THE MULTIPLICITY OF BIFURCATIONS FOR AREA-PRESERVING MAPS

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ABSTRACT

The multiplicity of generic bifurcations of periodic orbits of one-parameter families of area-preserving maps is computed. The numbers of bifurcation points (counting multiplicity) are computed at arbitrary period for the case of the Hénon family.

1. *Introduction and statement of results*

For a map f of a space X to itself, the *periodic points* of period q (or a factor) are the solutions of

$$f^q(\vec{x}) = \vec{x}, \quad \vec{x} \in X. \tag{1}$$

If X is a manifold and f is differentiable, then the roots are generically simple, that is, $\det(Df^q_{\vec{x}} - I) \neq 0$, where I is the identity. This is also true restricted to the class of C^∞ area-preserving maps of a surface, which is our special interest.

Given a family of maps f_μ, with parameter $\mu \in \mathbf{R}$, a *bifurcation point* at period q is defined by the equations

$$\begin{aligned} f^q_\mu(\vec{x}) &= \vec{x}, \\ \det(Df^q_{\mu,\vec{x}} - I) &= 0. \end{aligned} \tag{2}$$

Remarkably, for families of area-preserving maps f_μ, roots $(\vec{x}, \mu)$ of (2) are frequently not simple. They can have high multiplicity.

THEOREM 1. *At an elementary real n-furcation of a periodic point of least period p for a C^∞ one-parameter family of area-preserving maps, the multiplicity of the root of the bifurcation equations* (2) *with $q = np$ is $\mathcal{M}_n$, where*

$$\begin{aligned} \mathcal{M}_1 &= 1, \\ \mathcal{M}_2 &= 3, \\ \mathcal{M}_3 &= 8, \\ \mathcal{M}_n &= n^2 + 2, \quad n \geqslant 4. \end{aligned}$$

Elementary n-furcations are defined by Meyer [**10**]. The cases $n = 1$ and 2 are better known as *saddle-centre* and *period-doubling*, respectively. The multiplicity of a root can be quantified by tools from algebraic geometry, which we shall recall in Section 2. Theorem 1 is proved in Section 3.

Received 6 July 1992; revised 22 February 1993.

1991 *Mathematics Subject Classification* 58F22, 58F14, 58F05, 14C17.

Bull. London Math. Soc. 26 (1994) 382–394

Our next result is to compute how multiplicity of a bifurcation at period q appears at periods which are multiples of q.

THEOREM 2. *If a bifurcation point $(\vec{x}, \mu)$ at period q has multiplicity M, and all components of $f_\mu^{kq}(\vec{x}) = \vec{x}$ in $\mathbf{C}^3$ passing through $(\vec{x}, \mu)$ are solutions of $f_\mu^q(\vec{x}) = \vec{x}$, then regarded as a bifurcation point at period kq it also has multiplicity M.*

This is proved in Section 4.

For the area-preserving Hénon family,

$$f_\mu\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \mu - y - x^2 \\ x \end{pmatrix},$$

the periodic orbits, their bifurcations and their multiplicities can be computed explicitly up to period 4. The results are given in Section 5 and Figure 1. For the saddle-centre, period-doubling and 3-furcation, we find the same multiplicities as in Theorem 1, but the 4-furcation has multiplicity 22, rather than the generic result of 18. We conclude that the 4-furcation is non-elementary. This agrees with normal form calculations (Turchetti, private communication). That the 4-furcation is non-elementary can also be seen from the formulae for the period 4 orbits, which do not exhibit the same asymptotic scaling laws near the bifurcation as in the elementary cases.

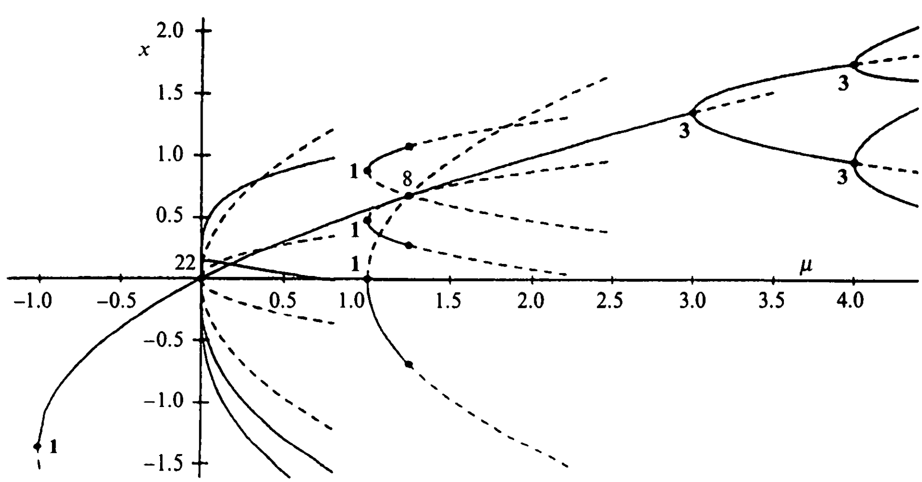

FIG. 1. Bifurcation diagram for the area-preserving Hénon family showing periodic orbits and the multiplicities of bifurcations, up to period 4.

For the area-preserving Hénon family, it is easy to compute the total number of periodic points of period q in $\mathbf{C}^2$ (counting according to multiplicity) at any given parameter value μ, because equation (1) can then be written as a system of q quadratic equations in q unknowns (Section 5). In Section 6 we recall how to use Bézout's theorem (see Section 2), to prove the following.

THEOREM 3. *For any parameter value μ in the Hénon map, the number of solutions of (1) in $\mathbf{C}^2$ is 2^q, counting multiplicity.*

Similarly, it is easy to compute the total number of roots of the bifurcation equation (2) at period q (counting multiplicity) for the Hénon family. In Section 7 we use Bézout's theorem to show that there are $q2^q$ solutions in complex projective space $\mathbf{CP}^3$. In this case, however, half the multiplicity for the solutions is at infinity.

THEOREM 4. *For each $q \in \mathbf{N}$, the total multiplicity in $\mathbf{C}^3$ of the equations for bifurcation at period q in the Hénon family is $q2^{q-1}$.*

This is proved in Section 7.

It follows that in the Hénon family, the bifurcation diagram of Figure 1 exhausts all the multiplicity for $q = 1, 2, 3, 4$. However, at $q = 5$, we know there are two 5-furcations from the fixed point and a period 5 saddle-centre **[9]**. Adding in the saddle-centre of period 1 and assuming that the 5-furcations are elementary, as seems to be the case numerically, this gives a total multiplicity of $2 \times 27 + 5 \times 1 + 1 = 60$, whereas $5 \times 2^4 = 80$. This suggests that there are four more period 5 saddle-centre bifurcations to be found. As already mentioned, the number of points of period 5 (or a factor) is $2^5 = 32$, that is, two fixed point solutions and six period 5 orbits. The known bifurcations at period 5 exhaust these, hence the four unknown saddle-centre bifurcations must occur in pairs; either they turn some pairs of real orbits into complex ones and back again; or they are complex conjugate pairs of purely complex bifurcations, whereby pairs of complex period 5 orbits collide and separate.

One aim of this work is to help decide whether or not the area-preserving Hénon family has any backward bifurcations, that is, bifurcations which generate real orbits as μ decreases. (We do not count the elementary 3-furcation and the case $|\kappa| < 1$ of the elementary 4-furcation as backward bifurcations as they destroy as many real periodic points of the relevant period as they create.) It is known that for $\mu < -1$ there are no real periodic orbits, while for μ large enough all 2^q fixed points of f_μ^q are real and simple **[3]**. Backward bifurcations occur in the dissipative Hénon map (with fixed dissipation); this is proved in **[8]** by considering homoclinic tangencies, but is, alternatively, a simple consequence of cusps in the saddle-node curves in parameter space **[7]**. But they do not appear in the one-dimensional limit; this is a consequence of a theorem of Douady and Hubbard which can be found in **[4]**. We conjecture that there are no backward bifurcations in the area-preserving case either: none have been seen, despite exhaustive searching up to period 20 by Davis **[2]**.

2. *Tools required from algebraic geometry*

Let $\mathscr{G}$ be a set of m polynomials (or forms) $\{G_1, \ldots, G_m\}$ in m (respectively, $m+1$) indeterminates. (Algebraic geometers use the word *form* to mean a homogeneous polynomial.) A *root* of $\mathscr{G}$ in $\mathbf{C}^m$ (respectively, $\mathbf{CP}^m$) is a simultaneous zero of $G_1, \ldots, G_m$. The multiplicity of a root counts the generic number of (complex) roots into which it breaks on perturbation of the equations. To define it formally, we first need the following two concepts (see, for example, **[5, 12]**).

DEFINITION 1. For $P \in \mathbf{C}^m$, the *local ring* $\mathcal{O}_P$ is the set of rational functions which are defined at P, with the natural ring structure.

DEFINITION 2. The *ideal* $(G_1, \ldots, G_m)$ generated over the ring of polynomials $\mathbf{C}[x_1, \ldots, x_m]$ is the set of finite sums $\sum_{i=1}^m F_i G_i$, $F_i \in \mathbf{C}[x_1, \ldots, x_m]$.

DEFINITION 3. The *multiplicity* of a root $P \in \mathbf{C}^m$ of $\mathscr{G}$ is

$$\mathscr{M}(\mathscr{G}, P) = \dim_{\mathbf{C}} \mathscr{O}_P/(G_1, \ldots, G_m). \tag{3}$$

All these definitions extend easily to the projective case, Definition 3 by use of affine coverings.

Note that the multiplicity could equally well be defined using convergent power series expansions instead of rational functions [6], which is more appropriate to our context.

To calculate the multiplicity of a root, it is simplest first to shift the root to the origin in affine space, and then to make use of the following properties of its multiplicity, which we just denote by $\mathscr{M}(\mathscr{G})$. We have expressed them as in [5], though only the case $m = 2$ is treated there. The general case is treated in [12], but we found the presentation less easy to understand.

First we need one more definition.

DEFINITION 4. The *tangent form* of a polynomial G at zero is its terms of lowest degree, and its degree is denoted by $m(G)$.

PROPERTY 1.

$$\mathscr{M}(\mathscr{G}) \geqslant \prod_{G \in \mathscr{G}} m(G),$$

with equality if zero is an isolated root of the set of tangent forms (we then say that they are *independent*).

PROPERTY 2. If $G_i = \prod_{j=1}^{J_i} G_{i,j}^{r_{i,j}}$, then

$$\mathscr{M}(\mathscr{G}) = \sum r_{1,\sigma(1)} \cdots r_{m,\sigma(m)} \mathscr{M}\{G_{1,\sigma(1)}, \ldots, G_{m,\sigma(m)}\},$$

where the sum runs over all maps $\sigma\colon \{1, \ldots, m\} \to \mathbf{N}$ with $\sigma(i) \leqslant J_i$.

PROPERTY 3. For all $A \in (G_2, \ldots, G_m)$,

$$\mathscr{M}(\mathscr{G}) = \mathscr{M}\{G_1 + A, G_2, \ldots, G_m\}.$$

Properties 1, 2 and 3 lead to an algorithm for computing the multiplicity of a root at zero, by reduction of $\mathscr{G}$ to a set of polynomials with independent tangent forms. Although the theory of multiplicity of roots is best developed in the context of polynomials, the definition extends to sufficiently smooth non-polynomial functions, by using their Taylor expansions up to the degree necessary to break degeneracy.

The remaining result needed from algebraic geometry is Bézout's theorem. A proof can be found for the case $m = 2$ in [5] and for the general case in [12].

BÉZOUT'S THEOREM. *Let $G_1, \ldots, G_m$ be polynomials of degrees $d_1, \ldots, d_m$ respectively in $\mathbf{CP}^m$. If the roots are isolated, then*

$$\sum_{\text{roots } P} \mathscr{M}(\mathscr{G}, P) = \prod_{i=1}^{m} d_i. \tag{4}$$

To apply this, we need a criterion for isolated roots. If $m = 2$, it is enough to show that G_1, G_2 do not have a common factor. In general, we need the following.

PROPOSITION [5]. *The set $V \subset \mathbf{C}^m$ of roots is finite if and only if the coordinate ring*

$$\mathbf{C}[V] = \mathbf{C}[x_1, \ldots, x_m]/(G_1, \ldots, G_m)$$

is finite-dimensional over $\mathbf{C}$.

This criterion could also be used to test for independence of the tangent forms in Property 1, though in every case that we deal with, their independence is clear by inspection.

3. *The multiplicities of generic bifurcations*

Proof of Theorem 1. At an n-furcation, the eigenvalues of a fixed point pass through $e^{\pm 2\pi i p/n}$, for some p coprime to n (except $p = 0$ in the case $n = 1$). Instead of using Meyer's normal forms for area-preserving maps near elementary n-furcations, we use Takens' normal forms. These are autonomous polynomial Hamiltonian flows for which there exists a (parameter-dependent) C^∞ coordinate change C such that the Taylor expansions of $C^{-1} f_\mu C$ and $\phi_1 \circ R_{p/n}$ agree exactly, where ϕ_1 is the time-1 map of the Hamiltonian flow and $R_{p/n}$ is rotation by $2\pi p/n$ about the origin [14] (also referred to in [1]). Actually, for $n \geqslant 3$, we rotate Takens' normal forms by π/n to make the analysis easier. The fixed points of f_μ^n then correspond to critical points of a family of Hamiltonians $H(x, y; \varepsilon)$. The problem reduces to determining the multiplicity of the solution $x = y = \varepsilon = 0$ for the bifurcation equations for critical points of H, namely

$$H_x = H_y = J = 0,$$

where

$$J = \det D^2 H.$$

To prove the theorem, we use Properties 1, 2 and 3 above to compute the multiplicity of $\{H_x, H_y, J\}$ at the origin for Takens' normal forms. Afterwards, we shall argue that the results do not change when we consider, instead, f_μ^n.

Case $n = 1$.

$$\begin{aligned} H(x, y; \varepsilon) &= y^2/2 - x^3/3 - \varepsilon x, \\ H_x &= -x^2 - \varepsilon, \\ H_y &= y, \\ J &= -2x. \end{aligned}$$

Property 1 gives $\mathcal{M}_1 = 1$.

Case $n = 2$.

$$\begin{aligned} H(x, y; \varepsilon) &= y^2/2 \pm x^4/4 + \varepsilon(x^2 + y^2), \\ H_x &= x(\pm x^2 + 2\varepsilon), \\ H_y &= y(1 + 2\varepsilon), \\ J &= (\pm 3x^2 + 2\varepsilon)(1 + 2\varepsilon); \\ \mathcal{M}_2 &= \mathcal{M}\{x, y, \pm 3x^2 + 2\varepsilon\} + \mathcal{M}\{\pm x^2 + 2\varepsilon, y, \pm 3x^2 + 2\varepsilon\} \\ &= 1 + \mathcal{M}\{2\varepsilon \pm x^2, y, \pm 2x^2\}, \end{aligned}$$

using, in turn, Properties 2, 1 and 3. Using Property 1 again, we obtain $\mathcal{M}_2 = 3$.

Case $n = 3$.

$$H(x,y;\varepsilon) = (x^3-3xy^2)+\varepsilon(x^2+y^2).$$

It is easy to check that the polynomials H_x, H_y, J are independent quadratic forms, and thus $\mathscr{M}_3 = 8$ (Property 1).

Case $n = 4$.

$$H(x,y;\varepsilon) = x^4-6x^2y^2+y^4+\kappa(x^2+y^2)^2+\varepsilon(x^2+y^2),$$

with $\kappa \neq \pm 1$;

$$\begin{aligned} H_x &= x[(4+4\kappa)\,x^2+(4\kappa-12)\,y^2+2\varepsilon],\\ H_y &= y[(4\kappa-12)\,x^2+(4+4\kappa)\,y^2+2\varepsilon],\\ J &= \begin{vmatrix} x^2 12(1+\kappa)+4y^2(\kappa-3)+2\varepsilon & 8(\kappa-3)\,xy \\ 8(\kappa-3)\,xy & 4x^2(\kappa-3)+12y^2(\kappa+1)+2\varepsilon \end{vmatrix}. \end{aligned}$$

It is then easy to see that

$$\mathscr{M}\{x,y,J\} = 2, \tag{5}$$

because we are allowed to ignore all terms with an x factor, thus factorising J. Similarly,

$$\begin{aligned} &\mathscr{M}\{x,(4\kappa-12)\,x^2+(4+4\kappa)\,y^2+2\varepsilon,J\}\\ &= \mathscr{M}\{x,(4+4\kappa)\,y^2+2\varepsilon,(4\kappa-12)\,y^2+2\varepsilon\}+\mathscr{M}\{x,(4+4\kappa)\,y^2+2\varepsilon,(12\kappa+12)\,y^2+2\varepsilon\}\\ &= \mathscr{M}\{x,4(1+\kappa)\,y^2+2\varepsilon,\,-16y^2\}+\mathscr{M}\{x,4(1+\kappa)\,y^2+2\varepsilon,8(1+\kappa)\,y^2\} = 4 \end{aligned}$$

provided $\kappa \neq -1$ (if $\kappa = -1$, then $\mathscr{M} = \infty$). By interchanging x and y, we obtain the same result for $\mathscr{M}\{(4+4\kappa)\,x^2+(4\kappa-12)\,y^2+2\varepsilon,y,J\}$.

To find the multiplicity of the final pair of factors, we work with the equivalent set x^2-y^2, $(4\kappa-12)\,x^2+(4+4\kappa)\,y^2+2\varepsilon$ and

$$J' = (12\kappa x^2+4\kappa y^2+2\varepsilon)(4\kappa x^2+12\kappa y^2+2\varepsilon)-(8\kappa-24)^2x^2y^2.$$

We eliminate the ε term from J' according to Property 3:

$$\begin{aligned} J'' &= (12\kappa x^2+4\kappa y^2)(4\kappa x^2+12\kappa y^2)-(8\kappa-24)^2x^2y^2\\ &\quad +[(4\kappa-12)\,x^2+(4+4\kappa)\,y^2][(4\kappa-12)\,x^2+(4+4\kappa)\,y^2\\ &\quad -4\kappa x^2-12\kappa y^2-12\kappa x^2-4\kappa y^2]\\ &= (144+96\kappa)\,x^4+(448\kappa-672)\,x^2y^2+(16-32\kappa)\,y^4. \end{aligned}$$

Then

$$\mathscr{M}\{x^2-y^2,(4\kappa-12)\,x^2+(4+4\kappa)\,y^2+2\varepsilon,J''\} = 8,$$

if $\kappa \neq 1$. (By substituting $x^2 = y^2$ in J'', we see that $\kappa = 1$ is the unique value for which x^2-y^2 divides J'', and $\mathscr{M}$ is then infinite.)

Finally, by Property 2:

$$\mathscr{M}_4 = 2+4+4+8 = 18.$$

Case $n \geqslant 5$.

$$H(x, y; \varepsilon) = \sum \binom{n}{p} x^{n-p}(iy)^p + (x^2+y^2)^2 + (x^2+y^2)\,\varepsilon,$$

where the sum runs over even p from 0 to n. Suppose n is even (the odd case is similar). Then

$$\begin{aligned} H_x &= \sum \binom{n}{p} (n-p)\, x^{n-p-1}(iy)^p + 4x(x^2+y^2) + 2x\varepsilon \\ &= x(A + 4(x^2+y^2) + 2\varepsilon), \end{aligned}$$

$$\begin{aligned} H_y &= B + 4y(x^2+y^2) + 2y\varepsilon, \\ H_{xx} &= C + 4(3x^2+y^2) + 2\varepsilon, \\ H_{xy} &= D + 8xy, \\ H_{yy} &= E + 4(x^2+3y^2) + 2\varepsilon, \end{aligned}$$

where A, C, D, E are forms in x, y of degree $n-2$, and B is a form in x, y of degree $n-1$. The form D has a factor of x, and thus we see that

$$\begin{aligned} \mathscr{M}\{x, H_y, J\} &= \mathscr{M}\{x, y(-n(iy)^{n-2} + 4y^2 + 2\varepsilon), (C + 4y^2 + 2\varepsilon)(E + 12y^2 + 2\varepsilon)\} \\ &= 1 + 1 + (n-2) + 2 \\ &= n + 2. \end{aligned}$$

To complete the calculation, we manipulate J according to Property 3 to eliminate ε:

$$\begin{aligned} J' &= CE + 16(3x^2+y^2)(x^2+3y^2) + 4(x^2+3y^2)\,E \\ &\quad + 4(3x^2+y^2)\,C - D^2 - 16xyD - 64x^2y^2 \\ &\quad + [A + 4(x^2+y^2)][A + 4(x^2+y^2) - (C+E) - 4(4x^2+4y^2)] \\ &= J_n + A^2, \end{aligned}$$

where

$$J_n = y^2E + x^2C - 2xyD - (x^2+y^2)\,A. \tag{6}$$

Using Property 3 again,

$$\mathscr{M}\{A + 4(x^2+y^2) + 2\varepsilon, H_y, J'\} = \mathscr{M}\{A + 4(x^2+y^2) + 2\varepsilon, B - Ay, J_n + A^2\}.$$

Now J_n is a form of degree n in x and y, and A^2 is a form of degree $2(n-2) > n$ in x and y. The following calculation shows that $B - Ay$ and J_n are independent. Write $z = x + iy$ and $\bar{z} = x - iy$ (though note that $\bar{z}$ is not necessarily the complex conjugate of z, as x and y may be complex). Then we may write

$$A = \frac{n}{2x}(z^{n-1} + \bar{z}^{n-1}), \tag{7}$$

$$B = \frac{in}{2}(z^{n-1} - \bar{z}^{n-1}). \tag{8}$$

Then

$$B - Ay = \frac{in}{2x}(z^n - \bar{z}^n). \tag{9}$$

If $B-Ay = 0$, then $z/\bar{z}$ must be an nth root of unity $e^{i\theta}$, $\theta = 2\pi j/n$ for some integer j. It follows that $y/x = \tan(\theta/2)$. Thus we can write

$$x = \alpha\cos(\theta/2), \quad y = \alpha\sin(\theta/2), \quad z = \alpha e^{i\theta/2}, \quad \bar{z} = \alpha e^{-i\theta/2} \tag{10}$$

for some $\alpha \in \mathbf{C}$. Now J_n can be written as

$$J_n = \frac{n(n-1)}{2}(z^{n-2}\bar{z}^2 + \bar{z}^{n-2}z^2) - \frac{nz\bar{z}}{2x}(z^{n-1}+\bar{z}^{n-1}). \tag{11}$$

Substituting equation (10), we obtain

$$J_n = \pm\alpha^n n((n-1)\cos 2\theta - 1), \tag{12}$$

the sign depending on the parity of j. Now for $n \geqslant 4$, $\cos(4\pi j/n)$ never takes the value $1/(n-1)$ (see the Appendix). Hence $J_n \neq 0$, except for the trivial case $\alpha = 0$. Thus $A-By$ and J_n are independent. We conclude that

$$\begin{aligned}\mathcal{M}_n &= \mathcal{M}\{x, H_y, J\} + \mathcal{M}\{A+4(x^2+y^2)+2\varepsilon, H_y, J'\}\\ &= (n+2)+(n-1)n\\ &= n^2+2.\end{aligned}$$

This completes the calculations of the multiplicities. To conclude the proof we must show that taking the time-n map of Takens' normal forms, applying a C^∞ coordinate change, and adding flat remainder terms does not change the results. The equations for bifurcation of fixed points of the time-n map of a Hamiltonian flow are identical to those for its critical points, provided the flow has no periodic orbits of period n (or a factor). Near enough to the origin, however, it is not hard to show that the above normal forms have no periodic orbits of period less than or equal to n, so this problem does not arise. C^∞ coordinate change does not change intersection numbers. Lastly, the addition of terms of high enough degree to independent tangent forms does not affect their intersection number, unless it is infinite. Thus the elementary n-furcations have the same multiplicities as computed above for Takens' normal forms.

4. *Multiplicity of non-primitive bifurcations*

Proof of Theorem 2. Recall that the multiplicity of a root at zero is

$$\mathcal{M}(\mathcal{G}) = \dim_{\mathbf{C}} \mathcal{O}/(G_1, \ldots, G_m),$$

where $\mathcal{O}$ is the local ring of rational functions defined at the origin.

Let

$$\begin{aligned}K &= (f_\mu^q(\vec{x}) - \vec{x}, \det|Df_{\mu,\vec{x}}^q - I|),\\ L &= (f_\mu^{kq}(\vec{x}) - \vec{x}, \det|Df_{\mu,\vec{x}}^{kq} - I|).\end{aligned}$$

It suffices to show that these ideals are equal in $\mathcal{O}$, because then $\mathcal{O}/K \cong \mathcal{O}/L$ by a linear isomorphism.

Modulo K,

$$L = \left(\det\left|\prod_{i=1}^{k} Df_\mu^q \circ f_\mu^{qi}(\vec{x}) - I\right|\right) = \det|(Df_{\mu,\vec{x}}^q)^k - I|) = 0$$

and we see $L \subset K$. To show $K \subset L$, similar arguments work when

$$(f_\mu^q(\vec{x}) - \vec{x}) = (f_\mu^{kq}(\vec{x}) - \vec{x}).$$

This is true in $\mathcal{O}$ if we assume there are no components of least period kq (or a factor strictly bigger than q) at the origin, because each solution of $f_\mu^q(\vec{x}) = \vec{x}$ is generically simple and a solution of $f_\mu^{kq}(\vec{x}) = \vec{x}$.

5. *Periods* 1, 2, 3 *and* 4 *in the area-preserving Hénon family*

We give the formulae for the periodic orbits for periods $q = 1,2,3,4$; they are easily derived using the equivalent form of the equations

$$\begin{gathered} x_i^2 + x_{i-1} + x_{i+1} - \mu = 0, \quad i = 1,\dots,q, \\ x_0 = x_q, \quad x_{q+1} = x_1, \end{gathered} \tag{13}$$

and using time-reversal symmetry.

$$\begin{aligned} q = 1:&\quad x_1 = (-1 \pm \sqrt{(1+\mu)}), \\ q = 2:&\quad (x_1, x_2) = (1 + \sqrt{(\mu-3)}, 1 - \sqrt{(\mu-3)}), \\ q = 3:&\quad (x_1, x_2, x_3) = (\pm \sqrt{(\mu-1)}, 1 \mp \sqrt{(\mu-1)}, \pm \sqrt{(\mu-1)}), \\ q = 4:&\quad (x_1, x_2, x_3, x_4) = (\sqrt{\mu}, \sqrt{\mu}, -\sqrt{\mu}, -\sqrt{\mu}), \\ &\quad (\pm \sqrt{\mu}, \sqrt{(\mu \mp 2\sqrt{\mu})}, \pm \sqrt{\mu}, -\sqrt{(\mu \mp 2\sqrt{\mu})}). \end{aligned}$$

Note, in particular, that the periodic points of periods 1, 2, 3 and 4 are isolated, for each fixed parameter value μ.

The bifurcations and their multiplicities up to period 4 are indicated on Figure 1, and again are easily derived from equation (13) plus the bifurcation equation

$$\begin{vmatrix} 2x_1 & 1 & & & 1 \\ 1 & 2x_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & 1 & 2x_q \end{vmatrix} = 0, \quad q > 2,$$

$$\begin{vmatrix} 2x_1 & 2 \\ 2 & 2x_2 \end{vmatrix} = 0, \quad q = 2,$$

$$2x_1 + 2 = 0, \quad q = 1.$$

Note, in particular, that the bifurcation points at periods 1, 2, 3 and 4 are isolated.

6. *The number of periodic points of period q in the Hénon map*

The result given in Theorem 3 is well-known: a proof is sketched by Simo **[13]**, for example. Another way to prove it would be along the lines of **[11]**, where Moser demonstrates a general result for even periods for polynomial area-preserving maps with polynomial inverse of the same degree, subject to a non-degeneracy condition, which one could check for the Hénon map. We give here a proof closer to that of Simo, whose advantage is that it will generalise easily to count the number of bifurcation points, which we are not aware that anyone has done before.

Proof of Theorem 3. We have done the easy cases $q = 1$ and 2 in Section 5, and so may fix $q > 2$, $\mu \in \mathbf{C}$ and consider the ideal

$$N = (x_1^2 + x_q + x_2 - \mu, \dots, x_q^2 + x_{q-1} + x_1 - \mu).$$

Let V be the set of roots, so that V is in one–one correspondence with the set of points of period q or a factor. Let D_m be the set of elements in the coordinate ring

$$\mathbf{C}[V] = \mathbf{C}[x_1, \ldots, x_q]/N$$

which have representatives in $\mathbf{C}[x_1, \ldots, x_q]$ of degree m or less. We first show $D_m \subset D_q$ for $m \geqslant q$. An inductive argument proves this, if we can show each monomial in $\mathbf{C}[x_1, \ldots, x_q]$ of degree greater than q is equivalent to a polynomial of smaller degree. This is elementary, as such terms must have a square factor and, for example, modulo N

$$x_1^2 = \mu - x_2 - x_q.$$

Then $\mathbf{C}[V] = D_q$; but D_q is spanned by residues of monomials of degree less than or equal to q, thus $\mathbf{C}[V]$ is a finite-dimensional vector space, and we may apply the proposition of Section 2 to $\mathbf{C}[V]$, to deduce that V is finite.

But then Bézout's theorem applied to the polynomials generating N, homogenised to extend to $\mathbf{CP}^q$, namely

$$x_{i-1} Z + x_i^2 + x_{i+1} Z - \mu Z^2,$$

with $x_q = x_0$, gives total multiplicity 2^q in $\mathbf{CP}^q$. None of these are at infinity ($Z = 0$), because the equations would then reduce to

$$x_1^2 = \ldots = x_q^2 = 0, \tag{14}$$

which have only the zero solution, which is not permitted in $\mathbf{CP}^q$.

7. *Total multiplicity for bifurcations in the Hénon family*

Proof of Theorem 4. Write the general homogeneous equations for a period q bifurcation in the Hénon family as

$$x_{i-1} Z + x_i^2 + x_{i+1} Z - \mu Z = 0, \quad i = 1, \ldots, q, \tag{15}$$

$$\begin{vmatrix} 2x_1 & Z & & & Z \\ Z & 2x_2 & Z & & \\ & \ddots & \ddots & \ddots & \\ & & & & Z \\ Z & & & Z & 2x_q \end{vmatrix} = 0, \quad q > 2, \tag{16}$$

$$\begin{vmatrix} 2x_1 & 2Z \\ 2Z & 2x_2 \end{vmatrix} = 0, \quad q = 2,$$

$$2x_1 + 2Z = 0, \quad q = 1.$$

If all the roots are isolated, then Bézout's theorem tells us that the total number in $\mathbf{CP}^{q+1}$ (counting multiplicity) is $q2^q$.

These equations have a unique solution $(x_1, \ldots, \mu, Z) = (0, \ldots, 1, 0)$ at projective infinity ($Z = 0$). Moving the solution to the origin of $\mathbf{C}^{q+1}$, by fixing $\mu = 1$ and using coordinates $(x_1, \ldots, x_q, Z)$, leaves (16) unchanged, but (15) becomes

$$x_{i-1} Z + x_i^2 + x_{i+1} Z - Z = 0, \quad i = 1, \ldots, q.$$

Apply Property 3, and subtract the first equation from the others, to obtain the set

$$x_q Z + x_1^2 + x_2 Z - Z = 0, \tag{17}$$

$$x_i^2 - x_1^2 + x_{i-1} Z - x_q Z + x_{i+1} Z - x_2 Z = 0, \quad 2 \leqslant i \leqslant q. \tag{18}$$

Then we have $q+1$ equations (16), (17) and (18) with minimal degrees q, 1 and 2 respectively, in $(x_1, \ldots, x_q, Z)$. They satisfy the case of equality of Property 1, hence there are $q2^{q-1}$ bifurcations at infinity.

Lastly, we have to prove that all the bifurcations at period q are isolated. We did this already for $q = 1, 2$ in Section 5. So fix $q > 2$, and define $\tilde{N}$ to be the ideal generated in $\mathbf{C}[x_1, \ldots, x_q, \mu]$ by the $q+1$ polynomials in (15), (16) above with $Z = 1$ (that is, we consider the bifurcations away from projective infinity), and $\tilde{V}$ to be the set of roots. We saw there was a single bifurcation at infinity, meaning it is sufficient to show that $\tilde{V}$, the set of affine bifurcations, is finite.

We first show that μ satisfies a monic polynomial over $\mathbf{C}[x_1, \ldots, x_q]$ of degree q. We start from equation (16). By the product rule for determinants (throughout we calculate modulo $\tilde{N}$),

$$0 = \left|\begin{pmatrix} x_1/2 & x_2 & & & x_q \\ x_1 & x_2/2 & x_3 & & \\ & \ddots & \ddots & \ddots & \\ & & & & x_q \\ x_1 & & & x_{q-1} & x_q/2 \end{pmatrix}\begin{pmatrix} 2x_1 & 1 & & & 1 \\ 1 & 2x_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & & & 1 \\ 1 & & & 1 & 2x_q \end{pmatrix}\right|$$

$$= \begin{vmatrix} \mu & \Lambda_1 & x_2 & & x_q & \Lambda_2 \\ \Lambda_3 & \mu & \Lambda_4 & x_3 & & x_1 \\ x_2 & \Lambda_5 & \ddots & \ddots & \ddots & \\ & \ddots & \ddots & & & \\ x_q & & & & & \\ \Lambda_{2q-1} & x_1 & & & \Lambda_{2q} & \mu \end{vmatrix},$$

where each Λ_i is linear, for example, $\Lambda_1 = x_1/2 + 2x_2^2 = x_1/2 + 2(\mu - x_1 - x_3)$. Hence, as each entry is linear or zero, we have

$$A_q \mu^q + \sum_{i=1}^{q-1} \mu^i h_i = 0, \tag{19}$$

where $h_i \in \mathbf{C}[x_1, \ldots, x_q]$, $\deg h_i = q - i$, and

$$A_q = \begin{vmatrix} 1 & 2 & & & 2 \\ 2 & 1 & 2 & & \\ & \ddots & \ddots & \ddots & \\ & & 2 & 1 & 2 \\ 2 & & & 2 & 1 \end{vmatrix}.$$

Now $A_q \neq 0$. One way to see this is that the spectrum of the matrix is

$$\{1 + 4\cos 2\pi k/q : k = 1, \ldots, q\},$$

so for the determinant to be non-zero it is sufficient that $\cos(2\pi k/q)$ never take the value $-\frac{1}{4}$. This is proved in the Appendix.

Now consider $\tilde{D}_m$, the elements of the coordinate ring

$$\mathbf{C}[\tilde{V}] = \mathbf{C}[x_1, \ldots, x_q, \mu]/\tilde{N}$$

which have representatives in $\mathbf{C}[x_1, \ldots, x_q, \mu]$ of degree m or less. This time we show that the coordinate ring lives in $\tilde{D}_{2q-1}$. Again, we need consider only monomials. If

Q is a monomial of degree larger than $2q-1$, then it must contain at least a factor x_i^2, some i, or a factor μ^q. In the first case, we can reduce the degree modulo $\tilde{N}$ as in Section 6. In the second case, we use equation (19) to reduce the degree. Hence for $m \geqslant 2q$, $\tilde{D}_{m-1} \subset \tilde{D}_m$. Consequently, $\mathbf{C}[\tilde{V}] = \tilde{D}_{2q-1}$, a finite-dimensional vector space, and we may apply the Proposition as before.

Appendix

LEMMA. *The only rational values that* $\cos 2\pi p/q$ *takes for rational* p/q *are* ± 1, $\pm\frac{1}{2}$ *and* 0.

Proof. Proofs can be given using algebraic number theory or Galois theory, or by examining recursion relations. We give an algebraic number theory proof. Write

$$c = \cos 2\pi p/q$$

and let

$$\lambda = e^{2\pi i p/q}$$

be the corresponding point on the unit circle. Then λ and its complex conjugate $\bar{\lambda}$ satisfy $\lambda^q = 1$, which is a monic polynomial over the integers. Thus λ and $\bar{\lambda}$ are 'algebraic integers'. The algebraic integers form a ring, thus in particular

$$2c = \lambda + \bar{\lambda}$$

is an algebraic integer. But the only rational algebraic integers are the integers, and $|c| \leqslant 1$, therefore the only rational values that c can take are ± 1, $\pm\frac{1}{2}$ and 0.

The Galois theory proof is based on the fact that λ also satisfies

$$\lambda^2 - 2c\lambda + 1 = 0, \tag{20}$$

which can be written as a quadratic polynomial over the integers if c is rational, and an incompatibility of this with $\lambda^q = 1$, if $q \neq 1, 2, 3, 4$ or 6.

The recursion relation proof is based on assuming c to be rational (not equal to ± 1, $\pm\frac{1}{2}$ or 0), and using equation (20) to express λ^q in the form $a_q\lambda + b_q$, with a_q, b_q rational, and showing that a_q can never be zero, so that the imaginary part of λ^q is never zero.

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