A Joint Adventure in Sasakian and Kähler Geometry

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Kähler Geometry

Let N be a smooth compact manifold of real dimension $2d_N$.

- If J is a smooth bundle-morphism on the real tangent bundle, $J : TN \to TN$ such that $J^2 = -Id$ and $\forall X, Y \in TN$

$$J(\mathcal{L}_X Y) - \mathcal{L}_X JY = J(\mathcal{L}_{JX} JY - J\mathcal{L}_{JX} Y),$$

  then (N, J) is a **complex manifold** with **complex structure** J.
- A Riemannian metric g on (N, J) is said to be a **Hermitian Riemannian metric** if

$$\forall X, Y \in TN, \quad g(JX, JY) = g(X, Y)$$

- This implies that $\omega(X, Y) := g(JX, Y)$ is a $J-$ invariant ($\omega(JX, JY) = \omega(X, Y)$) non-degenerate $2-$ form on N.
- If $d\omega = 0$, then we say that (N, J, g, ω) is a **Kähler manifold** (or **Kähler structure**) with **Kähler form** ω and **Kähler metric** g.
- The second cohomology class $[\omega]$ is called the **Kähler class**.
- For fixed J, the subset in $H^2(N, \mathbb{R})$ consisting of Kähler classes is called the **Kähler cone**.

Ricci Curvature of Kähler metrics:

Given a Kähler structure (N, J, g, ω), the Riemannian metric g defines (via the unique Levi-Civita connection ∇)

- the **Riemann curvature tensor** $R : TN \otimes TN \otimes TN \to TN$
- and the trace thereoff, the **Ricci tensor** $r : TN \otimes TN \to C^\infty(N)$

- This gives us the **Ricci form**, $\rho(X, Y) = r(JX, Y)$.
- The miracle of Kähler geometry is that $c_1(N, J) = [\frac{\rho}{2\pi}]$.

- If $\rho = \lambda\omega$, where λ is some constant, then we say that (N, J, g, ω) is **Kähler-Einstein** (or just **KE**).
- More generally, if
$$\rho - \lambda\omega = \mathcal{L}_V\omega,$$
where V is a holomorphic vector field, then we say that (N, J, g, ω) is a **Kähler-Ricci soliton** (or just **KRS**).
- KRS $\implies c_1(N, J)$ is positive, negative, or null.

Scalar Curvature of Kähler metrics:

Given a Kähler structure (N, J, g, ω), the Riemannian metric g defines (via the unique Levi-Civita connection ∇)

- the **scalar curvature**, $Scal \in C^\infty(N)$, where $Scal$ is the trace of the map $X \mapsto \tilde{r}(X)$ where $\forall X, Y \in TN, g(\tilde{r}(X), Y) = r(X, Y)$.
- If $Scal$ is a constant function, we say that (N, J, g, ω) is a constant scalar curvature Kähler metric (or just **CSC**).
- KE $\implies$ CSC (with $\lambda = \frac{Scal}{2d_N}$)
- Not all complex manifolds (N, J) admit CSC Kähler structures.
- There are generalizations of CSC, e.g. **extremal Kähler metrics** as defined by Calabi $(\mathcal{L}_{\nabla_g Scal} J = 0)$.
- Not all complex manifolds (N, J) admit extremal Kähler structures either.

Admissible Kähler manifolds/orbifolds

- Special cases of the more general (admissible) constructions defined by/organized by Apostolov, Calderbank, Gauduchon, and T-F.
- Credit also goes to **Calabi**, Koiso, Sakane, Simanca, Pedersen, Poon, Hwang, Singer, Guan, LeBrun, and others.
- Let ω_N be a primitive integral Kähler form of a CSC Kähler metric on (N, J).
- Let $\mathbb{1} \to N$ be the trivial complex line bundle.
- Let $n \in \mathbb{Z} \setminus \{0\}$.
- Let $L_n \to N$ be a holomorphic line bundle with $c_1(L_n) = [n\,\omega_N]$.
- Consider the total space of a projective bundle $S_n = \mathbb{P}(\mathbb{1} \oplus L_n) \to N$.
- Note that the fiber is $\mathbb{CP}^1$.
- S_n is called **admissible**, or an **admissible manifold**.

Admissible Kähler classes

- Let $D_1 = [\mathbb{1} \oplus 0]$ and $D_2 = [0 \oplus L_n]$ denote the "zero" and "infinity" sections of $S_n \to N$.
- Let r be a real number such that $0 < |r| < 1$, and such that $r\,n > 0$.
- A Kähler class on S_n, Ω, is **admissible** if (up to scale) $\Omega = \frac{2\pi n[\omega_N]}{r} + 2\pi PD(D_1 + D_2)$.
- In general, the **admissible cone** is a sub-cone of the Kähler cone.
- In each admissible class we can now construct explicit Kähler metrics g (called **admissible Kähler metrics**).
- We can generalize this construction to the log pair (S_n, Δ), where Δ denotes the branch divisor $\Delta = (1 - 1/m_1)D_1 + (1 - 1/m_2)D_2$.
- If $m = \gcd(m_1, m_2)$, then (S_n, Δ) is a fiber bundle over N with fiber $\mathbb{CP}^1[m_1/m, m_2/m]/\mathbb{Z}_m$.
- g is smooth on $S_n \setminus (D_1 \cup D_2)$ and has orbifold singularities along D_1 and D_2

Sasakian Geometry:

Sasakian geometry: odd dimensional version of Kählerian geometry and special case of **contact structure**.

A Sasakian structure on a smooth manifold M of dimension $2n+1$ is defined by a quadruple $\mathcal{S} = (\xi, \eta, \Phi, g)$ where

- η is **contact 1-form** defining a subbundle (contact bundle) in TM by $\mathcal{D} = \ker \eta$.
- ξ is the **Reeb vector field** of η [$\eta(\xi) = 1$ and $\xi \rfloor d\eta = 0$]
- Φ is an endomorphism field which annihilates ξ and satisfies $J = \Phi|_{\mathcal{D}}$ is a complex structure on the contact bundle ($d\eta(J\cdot, J\cdot) = d\eta(\cdot,\cdot)$)
- $g := d\eta \circ (\Phi \otimes \mathbb{1}) + \eta \otimes \eta$ is a Riemannian metric
- ξ is a Killing vector field of g which generates a one dimensional foliation $\mathcal{F}_\xi$ of M whose transverse structure is Kähler.
- (Let (g_T, ω_T) denote the transverse Kähler metric)
- $(dt^2 + t^2 g, d(t^2\eta))$ is Kähler on $M \times \mathbb{R}^+$ with complex structure I: $IY = \Phi Y + \eta(Y) t\frac{\partial}{\partial t}$ for vector fields Y on M, and $I(t\frac{\partial}{\partial t}) = -\xi$.

- If ξ is **regular**, the transverse Kähler structure lives on a smooth manifold (quotient of regular foliation $\mathcal{F}_\xi$).
- If ξ is **quasi-regular**, the transverse Kähler structure has orbifold singularities (quotient of quasi-regular foliation $\mathcal{F}_\xi$).
- If not regular or quasi-regular we call it **irregular**... (that's most of them)

Transverse Homothety:

- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is a Sasakian structure, so is $\mathcal{S}_a = (a^{-1}\xi, a\eta, \Phi, g_a)$ for every $a \in \mathbb{R}^+$ with $g_a = ag + (a^2 - a)\eta \otimes \eta$.
- So Sasakian structures come in rays.

Deforming the Sasaki structure:

In its contact structure isotopy class:

- $$\eta \to \eta + d^c\phi, \quad \phi \quad \text{is basic}$$
- This corresponds to a deformation of the transverse Kähler form
$$\omega_T \to \omega_T + dd^c\phi$$
in its Kähler class in the regular/quasi-regular case.
- "Up to isotopy" means that the Sasaki structure might have to been deformed as above.

In the Sasaki Cone:

- Choose a maximal torus T^k, $0 \leq k \leq n+1$ in the Sasaki automorphism group

$$\mathfrak{Aut}(\mathcal{S}) = \{\phi \in \mathcal{D}iff(M) \,|\, \phi^*\eta = \eta,\, \phi^* J = J,\, \phi^*\xi = \xi,\, \phi^* g = g\}.$$

- The unreduced Sasaki cone is

$$\mathfrak{t}^+ = \{\xi' \in \mathfrak{t}_k \,|\, \eta(\xi') > 0\},$$

where $\mathfrak{t}^k$ denotes the Lie algebra of T^k.

- Each element in $\mathfrak{t}^+$ determines a new Sasaki structure with the same underlying CR-structure.

Ricci Curvature of Sasaki metrics

- The Ricci tensor of g behaves as follows:
  - $r(X,\xi) = 2n\,\eta(X)$ for any vector field X
  - $r(X,Y) = r_T(X,Y) - 2g(X,Y)$, where X, Y are sections of $\mathcal{D}$ and r_T is the transverse Ricci tensor
- If the transverse Kähler structure is Kähler-Einstein then we say that the Sasaki metric is η-Einstein.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is η-Einstein iff its entire ray is η-Einstein ("η-Einstein ray")
- If the transverse Kähler-Einstein structure has positive scalar curvature, then exactly one of the Sasaki structures in the η-Einstein ray is actually Einstein (Ricci curvature tensor a rescale of the metric tensor). That metric is called Sasaki-Einstein.
- If $\mathcal{S} = (\xi, \eta, \Phi, g)$ is Sasaki-Einstein, then we must have that $c_1(\mathcal{D})$ is a torsion class (e.g. it vanishes).

- A Sasaki Ricci Soliton (SRS) is a transverse Kähler Ricci soliton, that is, the equation

$$\rho^T - \lambda\omega^T = \mathcal{L}_V \omega^T$$

  holds, where V is some transverse holomorphic vector field, and λ is some constant.
- So if V vanishes, we have an η-Einstein Sasaki structure.
- Our definition allows SRS to come in rays.
- We will say that $\mathcal{S} = (\xi, \eta, \Phi, g)$
  is η-Einstein / Einstein / SRS
  whenever it is
  η-Einstein / Einstein /SRS up to isotopy.

Scalar Curvature of Sasaki metrics

- The scalar curvature of g behaves as follows

$$Scal = Scal_T - 2n$$

- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has constant scalar curvature (CSC) if and only if the transverse Kähler structure has constant scalar curvature.
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ has CSC iff its entire ray has CSC ("CSC ray").
- CSC can be generalized to Sasaki Extremal (Boyer, Galicki, Simanca) such that
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal if and only if the transverse Kähler structure is extremal
- $\mathcal{S} = (\xi, \eta, \Phi, g)$ is extremal iff its entire ray is extremal ("extremal ray").
- We will say that $\mathcal{S} = (\xi, \eta, \Phi, g)$ is CSC/extremal whenever it is CSC/extremal up to isotopy.

The Join Construction

- The join construction of Sasaki manifolds (Boyer, Galicki, Ornea) is the analogue of Kähler products.
- Given quasi-regular Sasakian manifolds $\pi_i : M_i \to \mathcal{Z}_i$. Let $L = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}\xi_2$.
- Form (l_1, l_2)- join by taking the quotient by the action induced by L:

$$
\begin{array}{ccc}
M_1 \times M_2 & & \\
 & \searrow \pi_L & \\
\Big\downarrow \pi_{12} & & M_1 \star_{l_1,l_2} M_2 \\
 & \swarrow \pi & \\
\mathcal{Z}_1 \times \mathcal{Z}_2 & &
\end{array}
$$

- $M_1 \star_{l_1,l_2} M_2$ is a S^1-orbibundle (generalized Boothby-Wang fibration).
- $M_1 \star_{l_1,l_2} M_2$ has a natural quasi-regular Sasakian structure for all relatively prime positive integers l_1, l_2. Fixing l_1, l_2 fixes the contact orbifold. It is a smooth manifold iff $\gcd(\mu_1 l_2, \mu_2 l_1) = 1$, where μ_i is the order of the orbifold $\mathcal{Z}_i$.

Join with a weighted 3-sphere

- Take $\pi_2 : M_2 \to \mathcal{Z}_2$ to be the S^1-orbibundle

$$\pi_2 : S^3_{\mathbf{w}} \to \mathbb{CP}[\mathbf{w}]$$

  determined by a weighted S^1-action on S^3 with weights $\mathbf{w} = (w_1, w_2)$ such that $w_1 \geq w_2$ are relative prime.
- $S^3_{\mathbf{w}}$ has an extremal Sasakian structure.
- Let $M_1 = M$ be a regular CSC Sasaki manifold whose quotient is a compact CSC Kähler manifold N.
- Assume $\gcd(l_2, l_1 w_1 w_2) = 1$ (equivalent with $\gcd(l_2, w_i) = 1$).
-

$$\begin{array}{ccc} M \times S^3_{\mathbf{w}} & & \\ & \searrow \pi_L & \\ \Big\downarrow \pi_{12} & & M \star_{l_1,l_2} S^3_{\mathbf{w}} =: M_{l_1,l_2,\mathbf{w}} \\ & \swarrow \pi & \\ N \times \mathbb{CP}[\mathbf{w}] & & \end{array}$$

The w-Sasaki cone

- The Lie algebra $\mathfrak{aut}(\mathcal{S}_{l_1,l_2,\mathbf{w}})$ of the automorphism group of the join satisfies $\mathfrak{aut}(\mathcal{S}_{l_1,l_2,\mathbf{w}}) = \mathfrak{aut}(\mathcal{S}_1) \oplus \mathfrak{aut}(\mathcal{S}_{\mathbf{w}})$, mod $(L_{l_1,l_2,\mathbf{w}} = \frac{1}{2l_1}\xi_1 - \frac{1}{2l_2}\xi_2)$, where $\mathcal{S}_1$ is the Sasakian structure on M, and $\mathcal{S}_{\mathbf{w}}$ is the Sasakian structure on $S^3_{\mathbf{w}}$.
- The unreduced Sasaki cone $\mathfrak{t}^+_{l_1,l_2,\mathbf{w}}$ of the join $M_{l_1,l_2,\mathbf{w}}$ thus has a 2-dimensional subcone $\mathfrak{t}^+_{\mathbf{w}}$ is called the **w**-Sasaki cone.
- $\mathfrak{t}^+_{\mathbf{w}}$ is inherited from the Sasaki cone on S^3
- Each ray in $\mathfrak{t}^+_{\mathbf{w}}$ is determined by a choice of $(v_1, v_2) \in \mathbb{R}^+ \times \mathbb{R}^+$.
- The ray is quasi-regular iff $v_2/v_1 \in \mathbb{Q}$.
- $\mathfrak{t}^+_{\mathbf{w}}$ has a regular ray (given by $(v_1, v_2) = (1, 1)$) iff l_2 divides $w_1 - w_2$.

Motivating Questions

- Does $\mathfrak{t}^+_{\mathbf{w}}$ have a CSC/η-Einstein ray?

- What about extremal/Sasaki-Ricci solitons?

Key Proposition (Boyer, T-F)

Let $M_{l_1,l_2,\mathbf{w}} = M \star_{l_1,l_2} S^3_{\mathbf{w}}$ be the join as described above.
Let $\mathbf{v} = (v_1, v_2)$ be a weight vector with relatively prime integer components and let $\xi_{\mathbf{v}}$ be the corresponding Reeb vector field in the Sasaki cone $\mathfrak{t}^+_{\mathbf{w}}$.
Then the quotient of $M_{l_1,l_2,\mathbf{w}}$ by the flow of the Reeb vector field $\xi_{\mathbf{v}}$ is (S_n, Δ)
with $n = l_1\big(\frac{w_1 v_2 - w_2 v_1}{s}\big)$, where $s = \gcd(l_2, w_1 v_2 - w_2 v_1)$, and Δ is the branch divisor

$$\Delta = (1 - \frac{1}{m_1})D_1 + (1 - \frac{1}{m_2})D_2, \tag{1}$$

with ramification indices $m_i = v_i \frac{l_2}{s}$.

The Kähler class on the (quasi-regular) quotient

- is admissible up to scale.
- We can determine exactly which one it is.
- So we can test it for containing admissible KRS, KE, CSC, or extremal metrics.
- Hence we can test if the ray of $\xi_{\mathbf{v}}$ is (admissible and) η-Einstein/SRS/CSC/extremal (up to isotopy).
- By lifting the admissible construction to the Sasakian level (in a way so it depends smoothly on (v_1, v_2)), we can also handle the irregular rays.

Theorem A (Boyer, T-F)

- For each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in the 2-dimensional $\mathbf{w}$-Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding ray of Sasakian structures $\mathcal{S}_a = (a^{-1}\xi_{\mathbf{v}}, a\eta_{\mathbf{v}}, \Phi, g_a)$ has constant scalar curvature.
- Suppose in addition that the scalar curvature of N is non-negative. Then the $\mathbf{w}$-Sasaki cone is exhausted by extremal Sasaki metrics. In particular, if the Kähler structure on N admits no Hamiltonian vector fields, then the entire Sasaki cone of the join $M_{l_1,l_2,\mathbf{w}}$ can be represented by extremal Sasaki metrics.
- Suppose in addition that the scalar curvature of N is positive. Then for sufficiently large l_2 there are at least three CSC rays in the $\mathbf{w}$-Sasaki cone of the join $M_{l_1,l_2,\mathbf{w}}$.

Theorem B (Boyer, T-F)

Suppose N is positive Kähler-Einstein with Fano index $\mathcal{I}_N$ and

$$l_1 = \frac{\mathcal{I}_N}{\gcd(w_1 + w_2, \mathcal{I}_N)}, \qquad l_2 = \frac{w_1 + w_2}{\gcd(w_1 + w_2, \mathcal{I}_N)},$$

(ensures that $c_1(\mathcal{D})$ vanishes).

- Then for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ there exists a Reeb vector field $\xi_{\mathbf{v}}$ in the 2-dimensional $\mathbf{w}$-Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ such that the corresponding Sasakian structure $\mathcal{S} = (\xi_{\mathbf{v}}, \eta_{\mathbf{v}}, \Phi, g)$ is Sasaki-Einstein.
- Moreover, this ray is the only admissible CSC ray in the $\mathbf{w}$-Sasaki cone.
- In addition, for each vector $\mathbf{w} = (w_1, w_2) \in \mathbb{Z}^+ \times \mathbb{Z}^+$ with relatively prime components satisfying $w_1 > w_2$ every single ray in the 2-dimensional $\mathbf{w}$-Sasaki cone on $M_{l_1,l_2,\mathbf{w}}$ admits (up to isotopy) a Sasaki-Ricci soliton.

Remarks

- The Sasaki-Einstein structures were first found by the physicists Guantlett, Martelli, Sparks, Waldram.
- Starting from the join construction allows us to study the topology of the Sasaki manifolds more closely.
- When $N = \mathbb{CP}^1$, $M_{l_1,l_2,\mathbf{w}}$ are S^3-bundles over S^2. These were treated by Boyer and Boyer, Pati, as well as by E. Legendre.
- Our set-up, starting from a join construction, allows for cases where no regular ray in the **w**-Sasaki cone exists. If, however, the given **w**-Sasaki cone does admit a regular ray, then the transverse Kähler structure is a smooth Kähler Ricci soliton and the existence of an SE metric in some ray of the Sasaki cone is predicted by the work of Mabuchi and Nakagawa.

References

- **Apostolov, Calderbank, Gauduchon, and T-F.** Hamiltonian 2-forms in Kähler geometry, III *Extremal metrics and stability*, Inventiones Mathematicae 173 (2008), 547–601. For the "admissible construction" of Kähler metrics

- **Boyer and Galicki** Sasakian geometry, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2008.
- Other papers by **Boyer** et all. For the "join" of Sasaki structures

- **Boyer and T.-F.** The Sasaki Join, Hamiltonian 2-forms, and Constant Scalar Curvature (to appear in JGA, 2015) and references therein to our previous papers. For the details and proofs behind the statements in this talk.