Au\,dt = dW(t), \quad u(0) = U, \tag{1.1}$$

where the initial data $U \in L_2(0,1)$, $A := -\Delta$, the Laplacian scaled to be positive definite with domain $H^2(0,1) \cap H_0^1(0,1)$, and $W(t)$ is a Wiener process with covariance Q. For simplicity, we suppose that Q has eigenvalues $\alpha_j \geq 0$ corresponding to the eigenfunctions $\mathbf{e}_j := \sqrt{2}\sin(j\pi\cdot)$ of Δ; in other words,

$$W(t) = \sum_{j=1}^{\infty} \alpha_j^{1/2} \mathbf{e}_j \beta_j(t),$$

where $\beta_j(t)$ are independent and identically distributed Brownian motions. Equation (1.1) admits a unique mild solution for initial condition $U \in L_2(0,1)$, namely

$$u(t;U) = e^{-At}U + \int_0^t e^{-A(t-s)}\,dW(s),$$

where e^{-At} is the semigroup with infinitesimal generator $-A$. This theory is developed further in [2].

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We now define a simple discretisation of (1.1) based on the θ method in time and the standard three point approximation to the Laplacian. Consider a time step Δt and a grid size $\Delta x = 1/J$, some $J \in \mathbf{N}$. We construct a numerical approximation as follows. The Wiener process is approximated by truncating its Fourier expansion to $J-1$ terms. Let $\mathcal{P}_{J-1}$ denote the operator taking f to its first $J-1$ modes,

$$\mathcal{P}_{J-1} f = 2 \sum_{j=1}^{J-1} \langle f, \sin(\pi j \cdot) \rangle \sin(j\pi \cdot).$$

($\langle \cdot, \cdot \rangle$ denotes the $L_2(0,1)$ inner product.) Define the approximation to the Wiener process by

$$dB_{\Delta t}(n) := \int_{n\Delta t}^{(n+1)\Delta t} \mathcal{P}_{J-1} dW(s).$$

This gives an $L_2(0,1)$ function. The numerical method evaluates this function at the grid points $j\Delta x$ for $j = 1, \dots, J-1$. The initial condition chosen is $\mathbf{u}_0 = \mathcal{P}_{J-1} U$. Then, for $0 \le \theta \le 1$, we iterate

$$\mathbf{u}_{n+1} - \mathbf{u}_n + \frac{\Delta t}{\Delta x^2} A_\Delta ((1-\theta)\mathbf{u}_n + \theta \mathbf{u}_{n+1}) = \begin{pmatrix} dB_{\Delta t}(n)(\Delta x) \\ \vdots \\ dB_{\Delta t}(n)((J-1)\Delta x) \end{pmatrix} \tag{1.2}$$

where

$$A_\Delta = \begin{pmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}.$$

Let

$$\tilde{u}(n\Delta t; U) = \sum_{j=1}^{J-1} \tilde{u}_j \mathbf{e}_j, \tag{1.3}$$

where $\tilde{u}_j$ are chosen so that $\tilde{u}(n\Delta t; U)(j\Delta x)$ equals the jth component of $\mathbf{u}_n$ for $j = 1, \dots, J-1$.

The above numerical method has been studied in [12] for the problem of space-time white noise (case $Q = I$) in terms of strong convergence. Let $\mathbf{E}$ denote the average with respect to the law of $W(t)$. The conclusion is that for $T, \epsilon > 0$, there exists K_ϵ such that

$$(\mathbf{E}\, \|u(n\Delta t; U) - \tilde{u}(n\Delta t; U)\|^2)^{1/2} \le K_\epsilon \Delta x^{(1-\epsilon)/2} (1 + \|U\|) \Big(1 + \frac{1}{(n\Delta t)^{1-\epsilon}}\Big),$$

$0 < n\Delta t \le T$, as Δt, $\Delta x \to 0$ with $\nu := \Delta t/\Delta x^2$ constrained by $\nu(1-\theta) \le 1/4$ (the norm $\|\cdot\|$ is the standard $L_2(0,1)$ norm) (the stability condition can be relaxed to $\nu(1-2\theta) \le 1/2$ by using Lemma 2.4 of the present paper). Studies of similar methods for more general equations are available in [6, 7, 3, 4, 11] for space-time white noise and for more general noise terms in [8]. The studies indicate that standard approximations to (1.1) of the type described above converge in root mean square with rate $\Delta x^{1+(r-\epsilon)/2}$, where $-2 < r \le 0$ is determined by the following trace condition on the correlation operator

$$\sum_{j=1}^{\infty} \alpha_j j^r < \infty. \tag{1.4}$$

Results of this type have been extended to some nonlinear stochastic PDEs such as the Navier–Stokes equations. See for example, [8, 5].

In this paper, weak convergence is studied for the linear stochastic heat equation (1.1). For the most part, studies of weak convergence of numerical methods for (1.1) have been lacking in the literature, even though it is average properties that are often most interesting. Though we only tackle a linear equation, the technique of proof, the Kolmogorov equation, is used to understand weak convergence for numerical methods of SDEs in generality [13, 10, 9]. To the author's knowledge, this is the first paper to apply the Kolmogorov equation to the analysis of numerical methods for parabolic stochastic PDEs. It is certainly believable that the analysis can be extended to nonlinear PDEs to some extent, see for example an application to a nonlinear delay equation in [1]. The conclusion of our analysis is that subject to the trace condition (1.4) and for smooth initial data the numerical method (1.2) converges with rate $\Delta x^{2+r-\epsilon}$, each $\epsilon > 0$, with respect to the space of twice boundedly continuously differentiable test functions. That is, order $\Delta x^{1-\epsilon}$ for space time white noise (where (1.4) holds with $r < -1$). As in the finite dimensional situation, we observe that the rate of weak convergence is twice that of strong convergence.

THEOREM 1.1. *Let $u(t;U)$ (respectively, $\tilde{u}(t;U)$) denote a solution of* (1.1) *(resp., the trigonometric interpolant of the numerical solution* (1.2) *defined in* (1.3)) *corresponding to initial data $U \in L_2(0,1)$. Suppose that $\sum_{j=1}^{\infty} \alpha_j j^r < \infty$, for some $-2 < r \le 0$. For ϵ, $T > 0$ and a twice continuously boundedly differentiable function $\phi : L_2(0,1) \to \mathbf{R}$, there exists a constant $K > 0$ such that*

$$\left|\mathbf{E}\,\phi(u(n\Delta t;U)) - \mathbf{E}\,\phi(\tilde{u}(n\Delta t;U))\right| \le K\Delta x^{2+r-\epsilon}(1+\|U\|^2), \quad T = n\Delta t$$

as Δt, $\Delta x \to 0$ with $\nu = \Delta t/\Delta x^2$ fixed and $(1-2\theta)\nu < 1/2$.

The proof of this result is given in §3. We do not study the dependence of K on T in this theorem; further information can be gleaned from the proof of Lemma 3.4.

2 Background.

We will work on the space $L_2(0,1)$ with norm $\|f\| := (\int_0^1 f(x)^2 dx)^{1/2}$. Let H^p denote the Sobolev space of $L_2(0,1)$ functions with norm

$$\|u\|_{H^p} := \Big(\sum_{j=1}^{\infty} j^{2p} \langle u, \mathbf{e}_j \rangle^2 \|\mathbf{e}_j\|^2 \Big)^{1/2}.$$

We denote the projection operator from $L_2(0,1)$ to the first J eigenfunctions by $\mathcal{P}_J$; that is, $\mathcal{P}_J u = \sum_{j=1}^{J} \langle u, \mathbf{e}_j \rangle \mathbf{e}_j$. Fractional powers are denoted A^γ so that $A^\gamma u = \sum_{j=1}^{\infty} (j\pi)^{2\gamma} \langle u, \mathbf{e}_j \rangle \mathbf{e}_j$. Throughout the paper, we will make use of a generic constant K, which will be independent of the initial data U and the smoothing parameter k.

Before proving the main result, we develop an abstract framework for our problem and give some basic results useful in the proof. The next Lemma is well known and describes a smoothing property of the heat semigroup:

LEMMA 2.1. *For $\gamma > 0$, there exists a constant $C_\gamma > 0$ with*

$$\|A^\gamma e^{-At} \mathbf{e}_j\| \le C_\gamma t^{-\gamma} \|\mathbf{e}_j\|, \qquad t > 0.$$

The following describes conditions on Q for the boundedness of the solution of (1.1) in H^p.

LEMMA 2.2. *Consider initial data $U \in L_2(0,1)$. Suppose that the eigenvalues α_j of Q obey $\sum_{j=1}^{\infty} j^r \alpha_j < \infty$, for some $r > -2$. For $0 \le p \le (2+r)/2$ and $T > 0$, there exists $K > 0$ with*

$$\Big(\mathbf{E}\, \|u(t;U)\|_{H^p}^2 \Big)^{1/2} \le K \Big[\frac{1}{t^{p/2}} \|U\| + 1 \Big], \quad 0 \le t \le T.$$

PROOF. The solution

$$u(t;U) = e^{-At} U + \int_0^t e^{-A(t-s)}\, dW(s)$$

so that

$$\begin{aligned}
\Big(\mathbf{E}\, \|u(t;U)\|_{H^p}^2 \Big)^{1/2} &\le \frac{K}{t^{p/2}} \|U\| + \Big(\sum_{j=1}^{\infty} \int_0^t j^{2p} e^{-2j^2\pi^2(t-s)} \alpha_j \, ds \Big)^{1/2} \\
&\le \frac{K}{t^{p/2}} \|U\| + \Big(\sum_{j=1}^{\infty} j^{2p} \frac{1 - e^{-2j^2\pi^2 t}}{2j^2\pi^2} \alpha_j \Big)^{1/2} \\
&\le \frac{K}{t^{p/2}} \|U\| + K \Big(\sum_{j=1}^{\infty} j^{2p-2-r} (1 - e^{-2j^2\pi^2 t})(j^r \alpha_j) \Big)^{1/2} \\
&\le \frac{K}{t^{p/2}} \|U\| + K \Big(\sum_{j=1}^{\infty} j^r \alpha_j \Big)^{1/2}
\end{aligned}$$

if $2p - 2 - r \le 0$. □

The Laplacian operator A is unbounded. We will frequently approximate A by a bounded approximation A_k, defined as follows: Let $A_k := \mathcal{P}_k A$; that is, A_k is the operator where $\mathbf{e}_j$ has eigenvalue $j^2\pi^2$ for $j = 1, \dots, k$ and eigenvalue 0 for $j = k+1, \dots$. By use of this approximation, we find strong solutions of an SDE that converge to the mild solutions of (1.1) and that yield to the Itô formula (see [2]).

LEMMA 2.3. *For initial data $U \in L_2(0,1)$, consider the mild solution $u(t;U)$ of*

$$du = Au\,dt + dW(t), \quad u(0) = U,$$

and the strong solution $u^k(t;U)$ of

$$du^k = A_k u^k\,dt + \mathcal{P}_k\,dW(t), \quad u^k(0) = \mathcal{P}_k U. \tag{2.1}$$

Then, for $p \geq 2$,

$$\sup_{0\le t\le T} \mathbf{E}\,\|u(t;U) - u^k(t;U)\|^p \to 0, \qquad \text{as } k \to \infty.$$

PROOF. This is elementary. □

We wish to express the numerical method first as a difference equation on $L_2(0,1)$ and then write down an interpolant of the numerical solution that solves a stochastic evolution equation on $L_2(0,1)$. Consider the following difference equation on $L_2(0,1)$

$$\tilde{\mathbf{u}}_{n+1} - \tilde{\mathbf{u}}_n + \frac{\Delta t}{\Delta x^2}\tilde{A}\Big[(1-\theta)\tilde{\mathbf{u}}_n + \theta\tilde{\mathbf{u}}_{n+1}\Big] = dB_{\Delta t}(n), \quad \tilde{\mathbf{u}}_0 = \mathcal{P}_{J-1}\mathbf{U}.$$

Here $\tilde{A}$ is defined by

$$\tilde{A}\mathbf{e}_j = \lambda_j \mathbf{e}_j, \quad j = 0, 1, \dots$$

where $\lambda_j = \lambda_{j+nJ} = 4\sin^2(j\pi\Delta x/2)$ for $n \in \mathbf{Z}$. The solutions $\tilde{\mathbf{u}}_n$ of this iteration evaluated at $j\Delta x$ for $j = 1, \dots, J-1$ agree with the solution of the numerical method (1.2) (see [12] for further details). This equation can be rearranged to achieve

$$\tilde{\mathbf{u}}_{n+1} = (I - \mathcal{C}\Delta t)\tilde{\mathbf{u}}_n + Q_{\Delta x} dB_{\Delta t}(n)$$

where

$$\mathcal{C} = \frac{1}{\Delta t}\Big(I - \Big[I + \frac{\Delta t}{\Delta x^2}\theta\tilde{A}\Big]^{-1}\Big[I - \frac{\Delta t}{\Delta x^2}(1-\theta)\tilde{A}\Big]\Big), \qquad Q_{\Delta x} = \Big[I + \frac{\Delta t}{\Delta x^2}\theta\tilde{A}\Big]^{-1}.$$

The eigenvalue of $\mathcal{C}$ corresponding to the eigenfunction $\mathbf{e}_j$ is

$$\mu_j := \frac{1}{\Delta t}\Big(1 - \frac{1 - \nu(1-\theta)\lambda_j}{1 + \nu\theta\lambda_j}\Big) = \frac{1}{\Delta t}\frac{\nu\lambda_j}{1 + \nu\theta\lambda_j}, \quad \nu = \Delta t/\Delta x^2.$$

Clearly μ_j are increasing for $j = 1, \dots, J-1$. The corresponding eigenvalue of $Q_{\Delta x}$ is

$$\frac{1}{1+\theta\nu\lambda_j}, \quad j = 1, \dots.$$

The continuous interpolant, which we denote by $\tilde{u}$, is the solution of

$$d\tilde{u}(t,U) = -\mathcal{C}\tilde{u}(\hat{t},U)\,dt + d\tilde{W}(t), \quad \tilde{u}(0;U) = \mathcal{P}_{J-1}U, \tag{2.2}$$

where $\hat{t} = \max\{n\Delta t \colon n\Delta t \le t, n = 0, 1, \dots\}$ and $\tilde{W}(t)$ is a Wiener process on $L_2(0,1)$ with covariance, $\tilde{Q}$, defined by

(2.3)

$$\tilde{Q}\mathbf{e}_j = \tilde{\alpha}_j\mathbf{e}_j, \quad \tilde{\alpha}_j = \frac{\alpha_j}{(1+\theta\nu\lambda_j)^2} \text{ for } j = 1, \dots, J-1, \quad \tilde{\alpha}_j = 0 \text{ for } j = J, \dots.$$

Clearly, $\tilde{\alpha}_j \le \alpha_j$. Note that $\tilde{u}(n\Delta t; U)$ agrees with $\tilde{\mathbf{u}}_n$ and hence with the trigonometric interpolant defined in (1.3).

We describe some important properties of this approximation in the next two Lemmas. The first Lemma deals with the approximations we have made from the Laplacian A and the covariance operator Q:

Lemma 2.4. *Consider $0 \le \theta \le 1$ and $\nu = \Delta t/\Delta x^2$ fixed.*

1. *For $j = 1, \dots, J-1$ and $\gamma > 0$,*

$$\|A^{-\gamma}(Q-\tilde{Q})\mathbf{e}_j\| \le (2\theta\nu\pi^{2(1-\gamma)})\alpha_j j^{2(1-\gamma)}\Delta x^2\|\mathbf{e}_j\|$$

and for $j = J, \dots,$

$$\|A^{-\gamma}(Q-\tilde{Q})\mathbf{e}_j\| \le \frac{2}{\pi^{2\gamma}}\alpha_j j^{-2\gamma}\|\mathbf{e}_j\|.$$

2. *There exists $\sigma > 0$ such that*

$$\pi^2 - \sigma^2\Delta x^2 j^2 \le \mu_j/j^2 \le \pi^2, \quad j = 1, \dots, J-1.$$

Thus, $j^2\pi^2 - \mu_j \le \sigma^2\Delta x^2 j^2$.

3. *If $\nu(1-\theta) < 1/4$, then for all $\Delta t > 0$ we have $0 \le (1-\mu_j\Delta t) \le e^{-\mu_j\Delta t}$. If $\nu(1-2\theta) < 1/2$, there exists $c > 0$ such that $|1-\mu_j\Delta t| \le e^{-c\,j^2\Delta t}$ for $j = 1, \dots, J-1$ as $\Delta t \to 0$ with ν fixed.*

Proof.

1. Notice that

$$1 - \frac{1}{(1+\theta\nu\lambda_j)^2} = \Big(1 - \frac{1}{1+\theta\nu\lambda_j}\Big)\Big(1 + \frac{1}{1+\theta\nu\lambda_j}\Big) \le 2\Big(1 - \frac{1}{1+\theta\nu\lambda_j}\Big).$$

Now for $j = 1, \dots, J-1$, from (2.3)

$$\begin{aligned}\|A^{-\gamma}(Q-\tilde{Q})\mathbf{e}_j\| \leq & \frac{2\alpha_j}{(j^2\pi^2)^\gamma}\Big(1-\frac{1}{1+\theta\nu\lambda_j}\Big) = \frac{2\alpha_j}{j^{2\gamma}\pi^{2\gamma}}\frac{\theta\nu\lambda_j}{1+\theta\nu\lambda_j}\\ =&\frac{2\nu\alpha_j}{j^{2\gamma}\pi^{2\gamma}}\frac{4\sin^2(j\pi\Delta x/2)}{1+\theta\nu\lambda_j}\\ \leq&\frac{2\alpha_j\nu}{j^{2\gamma}\pi^{2\gamma}}4(j\pi\Delta x/2)^2 = 2\alpha_j\nu j^{2(1-\gamma)}\pi^{2(1-\gamma)}\Delta x^2.\end{aligned}$$

A similar argument applies for $j = J, J+1, \dots$.

2. This is Lemma 2.3 of [12].
3. The first part is contained in [12]. For the second part,

$$|1-\mu_j\Delta t| = e^{\ln|1-\mu_j\Delta t|} = e^{(\ln|1-\mu_j\Delta t|/j^2\Delta t)j^2\Delta t}.$$

Now for $\nu(1-2\theta) < 1/2$, $|1-\mu_j\Delta t|$ is uniformly bounded less than one and $j^2\Delta t$ for $j = 1, \dots, J-1$ is uniformly bounded above. Hence, there exist $c > 0$ such that $|1-\mu_j\Delta t| \leq e^{-cj^2\Delta t}$, $j = 1, \dots, J-1$.

□

The next Lemma gives boundedness of the interpolated numerical solution in H^p:

LEMMA 2.5. *Let $\nu := \Delta t/\Delta x^2$ and suppose that $\nu(1-2\theta) < 1/2$. For $p \geq 0$, the numerical interpolant $\tilde{u}(t;U)$ obeys for $t > 0$*

$$\begin{aligned}\mathbf{E}\Big[\|\tilde{u}(t;U)\|^2_{H^p}\Big]^{1/2} &\leq K\frac{1}{t^{p/2}}\|U\| + K(1+\Delta x^{1+r/2-p}),\\ \mathbf{E}\Big[\|\tilde{u}(t;U)-\tilde{u}(\hat{t};U)\|^2_{H^p}\Big]^{1/2} &\leq K\Delta t\frac{1}{t^{p/2}}\|U\| + K\Delta x^{1+\min\{0,r/2-p\}},\end{aligned}$$

for a constant K independent of U, uniformly as $\Delta t, \Delta x \to 0$ with ν fixed.

PROOF. Start with

$$(\mathbf{E}\,\|\tilde{u}(t;U)\|^2_{H^p})^{1/2} \leq (\mathbf{E}\,\|\tilde{u}(t;U)-\tilde{u}(\hat{t};U)\|^2_{H^p})^{1/2} + (\mathbf{E}\,\|\tilde{u}(\hat{t};U)\|^2_{H^p})^{1/2}. \tag{2.4}$$

We estimate the $\tilde{u}(t;U)-\tilde{u}(\hat{t};U)$ term at the end of the proof. There are two terms to estimate in $\tilde{u}(\hat{t};U)$: the part resulting from the initial data: for $t > \Delta t$, we have by Lemma 2.4

$$\|(I-\mathcal{C}\Delta t)^{\hat{t}/\Delta t}U\|^2_{H^p} = \sum_{j=1}^{J-1} j^{2p}(1-\mu_j\Delta t)^{2\hat{t}/\Delta t}U_j^2 \leq \sum_{j=1}^{J-1} j^{2p}e^{-2cj^2\hat{t}}U_j^2.$$

Now, using $1/\hat{t} \leq 2/t$ for $t > \Delta t$ and Lemma 2.1, we have for $p > 0$

$$\|(I-\mathcal{C}\Delta t)^{\hat{t}/\Delta t}U\|^2_{H^p} \leq K\frac{1}{\hat{t}^p}\|U\|^2 \leq K\frac{1}{t^p}\|U\|^2.$$

For $0 \le t < \Delta t$, using $\tilde u(\hat t; U) = U$ and $\Delta t/\Delta x^2$ fixed, the following holds

$$\|U\|_{H^p}^2 = \sum_{j=1}^{J-1} j^{2p} U_j^2 \le K\ J^{2p}\|U\|^2 \le K\Delta t^{-p}\|U\|^2 \le K t^{-p}\|U\|^2.$$

One can show that $-1 < 1-\mu_j\Delta t < 1$ for $\nu(1-2\theta) < 1/2$ and that

$$1-\tfrac{1}{2}\mu_j\Delta t = \frac{1+\nu\sin^2(j\pi\Delta x/2)(4\theta-2)}{1+4\theta\nu\sin^2(j\pi\Delta x/2)} \ge \frac{1-|4\theta-2|\nu}{1+4\theta\nu} =: K_{\nu,\theta} > 0.$$

The second part of the solution $\tilde u(\hat t; U)$ is the stochastic integral

$$\sum_{i=0}^{n-1}\int_{i\Delta t}^{(i+1)\Delta t} (I-\mathcal{C}\Delta t)^i \, d\tilde W(s), \qquad t = n\Delta t.$$

This is bounded in $\mathbf{E}\|\cdot\|_{H^p}^2$ by $\displaystyle\sum_{i=0}^{n-1}\sum_{j=1}^{J-1}\tilde\alpha_j j^{2p}(1-\mu_j\Delta t)^{2i}\Delta t$

(as the eigenvalues $\tilde\alpha_j \le \alpha_j$)

$$\le \sum_{i=0}^{n-1}\sum_{j=1}^{J-1}\alpha_j j^{2p}(1-\mu_j\Delta t)^{2i}\Delta t$$

$$\le \sum_{j=1}^{J-1}\alpha_j j^{2p}\frac{1-(1-\mu_j\Delta t)^{2n}}{1-(1-\mu_j\Delta t)^2}\Delta t$$

(using $1-(1-\mu_j\Delta t)^2 = 2\Delta t\mu_j(1-\frac{1}{2}\mu_j\Delta t)$)

$$\le \sum_{j=1}^{J-1}(\alpha_j j^r)\frac{j^{2p-r}}{2K_{\theta,\nu}\mu_j}.$$

If $Y = \sin^2(j\pi\Delta x/2)/(\Delta x/2)^2$, then it is easy to show $4j^2 \le Y \le j^2\pi^2$ for $j = 1,\dots,J-1$. This gives

$$\mu_j = \frac{Y}{1+\theta\Delta t Y} \ge \frac{4j^2}{1+\theta\Delta t j^2\pi^2} \ge \frac{4j^2}{1+\theta\nu\pi^2}.$$

Hence,

$$\sum_{j=1}^{J-1}(\alpha_j j^r)\frac{j^{2p-r}}{\mu_j} \le K\sum_{j=1}^{J-1}(\alpha_j j^r) j^{2p-r-2},$$

which is uniformly bounded in the limit $\Delta t, \Delta x \to 0$ with $(1-2\theta)\nu < 1/2$ if $2p-r \le 2$ and grows like Δx^{2+r-2p} if $2p-r > 2$. We have shown that

$$\mathbf{E}\Big[\|\tilde u(\hat t; U)\|_{H^p}^2\Big]^{1/2} \le K\frac{1}{t^{p/2}}\|U\| + K(1+\Delta x^{1+r/2-p}), \tag{2.5}$$

To complete the proof, consider

$$\tilde{u}(t;U) - \tilde{u}(\hat{t};U) = (I - \mathcal{C}\Delta t)\tilde{u}(\hat{t};U)(t-\hat{t}) + (\tilde{W}(t) - \tilde{W}(\hat{t})).$$

Then, for $\hat{t} > 0$, as $|1 - \mu_j \Delta t| < 1$

$$\begin{aligned}\left(\mathbf{E}\,\|\tilde{u}(t;U) - \tilde{u}(\hat{t};U)\|_{H^p}^2\right)^{1/2} &\le \left(\mathbf{E}\,\|\tilde{u}(\hat{t};U)\|_{H^p}^2\right)^{1/2}\Delta t + \left[\sum_{j=1}^{J-1}(\alpha_j j^r) j^{2p-r}\Delta t\right]^{1/2} \\ &\le \left(\mathbf{E}\,\|\tilde{u}(\hat{t};U)\|_{H^p}^2\right)^{1/2}\Delta t + K\Delta t^{1/2} J^{\max\{0,p-r/2\}}.\end{aligned}$$

With (2.5) and (2.4), this completes the proof. ☐

3 Proof of Theorem 1.1.

We introduce the Kolmogorov equation for the stochastic evolution equation (1.1). The background theory is developed in Da Prato and Zabczyk [2], where further references are also given.

THEOREM 3.1. *Let $\phi : L_2(0,1) \to \mathbf{R}$ be twice continuously Frechet differentiable with bounded derivatives. The function $v^k(t,X) := \mathbf{E}\,\phi(u^k(t;X))$, where u^k is defined in* (2.1)*, is once differentiable in time and twice differentiable in space and satisfies*

$$v_t^k(t,X) = \tfrac{1}{2}\operatorname{Tr}\left[v_{XX}^k(t,X)Q\mathcal{P}_k\right] - v_X^k(t,X)A_k X.$$

Moreover, the derivatives v_t^k, v_X^k, and v_{XX}^k are uniformly continuous on bounded subsets of $\mathbf{R}^+ \times L_2(0,1)$.

PROOF. The truncation u^k is finite dimensional and so the Kolmogorov equation is simply the usual Kolmogorov equation written on an infinite dimensional space. ☐

PROOF. (of Theorem 1.1) Let $v^k(t,X) := \mathbf{E}\,(\phi(u^k(T-t;X))$ for $t \ge 0$ and $X \in L_2(0,1)$, and

$$\mathcal{L}^k v(t,X) := v_t(t,X) + \tfrac{1}{2}\operatorname{Tr}\left[v_{XX}(t,X)Q\mathcal{P}_k\right] - v_X(t,X)A_k X.$$

After reversing time, Theorem 3.1 states that $\mathcal{L}^k v^k(t,X) = 0$ and that v^k satisfies the hypothesis of Itô's formula. Apply the Itô formula to the approximations $\tilde{u}$ defined in (2.2) and then take averages to get

$$\begin{aligned}&v^k(T,\tilde{u}(T;U)) - v^k(0,\tilde{u}(0;U)) \\ &\quad= \mathbf{E}\bigg[\int_0^T \Big\{ - v_X^k(s,\tilde{u}(s;U))\mathcal{C}\tilde{u}(\hat{s};U) + \tfrac{1}{2}\operatorname{Tr}\left[v_{XX}^k(s,\tilde{u}(s;U))\tilde{Q}\right] \\ &\qquad + v_t^k(s,\tilde{u}(s;U))\Big\}\,ds\bigg]\end{aligned}$$

(subtracting off $0 = \mathcal{L}^k v^k(s,\tilde{u}(s;U))$)

$$= \mathbf{E}\bigg[\int_0^T \tfrac{1}{2}\operatorname{Tr}\Big[v^k_{XX}(s,\tilde u(s;U))\tilde Q\Big] - \tfrac{1}{2}\operatorname{Tr}\Big[v^k_{XX}(s,\tilde u(s;U))Q\mathcal{P}_k\Big] - v^k_X(s,\tilde u(s;U))\mathcal{C}\tilde u(\hat s;U) + v^k_X(s,\tilde u(s;U))A_k\tilde u(s;U)\,ds\bigg].$$

Clearly,

$$\mathbf{E}\,\phi(\tilde u(T;U)) - \mathbf{E}\,\phi(u^k(T;U)) = v^k(T,\tilde u(T;U)) - v^k(0,\tilde u(0;U))$$

and hence

$$\Big|\mathbf{E}\,\phi(\tilde u(T;U)) - \mathbf{E}\,\phi(u^k(T;U))\Big| \le \bigg|\mathbf{E}\int_0^T \tfrac{1}{2}\operatorname{Tr}\Big[v^k_{XX}(s,\tilde u(s;U))(\tilde Q - Q\mathcal{P}_k)\Big] + v^k_X(s,\tilde u(s;U))(A_k-\mathcal{C})\tilde u(\hat s;U) + v^k_X(s,\tilde u(s;U))A_k(\tilde u(s;U)-\tilde u(\hat s;U))\,ds\bigg|. \tag{3.1}$$

Now, we have that

$$v^k_X(s;U) = \mathbf{E}\,\phi'(u^k(T-s;U))u^k_X(T-s;U)$$

and

$$v^k_{XX}(s;U)(\xi_1,\xi_2) = \mathbf{E}\,\phi''(u^k(T-s;U))(u^k_X(T-s;U)\xi_1, u^k_X(T-s;U)\xi_2) + \phi'(u^k(T-s;U))u^k_{XX}(T-s;U)(\xi_1,\xi_2).$$

Because we are working on a linear equation (2.1),

$$u^k_X(s;U)\xi = e^{-A_k s}\xi, \qquad u^k_{XX}(s;U) = 0.$$

Thus,

$$v^k_X(s;U) = \mathbf{E}\,\phi'(u^k(T-s;U))e^{-A_k(T-s)} \tag{3.2}$$

and

$$v^k_{XX}(s;U)(\xi_1,\xi_2) = \mathbf{E}\,\phi''(u^k(T-s;U))(e^{-A_k(T-s)}\xi_1, e^{-A_k(T-s)}\xi_2). \tag{3.3}$$

Consider the first term on the right hand side of (3.1). As ϕ'' is bounded,

$$\begin{aligned}
&\operatorname{Tr}\Big[v^k_{XX}(s,\tilde u(s;U))(\tilde Q - Q\mathcal{P}_k)\Big]\\
&= \sum_{j=1}^{\infty}\Big\langle \mathbf{E}\,\phi''(u^k(T-s;\tilde u(s;U)))e^{-A_k(T-s)}(\tilde Q - Q\mathcal{P}_k)\mathbf{e}_j, e^{-A_k(T-s)}\mathbf{e}_j\Big\rangle\\
&\le K\sum_{j=1}^{\infty}\Big\langle e^{-A_k(T-s)}(\tilde Q - Q\mathcal{P}_k)\mathbf{e}_j, e^{-A_k(T-s)}\mathbf{e}_j\Big\rangle\\
&= K\operatorname{Tr} e^{-2A_k(T-s)}(\tilde Q - Q\mathcal{P}_k).
\end{aligned}$$

Take k large enough that $k > J$. Then, using Lemma 3.3 and the condition $r \le 0$, we have for each $\epsilon > 0$, a K such that

$$\operatorname{Tr}\Big[v^k_{XX}(s,\tilde u(s;U))(\tilde Q - Q\mathcal{P}_k)\Big] \le K\frac{1}{(T-s)^{1-\epsilon}}\Delta x^{2+\min\{0,r-2\epsilon\}} \le K\frac{1}{(T-s)^{1-\epsilon}}\Delta x^{2+r-2\epsilon}. \tag{3.4}$$

Consider the second term in (3.1). Using the boundedness of ϕ' and (3.2),

$$|v^k_X(s,U)(\mathcal{C}-A_k)\tilde u(\hat s;U)| \le K\|e^{-A_k(T-s)}(\mathcal{C}-A_k)\tilde u(\hat s;U)\|.$$

From Lemma 3.2, we see that

$$|v^k_X(s,U)(\mathcal{C}-A_k)\tilde u(\hat s;U)| \le K\frac{1}{(T-s)^{1-\epsilon}}\Delta x^{2-2\epsilon}\cdot\|\tilde u(\hat s;U)\|. \tag{3.5}$$

Then, using Lemma 2.5 with $r > -2$,

$$|\mathbf{E}\,v^k_X(s,U)(\mathcal{C}-A_k)\tilde u(\hat s;U)| \le K\frac{1}{(T-s)^{1-\epsilon}}\Delta x^{2-2\epsilon}(1+\|U\|). \tag{3.6}$$

The integral of the third term in (3.1) is bounded by Lemma 3.4. Integrating the terms (3.4) and, (3.6), and adding to that in Lemma 3.4, we have

$$|\mathbf{E}\,\phi(\tilde u(T;U)) - \mathbf{E}\,\phi(u^k(T;U))| \le K\Delta x^{2+r-2\epsilon}(1+\|U\|^2).$$

The constant K is independent of k and is uniform in the limit $\Delta t, \Delta x \to 0$ with $\Delta t/\Delta x^2 = \nu$ fixed subject to the stability condition $\nu(1-2\theta) < 1/2$. Use the convergence of $u^k \to u$ in the sense of Lemma 2.3 with the continuity of ϕ, to complete the proof. □

Lemma 3.2. *Consider $\nu = \Delta t/\Delta x^2$ fixed. For all $\epsilon > 0$, there exists $K > 0$ such that for $k > J$*

$$\|e^{-sA_k}(\mathcal{C}-A_k)X\| \le \frac{K}{s^{1-\epsilon}}\Delta x^{2-2\epsilon}\|X\|, \qquad s > 0, \quad X \in L_2(0,1).$$

Proof. The eigenvalues of $\mathcal{C}$ are μ_j and of A_k $j^2\pi^2$ with corresponding eigenfunctions $\mathbf{e}_j$. Thus,

$$\begin{aligned}
\|e^{-sA_k}(\mathcal{C}-A_k)X\|^2 &= \sum_{j=1}^{J-1} e^{-2j^2\pi^2 s}X_j^2(j^2\pi^2-\mu_j)^2 \\
\text{(using Lemma 2.4)} \qquad &\le \sum_{j=1}^{J-1} e^{-2j^2\pi^2 s}X_j^2(\sigma^2\Delta x^2 j^2)^2 \\
\text{(using Lemma 2.1)} \qquad &\le K\sum_{j=1}^{J-1} s^{-\gamma}j^{-2\gamma}X_j^2(\sigma^2\Delta x^2 j^2)^2
\end{aligned}$$

$$\begin{aligned}
&\le K \sum_{j=1}^{J-1} s^{-\gamma} j^{-2\gamma+4} X_j^2 \Delta x^4 \\
&\le K s^{-\gamma} \Delta x^4 J^{\max(0,-2\gamma+4)} \|X\|^2 \\
&\le K s^{-\gamma} \Delta x^{\min(4,2\gamma)} \|X\|^2.
\end{aligned}$$

Put $\gamma = 2 - 2\epsilon$; then

$$\|e^{-A_k s}(\mathcal{C} - A_k)X\|^2 \le K s^{-2+2\epsilon} \Delta x^{\min\{4,4-4\epsilon\}} \|X\|^2.$$

□

Lemma 3.3. *Assume the eigenvalues of the correlation operator Q satisfy $\sum_{j=1}^{\infty} j^r \alpha_j < \infty$. Consider $\nu := \Delta t/\Delta x^2$ fixed. For all $\epsilon > 0$, there exists $K > 0$ such that for $k > J$*

$$\operatorname{Tr} e^{-2A_k s}(\tilde{Q} - Q\mathcal{P}_k) \le K \frac{1}{s^{1-\epsilon}} \Delta x^{2+\min(0,r-2\epsilon)}, \quad s > 0.$$

Proof. Let $\gamma > 0$. Then, as $k > J$,

$$\operatorname{Tr} e^{-2A_k s}(Q\mathcal{P}_k - \tilde{Q}) = \sum_{j=1}^{k} (\pi j)^{2\gamma} e^{-2j^2\pi^2 s} \frac{\|A^{-\gamma}(Q\mathcal{P}_k - \tilde{Q})\mathbf{e}_j\|}{\|\mathbf{e}_j\|}$$

(using Lemma 2.4 and Lemma 2.1)

$$\begin{aligned}
&\le K \sum_{j=1}^{J-1} s^{-\gamma} \alpha_j j^{2(1-\gamma)} \Delta x^2 + K \sum_{j=J}^{k} s^{-\gamma} j^{-2\gamma} \alpha_j \\
&\le K \sum_{j=1}^{J-1} s^{-\gamma} (j^r \alpha_j) j^{-r+2(1-\gamma)} \Delta x^2 + K \sum_{j=J}^{k} s^{-\gamma} j^{-2\gamma-r} (j^r \alpha_j) \\
&\le K s^{-\gamma} \left[J^{\max(0,2-2\gamma-r)} \Delta x^2 + J^{-2\gamma-r} \right] \\
&\le K s^{-\gamma} \Delta x^{\min(2,2\gamma+r)}.
\end{aligned}$$

Finally, put $\gamma = 1 - \epsilon$, to complete the proof. □

Lemma 3.4. *Let $\nu := \Delta t/\Delta x^2$ and suppose that $\nu(1-2\theta) < 1/2$. Let $T > 0$, $-2 < r \le 0$, and $U \in L_2(0,1)$. Let $v^k(t,U) = \mathbf{E}\,\phi(u^k(t;U))$ for $0 \le t \le T$ and a function ϕ: $L_2(0,1) \to \mathbf{R}$ with two bounded derivatives. For $\epsilon > 0$, there exists $K > 0$ (independent of U) such that*

$$\int_0^T |\mathbf{E}\, v_X^k(s, \tilde{u}(s;U)) A_k(\tilde{u}(s;U) - \tilde{u}(\hat{s};U))|\, ds \le K(1 + \|U\|^2)\Delta x^{2+r-2\epsilon}.$$

PROOF. Let $\delta = \tilde{u}(s;U) - \tilde{u}(\hat{s};U)$. First note that by (3.2)

$$|\mathbf{E}\, v_X^k(s, \tilde{u}(s;U))A_k\delta| \le K\|e^{-A_k(T-s)}A_k^{1-\epsilon}\| \cdot (\mathbf{E}\,\|A_k^{\epsilon}\delta\|^2)^{1/2}.$$

Using Lemma 2.5 with $p = 2\epsilon$, we have

$$|\mathbf{E}\, v_X^k(s, \tilde{u}(s;U))A_k\delta| \le \frac{K}{(T-s)^{1-\epsilon}}\left[\Delta t \frac{1}{s^{\epsilon}}\|U\| + \Delta x^{1+\min\{0, r/2-2\epsilon\}}\right].$$

This estimate is not enough to complete the proof, and so further investigation is given below. The estimate can be used on the interval $0 \le s < \Delta t$ as an extra Δt is introduced when integrating. Thus to simplify arguments below we assume $s \ge \Delta t$ in the following analysis.

Let $\mathcal{F}_s$ be the σ-algeba generated by $W(s)$ and use the notation $\mathbf{E}\,[\cdot \mid \mathcal{F}_s]$ to denote conditional expectations with respect to $\mathcal{F}_s$. By the intermediate value theorem,

$$\begin{aligned} v_X^k(s, \tilde{u}(s;U))A_k\delta =& v_X^k(s, \tilde{u}(\hat{s};U))A_k\delta + \Big[v_X^k(s, \tilde{u}(s;U)) - v_X^k(s, \tilde{u}(\hat{s};U))\Big]A_k\delta \\ =& v_X^k(s, \tilde{u}(\hat{s};U))A_k\delta + v_{XX}^k(s, Z_s)(\delta, A_k\delta), \end{aligned}$$

where $Z_s := \tilde{u}(\hat{s};U) + z_1\,\delta$, some $0 \le z_1 \le 1$, and is $\mathcal{F}_s$ measurable. Similarly

$$\begin{aligned} \mathbf{E}\Big[v_X^k(s, \tilde{u}(\hat{s};U))A_k\delta \,\Big|\, \mathcal{F}_{\hat{s}}\Big] =& v_X^k(s, \tilde{u}(\hat{s};U))\mathbf{E}\Big[A_k\delta \,\Big|\, \mathcal{F}_{\hat{s}}\Big] \\ =& v_X^k(s, 0)\mathbf{E}\Big[A_k\delta \,\Big|\, \mathcal{F}_{\hat{s}}\Big] \\ &+ v_{XX}^k(s, Z_{\hat{s}})\Big(\tilde{u}(\hat{s};U), \mathbf{E}\Big[A_k\delta \,\Big|\, \mathcal{F}_{\hat{s}}\Big]\Big), \end{aligned}$$

where $Z_{\hat{s}} := z_2\,\tilde{u}(\hat{s};U)$, some $0 \le z_2 \le 1$, and is $\mathcal{F}_{\hat{s}}$ measurable. Then,

$$\begin{aligned} \mathbf{E}\, v_X^k(s, \tilde{u}(s;U))A_k\delta =& v_X(s,0)A_k\mathbf{E}\,\delta + \mathbf{E}\Big[v_{XX}^k(s, Z_{\hat{s}})\Big(\tilde{u}(\hat{s};U), A_k\mathbf{E}\Big[\delta \,\Big|\, \mathcal{F}_{\hat{s}}\Big]\Big)\Big] \\ &+ \mathbf{E}\Big[v_{XX}^k(s, Z_s)\Big(\delta, A_k\delta\Big)\Big]. \end{aligned} \tag{3.7}$$

We deal with the three terms on the right hand side separately. First note that by Lemma 2.4 for $\epsilon > 0$,

$$\|\mathcal{C}(I - \mathcal{C}\Delta t)^{\hat{s}/\Delta t}\mathbf{e}_j\|_{H^{2\epsilon}} = j^{2\epsilon}\mu_j(1 - \mu_j\Delta t)^{\hat{s}/\Delta t} \le j^{2(1+\epsilon)}\pi^2 e^{-cj^2\hat{s}} \le \frac{K}{\hat{s}^{1+\epsilon}}.$$

From (2.2),

$$\mathbf{E}\,\delta = -\,\mathcal{C}\mathbf{E}\,\tilde{u}(\hat{s}, U)(s - \hat{s}) = -\mathcal{C}(I - \mathcal{C}\Delta t)^{\hat{s}/\Delta t}\mathcal{P}_{J-1}U\ (s - \hat{s}).$$

Using (3.2), this gives the following estimate for the first term in (3.7):

$$\Big|\, v_X^k(s,0)A_k\mathbf{E}\,\delta\, \Big| \le K\|A_k^{1-\epsilon}e^{-A_k(T-s)}\| \cdot \|\mathbf{E}\,\delta\|_{H^{2\epsilon}} \le \frac{K}{(T-s)^{1-\epsilon}}\|U\|\frac{\Delta t}{\hat{s}^{1+\epsilon}}.$$

Then integrating,

$$\int_{\Delta t}^{T} \left| v_X^k(s,0) A_k \mathbf{E}\, \delta \right| \; ds \le K \; \|U\| \; \Delta t^{1-\epsilon}.$$

From (2.2)

$$\mathbf{E}\Big[\delta \Big| \mathcal{F}_{\hat{s}}\Big] = \mathbf{E}\Big[\tilde{u}(s;U) - \tilde{u}(\hat{s};U) | \mathcal{F}_{\hat{s}}\Big] = -\mathcal{C}\tilde{u}(\hat{s};U)(s-\hat{s}).$$

Because v_{XX} is symmetric and ϕ'' bounded and by (3.3), we have the following for $p>0$

$$|v_{XX}^k(s,Z)(\xi_1,\xi_2)| \le K \|A_k^{p/2} e^{-A_k(T-s)}\xi_1\| \cdot \|A_k^{-p/2} e^{-A_k(T-s)}\xi_2\|.$$

Estimate the second term in (3.7) as follows: for any $\epsilon > 0$,

$$\begin{aligned} &\Big| \mathbf{E}\Big[v_{XX}^k(s,Z_{\hat{s}}) \Big(\tilde{u}(\hat{s};U), A_k \mathbf{E}\Big[\delta \;\Big|\; \mathcal{F}_{\hat{s}}\Big]\Big)\Big] \Big| \\ &\quad \le K(\mathbf{E}\, \|A_k^{p/2} \tilde{u}(\hat{s};U)\|^2)^{1/2} (\mathbf{E}\, \|e^{-A_k(T-s)} A_k^{1-p/2} \mathcal{C}\tilde{u}(\hat{s};U)\|^2)^{1/2} \; \Delta t \\ &\quad \le K(\mathbf{E}\, \|A_k^{p/2} \tilde{u}(\hat{s};U)\|^2)^{1/2} \|A_k^{1-\epsilon} e^{-A_k(T-s)}\| \cdot (\mathbf{E}\, \|A_k^{1-p/2+\epsilon} \tilde{u}(\hat{s};U)\|^2)^{1/2} \Delta t. \end{aligned}$$

Now, by Lemma 2.5, for $0 \le p \le 1 + r/2$,

$$\begin{aligned} (\mathbf{E}\, \|A_k^{p/2} \tilde{u}(\hat{s};U)\|^2)^{1/2} &\le K\Big[\frac{1}{\hat{s}^{p/2}} \|U\| + 1\Big] \\ (\mathbf{E}\, \|A_k^{1-p/2+\epsilon} \tilde{u}(\hat{s};U)\|^2)^{1/2} &\le K\Big[\frac{1}{\hat{s}^{1-p/2+\epsilon}} \|U\| + \Delta x^{1+r/2-2+p-2\epsilon}\Big]. \end{aligned}$$

Set $p = 1 + r/2$ so that $1 + r/2 - 2 + p - 2\epsilon = r - 2\epsilon$. Then, taking the previous three together,

$$\begin{aligned} &\Big| \mathbf{E}\Big[v_{XX}(s,Z_{\hat{s}}) \Big(\tilde{u}(\hat{s};U), A_k \mathbf{E}\Big[\delta \;\Big|\; \mathcal{F}_{\hat{s}}\Big]\Big)\Big] \Big| \\ &\quad \le \frac{K}{(T-s)^{1-\epsilon}} \Big(\frac{\|U\|}{\hat{s}^{p/2}} + 1 \Big) \Big[\frac{1}{\hat{s}^{1-p/2+\epsilon}} \|U\| + \Delta x^{r-2\epsilon} \Big] \Delta t \\ &\quad \le \frac{K}{(T-s)^{1-\epsilon}} \Big(\frac{\|U\|^2}{\hat{s}^{1+\epsilon}} \Delta t + \|U\| \Delta t \Big(\frac{1}{\hat{s}^{1-p/2+\epsilon}} + \frac{\Delta x^{r-2\epsilon}}{\hat{s}^{p/2}} \Big) + \Delta t \Delta x^{r-2\epsilon} \Big). \end{aligned}$$

As $-2 < r \le 0$ and $0 < p \le 1$, integration yields

$$\int_{\Delta t}^{T} \Big| \mathbf{E}\Big[v_{XX}(s,Z_{\hat{s}}) \Big(\tilde{u}(\hat{s};U), A_k \mathbf{E}\Big[\delta \;\Big|\; \mathcal{F}_{\hat{s}}\Big]\Big)\Big] \Big| \; ds \le K(1 + \|U\|^2) \Delta x^{2+r-2\epsilon}.$$

For the third term in (3.7), by using Lemma 2.5,

$$\begin{aligned} \Big| \mathbf{E}\Big[v_{XX}^k(s,Z_s) \Big(\delta, A_k \delta \Big) \Big] \Big| &\le K(\mathbf{E}\, \|\delta\|^2)^{1/2} \cdot (\mathbf{E}\, \|A_k^{1-\epsilon} e^{-A_k(T-s)} A_k^{\epsilon} \delta\|^2)^{1/2} \\ &\le \frac{K}{(T-s)^{1-\epsilon}} \Big[\Delta t \; \|U\| + \Delta x^{1+\min\{0,r/2\}} \Big] \\ &\quad \cdot \Big[\frac{\Delta t \|U\|}{s^{\epsilon}} + \Delta x^{1+\min\{0,r/2-2\epsilon\}} \Big] \\ &\le \frac{K}{(T-s)^{1-\epsilon}} \frac{1}{s^{\epsilon}} \Delta x^{2+\min\{0,r-2\epsilon\}} (\|U\|^2 + 1). \end{aligned}$$

After integrating and using $r \leq 0$, we conclude that

$$\int_0^T \left| \mathbf{E}\Big[v^k_{XX}(s, Z_s)\Big(\delta, A_k\delta\Big)\Big] \right| \, ds \leq K(1 + \|U\|^2)\Delta x^{2+r-2\epsilon}.$$

This completes the proof. □

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