Domain Decomposition and Upscaling

Robert Scheichl
(joint work with Ivan Graham, Clemens Pechstein, Eero Vainikko, Jan Van lent)

In this talk we discuss the use of domain decomposition parallel iterative solvers for highly heterogeneous problems of flow in porous media, in both the deterministic and (Monte-Carlo simulated) stochastic cases. We are particularly interested in the case of highly unstructured coefficient variation where standard periodic or stochastic homogenisation theory is not applicable, and where there is no a priori scale separation. We will restrict attention to the important model elliptic problem

$$-\nabla \cdot (K \nabla u) \;=\; f \;, \tag{1}$$

in a bounded polygonal or polyhedral domain $\Omega \subset \mathbb{R}^d$, $d = 2, 3$, with suitable boundary data on the boundary $\partial\Omega$. The $d \times d$ coefficient tensor $K(x)$ is assumed symmetric positive definite, but may vary over many orders of magnitude in an unstructured way on Ω. Many examples arise in groundwater flow and oil reservoir modelling, e.g. in the context of the SPE10 benchmark or in Monte Carlo simulations of stochastic models for strong heteoregeneities [3] (see Figure 1).

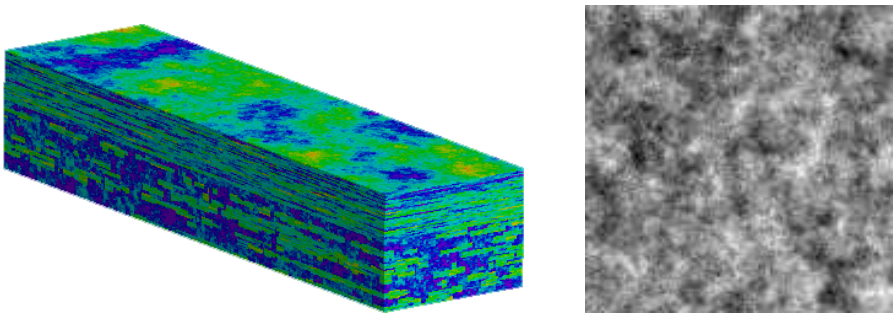

Figure 1. Typical coefficients: Society of Petroleum Engineer benchmark SPE10 (left); lognormal random field (right).

Let $\mathcal{T}^h$ be a conforming shape-regular simplicial mesh on Ω and let $\mathcal{S}^h(\Omega)$ denote the space of continuous piecewise linear finite elements on $\mathcal{T}^h$. The finite element discretisation of (1) in $\mathcal{V}^h$ (the N-dimensional subspace of functions in $\mathcal{S}^h(\Omega)$ which vanish on essential boundaries), yields the linear system:

$$A\mathbf{u} = \mathbf{f} \;. \tag{2}$$

It is well-known that the size of this system grows like $\mathcal{O}(h^{-d})$, as $\mathcal{T}^h$ is refined, and that the condition number $\kappa(A)$ of A worsens like $\mathcal{O}(h^{-2})$. Moreover the conditioning of A also depends on the heterogeneity (characterised by the range and the variability of K) and the anisotropy (characterised by the maximum ratio of the largest to the smallest eigenvalue of $K(x)$ at any point $x \in \Omega$). It is of interest to find solvers for (2) which are robust to changes in the mesh width h as

well as to heterogeneity and anisotropy in K. For the remainder we assume that K is only "mildly" anisotropic, i.e. that the ratio of the largest to the smallest eigenvalue of $K(x)$ is uniformly bounded from above by a benign constant of $\mathcal{O}(1)$, and concentrate on spatial heterogeneity in the coefficient tensor K.

When the smallest scale ε, at which the coefficient tensor $K(x)$ varies, is very small it may not be feasible to solve (1) on a mesh of size $h = \mathcal{O}(\varepsilon)$ with standard solvers, and it may be necessary to scale up the equation to a coarser computational grid of size $H \gg \varepsilon$. A large number of computational methods have been suggested over the years in the engineering literature on how to derive such an upscaled equation numerically (see e.g. the review [16]). More recently this area has also started to attract the attention of numerical analysts, who have started to try to analyse the approximation properties of such upscaling or multiscale techniques theoretically. Among the methods that have been suggested and analysed are the Variational Multiscale Method [9], the Multiscale Finite Element Method [8], the Multiscale Finite Volume Method [10]. However, the existing theory is restricted to periodic fine scale variation or to ergodic random variation. No theory is yet available that gives a comprehensive analysis of the dependency of the accuracy of the upscaled solution on the coefficient variation in the general case.

Moreover, if the coefficient varies arbitrarily throughout Ω and there is no scale separation into a fine $\mathcal{O}(\varepsilon)$ scale variation and a coarse $\mathcal{O}(H)$ scale variation, then all these methods require the solution of local "cell" problems, of size $\mathcal{O}((H/\varepsilon)^d)$, in each cell or element of the coarse mesh, i.e. $\mathcal{O}(H^{-d})$ problems. Thus, even if we assume that the local problems can be solved with optimal (linear) complexity, the total computational cost of the method is $\mathcal{O}(\varepsilon^{-d})$. In practice the complexity may actually be worse. A huge advantage is of course the fact that the cell problems are all completely independent from each other. This means that they can be solved very efficiently on a modern multiprocessor machine. This makes this method so attractive to scale up a physical problem, especially if the upscaled matrix can be used for several right hand sides, within a two-phase flow simulation, or for several time steps in a time-dependent simulation.

A viable and attractive alternative is the use of parallel multilevel iterative solvers, such as multigrid or domain decomposition, for the original fine scale problem (2) on the "subgrid" $\mathcal{T}^h$ where $h = \mathcal{O}(\varepsilon)$. These are known to lead to a similar overall computational complexity $\mathcal{O}(\varepsilon^{-d})$ and, especially in the case of domain decomposition, are designed to scale optimally on modern multiprocessor machines. That is, at (asymptotically) the same cost as using any of the above upscaling procedures, we can obtain the fine-scale solution with guaranteed and quantifiable approximation properties. However, previously no theory was available that guarantees the robustness of these multilevel iterative solvers to heterogeneities in the coefficient, and indeed most of these methods are not robust in their unmodified form. The most successful, completely robust method for (2) is algebraic multigrid (AMG), originally introduced in [2, 17]. Many different versions of AMG have emerged since, but unfortunately no theory exists that proves the (observed) robustness of any of these methods to arbitrary spatial variation

of $K(x)$. The robustness of geometric multigrid for "layered media" in which discontinuities in K are simple interfaces that can be resolved by the coarsest mesh has recently been proved in [21]. Some ideas towards a theory for more general coefficients can be found in [1]. The related BoxMG by Dendy has recently also been used in the context of numerical upscaling in [13, 11, 12].

The situation is different for domain decomposition methods. There are many papers (with rigorous theory) which solve (2) for "layered media" in which discontinuities in K are simple interfaces that can be resolved by the subdomain partitioning and the coarse mesh (see e.g. [19] and the references therein). However, until recently there was no rigorously justified method for general heterogeneous media. In a series of papers [5, 6, 7, 14, 15, 18, 20] we have started to develop new theoretical tools to analyse domain decomposition methods for (2) (which have inherent robustness with respect to h). This analysis indicates explicitly how subdomains and coarse solves should be designed in order to achieve robustness also with respect to heterogeneities. It does not require periodicity and does not appeal to homogenisation theory. Although the analysis in [14, 15] on nonoverlapping FETI-type methods [4] is also of large current interest, in the talk we will focus on the theory for two-level overlapping Schwarz methods in [5, 6, 7, 18, 20], since it gives a clearer picture of the synergies between domain decomposition and numerical upscaling. In particular, we will highlight the important concept of a certain energy minimising property of the coarse space which has yet got to be fully understood in the context of numerical upscaling.

To give a brief indication of the kind of results presented in [5, 6, 7, 18, 20] let us assume that we have a finite overlapping covering of Ω by (open) subdomains $\{\Omega_i : i = 1, \ldots, s\}$. Let us assume that the diameter of a typical subdomain Ω_i is of size $\mathcal{O}(H)$ and that the minimum overlap of Ω_i with the neighbouring domains is of size $\mathcal{O}(\delta)$.

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