Meromorphic connections and the Stokes groupoids

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Based on arXiv:1305.7288 (*Crelle* 2015)
with Marco Gualtieri and Songhao Li

Warmup

Exercise

Find the flat sections of the connection

$$\nabla = d - \begin{pmatrix} 1 & -z \\ 0 & 0 \end{pmatrix} \frac{dz}{z^2}$$

on the trivial bundle $\mathcal{E} = \mathcal{O}_X^{\oplus 2}$ *over the curve* $X = \mathbb{C}$.

i.e. find a fundamental matrix solution of the ODE

$$\frac{d\psi}{dz} = \begin{pmatrix} z^{-2} & -z^{-1} \\ 0 & 0 \end{pmatrix} \psi$$

NB: Pole of order two, i.e. $\nabla : \mathcal{E} \to \Omega^1_X(D) \otimes \mathcal{E}$, where $D = 2 \cdot \{0\} \subset X$.

Solution method

Goal: flat sections of

$$\nabla = d - \begin{pmatrix} 1 & -z \\ 0 & 0 \end{pmatrix} \frac{dz}{z^2}$$

Strategy: Find a gauge transformation ϕ taking ∇ to the simpler *diagonal* connection

$$\nabla_0 = \phi^{-1}\nabla\phi = d - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{dz}{z^2}$$

Solutions of ∇_0 are easily found:

$$\psi_0 = \begin{pmatrix} e^{-1/z} & 0 \\ 0 & 1 \end{pmatrix}.$$

Then we can write

$$\psi = \phi\psi_0.$$

The gauge transformation

Want:

$$\phi^{-1}\left(d - \begin{pmatrix}1 & -z\\ 0 & 0\end{pmatrix}\frac{dz}{z^2}\right)\phi = d - \begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\frac{dz}{z^2}$$

Guess form for ϕ:

$$\phi = \begin{pmatrix}1 & f(z)\\ 0 & 1\end{pmatrix} \text{ a solution} \qquad \iff \qquad z^2\frac{df}{dz} = f - z.$$

Solution has series expansion

$$f(z) = \sum_{n\geq 0} n!\, z^{n+1}.$$

DIVERGES!!!!!

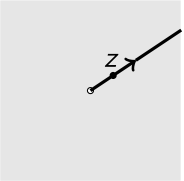
All is not lost

Borel summation/multi-summation: recover solutions from divergent series (É. Borel, Écalle, Ramis, Sibuya, ...)

The essential idea:

$$\sum_{n=0}^{\infty} a_n z^{n+1} = \sum_{n=0}^{\infty} a_n \left(\frac{1}{n!} \int_0^{\infty} t^n e^{-t/z} \, dt \right)$$
$$= \int_0^{\infty} \left(\sum_{n=0}^{\infty} \frac{a_n t^n}{n!} \right) e^{-t/z} \, dt$$

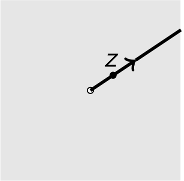
and the new series (the Borel transform) is more likely to converge.



Our example

$$\sum_{n=0}^{\infty} n! z^{n+1} = \int_0^{\infty} \left(\sum_{n=0}^{\infty} t^n \right) e^{-t/z} dt = \int_0^{\infty} \frac{e^{-t/z}}{1-t} dt$$

Stokes phenomenon: sums for $Im(z) > 0$ and $Im(z) < 0$ differ:



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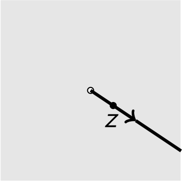


$= 2\pi i \text{Res} = -2\pi i e^{-1/z}$

NB: this comes from the other solution of ODE.

Resummation, cont.

(Nearly) equivalent: Weight the partial sums:

$$\sum_{n=0}^{\infty} a_n z^{n+1} = \lim_{\mu \to \infty} e^{-\mu} \sum_{n=0}^{\infty} \left(\frac{\mu^n}{n!} \sum_{k=0}^{n} a_k z^{k+1} \right)$$

Pros and cons

Success: solution of the ODE with the divergent series as an **asymptotic expansion**; truncating the series gives a good approximation for small z

The Stokes phenomenon: "correct" sum of the series varies from sector to sector (wall crossing) — patched by "generalized monodromy data"

Drawbacks: the procedure is a bit ad hoc:

- Correct weights depend on order of pole and "irregular type"
- Not directly applicable to related and important situations
  - WKB approximation (aka λ-connections)
  - Normal forms in dynamical systems
  - Perturbative QFT

Leads to even more complicated theory of "resurgence" (Écalle)

The problem

Question

What is the geometry of these resummation procedures?

Answer (Gualtieri–Li–P.)

It is governed by a very natural **Lie groupoid***.*

Viewpoint

A holomorphic flat connection $\nabla : \mathcal{E} \to \Omega^1_X \otimes \mathcal{E}$ gives an action of vector fields by derivations

$$\begin{aligned} \mathcal{T}_X \times \mathcal{E} &\to \mathcal{E} \\ (\eta, \psi) &\mapsto \nabla_\eta \psi, \end{aligned}$$

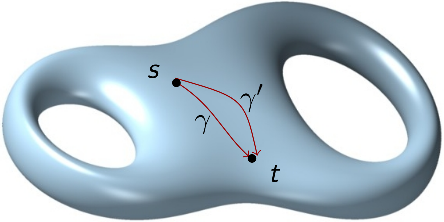
compatible with Lie brackets:

$$\nabla_\eta \nabla_\xi - \nabla_\xi \nabla_\eta = \nabla_{[\eta,\xi]}$$

Slogan:

$$\{\text{holomorphic flat connections}\} = \{\text{representations of } \mathcal{T}_X\}.$$

Parallel transport



- Solve the ODE $\nabla\psi = 0$ along a path $\gamma : [0,1] \to X$ from s to t
- Get the **parallel transport**

$$\Psi(\gamma) : \mathcal{E}|_s \to \mathcal{E}|_t$$

- If γ, γ' are homotopic, then $\Psi(\gamma) = \Psi(\gamma')$.

The fundamental groupoid

- Domain for parallel transport is the **fundamental groupoid:**

$$\Pi_1(X) = \{\text{paths } \gamma : [0,1] \to X\}/(\text{end-point-preserving homotopies})$$

- Source and target $s, t : \Pi_1(X) \to X$

$$s(\gamma) = \gamma(0) \qquad\qquad t(\gamma) = \gamma(1)$$

- Product: concatenation of paths, defined when endpoints match
- Identities: constant paths, one for each $x \in X$
- Inverses: reverse directions

Lemma

$\Pi_1(X)$ has a unique manifold structure such that $(s, t) : \Pi_1(X) \to X \times X$ is a local diffeomorphism. Thus $\Pi_1(X)$ is a **(complex) Lie groupoid***.*

Example: the fundamental groupoid of $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$

- We have an isomorphism

$$\mathbb{C} \times \mathbb{C}^* \cong \Pi_1(\mathbb{C}^*)$$
$$(\lambda, z) \mapsto [\gamma_{\lambda,z}]$$

  - Source and target:

$$s(\lambda, z) = z \qquad t(\lambda, z) = e^{\lambda} z$$

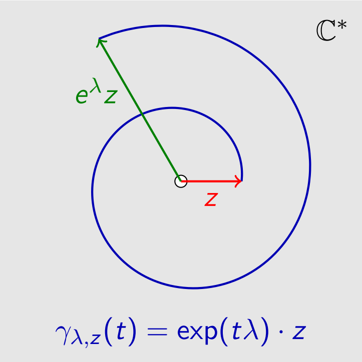
  - Identities:

$$i(z) = (0, z)$$

  - Product:

$$(\lambda, z)(\lambda', z') = (\lambda + \lambda', z')$$

  defined whenever $z = e^{\lambda'} z'$.



Parallel transport as a representation

- Parallel transport of holomorphic connection ∇ is an isomorphism of bundles on $\Pi_1(X)$:

$$\Psi : s^*\mathcal{E} \to t^*\mathcal{E}$$

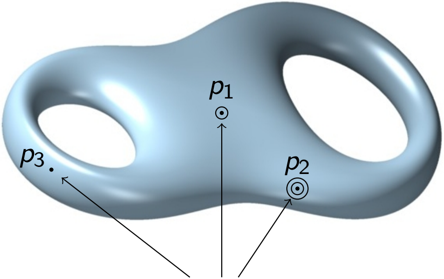
- If ψ is a fundamental solution, then $\Psi = t^*\psi \cdot s^*\psi^{-1}$
- It's a **representation of** $\Pi_1(X)$:

$$\Psi(\gamma_1\gamma_2) = \Psi(\gamma_1)\Psi(\gamma_2) \qquad \Psi(\gamma^{-1}) = \Psi(\gamma)^{-1} \qquad \Psi(1_x) = 1$$

- Version of the **Riemann–Hilbert correspondence**:

$$\{\text{representations of } \mathcal{T}_X\} \underset{\text{Differentiation}}{\overset{\text{Integration}}{\rightleftarrows}} \{\text{representations of } \Pi_1(X)\}$$

Meromorphic connections



$D = k_1 \cdot p_1 + \cdots + k_n \cdot p_n$ an effective divisor ($k_i \in \mathbb{N}$)

- Meromorphic connection $\nabla : \mathcal{E} \to \Omega^1_X(D) \otimes \mathcal{E}$
- In a local coordinate z near p_i

$$\nabla\psi = dz \otimes \left(\frac{d\psi}{dz} - \frac{A(z)}{z^{k_i}}\psi \right).$$

- **Can't define parallel transport for paths that intersect D**

Lie-theoretic perspective

- $\mathcal{T}_X(-D)$ the sheaf of vector fields vanishing on D.
  - Locally free (a vector bundle). Near a point $p \in D$, we have

$$\mathcal{T}_X(-D) \cong \langle z^k \partial_z \rangle$$

  - Anchor map

$$a : \mathcal{T}_X(-D) \to \mathcal{T}_X$$

  - Closed under Lie brackets
- Thus, $\mathcal{T}_X(-D)$ is a very simple example of a **Lie algebroid**
- Pairing with $\nabla : \mathcal{E} \to \Omega^1_X(D) \otimes \mathcal{E}$ gives a *holomorphic* action

$$\mathcal{T}_X(-D) \times \mathcal{E} \to \mathcal{E}.$$

Lie-theoretic perspective

Slogan:

$$\{\text{flat connections on } X \text{ with poles } \leq D\} = \{\text{representations of } \mathcal{T}_X(-D)\}$$

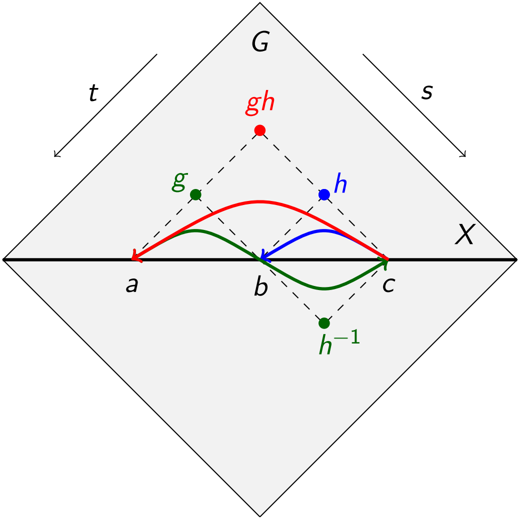
Consequence: The correct domain for the solutions is the **Lie groupoid** that "integrates" $\mathcal{T}_X(-D)$.

Lie groupoids (Ehresmann, Pradines 50–60s)

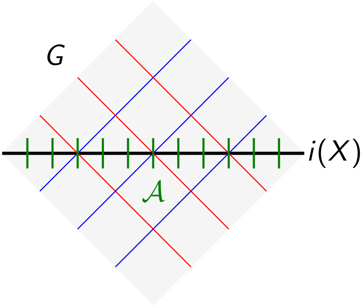
A Lie groupoid $G \rightrightarrows X$ is

1. A manifold X of **objects**
2. A manifold G of **arrows**
3. Maps $s, t : G \to X$ indicating the **source** and **target**
4. Composition of arrows whose endpoints match
5. An identity arrow for each object $i : X \hookrightarrow G$
6. Inversion $\cdot^{-1} : G \to G$.

satisfying associativity, etc.



Infinitesimal counterpart: Lie algebroids



Vector bundle $\mathcal{A} = \mathcal{N}_{i(X),G}$ with Lie bracket

$$[\cdot,\cdot] : \mathcal{A} \times \mathcal{A} \to \mathcal{A}$$

on sections and anchor map $a : \mathcal{A} \to \mathcal{T}_X$ satisfying the Leibniz rule

$$[\xi, f\eta] = (\mathcal{L}_{a(\xi)} f)\eta + f[\xi,\eta].$$

Examples

G	$\mathcal{A}$
$G \rightrightarrows \{pt\}$ a Lie group	$\mathfrak{g}$ its Lie algebra
$H \times X \rightrightarrows X$ group action	$\mathfrak{h} \to \mathcal{T}_X$ infinitesimal action
$\Pi_1(X)$	$\mathcal{T}_X$
$Pair(X) = X \times X \rightrightarrows X$	$\mathcal{T}_X$

Algebroid representations

A **representation of** $\mathcal{A}$ is a flat $\mathcal{A}$-connection, i.e. an operator

$$\nabla : \mathcal{E} \to \mathcal{A}^\vee \otimes \mathcal{E}$$

satisfying

$$\nabla(f\psi) = (a^\vee df) \otimes \psi + f\nabla\psi$$

and having zero curvature in $\bigwedge^2 \mathcal{A}^\vee \otimes End\mathcal{E}$.

Examples:

1. For $X = \{*\}$ and $\mathcal{A} = \mathfrak{g}$: finite-dimensional $\mathfrak{g}$-reps
2. For $\mathcal{A} = \mathcal{T}_X$: have $\mathcal{A}^\vee = \Omega^1_X$ and ∇ a usual flat connection
3. **For $\mathcal{T}_X(-D)$: have $\mathcal{A}^\vee = \Omega^1_X(D)$, and ∇ a meromorphic flat connection with poles bounded by D**
4. Logarithmic connections, λ-connections, connections with central curvature, Poisson modules (= “semi-classical” bimodules), ...

Parallel transport for algebroid connections

An $\mathcal{A}$-**path** is a Lie algebroid homomorphism

$$\Gamma : \mathcal{T}_{[0,1]} \to \mathcal{A}$$

$\mathcal{A}$-connections pull back to usual connections on $[0,1]$.

Thus, parallel transport is defined on the **fundamental groupoid of** $\mathcal{A}$:

$$\Pi_1(\mathcal{A}) = \frac{\{\mathcal{A}\text{-paths}\}}{\{\mathcal{A}\text{-homotopies}\}}$$

Examples:

- For $\mathcal{A} = \mathfrak{g}$ a Lie algebra, get $\Pi_1(\mathfrak{g}) = G$, the simply-connected group
- For $\mathcal{A} = \mathcal{T}_X$, get $\Pi_1(\mathcal{T}_X) = \Pi_1(X)$.

Integrability of algebroids (analogue of Lie III)

The **Crainic–Fernandes theorem** (Annals 2003) gives necessary and sufficient conditions for $\Pi_1(\mathcal{A})$ to have a smooth structure, making it a Lie groupoid.

Parallel transport of $\mathcal{A}$-connections along $\mathcal{A}$-paths gives:

$$\{\text{representations of } \mathcal{A}\} \underset{\text{Differentiation}}{\overset{\text{Integration}}{\rightleftarrows}} \{\text{representations of } \Pi_1(\mathcal{A})\}$$

Theorem (Debord 2001)

If $\mathcal{A} \to \mathcal{T}_X$ is an embedding of sheaves, then $\mathcal{A}$ is integrable.

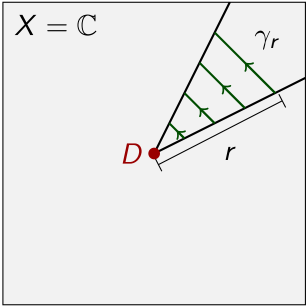
Applied to $\mathcal{T}_X(-D)$

- Get a Lie groupoid $\Pi_1(X, D)$, functorial in X and D
- Two types of algebroid paths:
  - Usual paths in $X \setminus D$, so we have open dense

$$\Pi_1(X \setminus D) \hookrightarrow \Pi_1(X, D)$$

  - Boundary: a one-dimensional Lie group of loops at each $p \in D$

$$\bigsqcup_{p \in D} (T_p^* X)^{mult(p)-1} \hookrightarrow \Pi_1(X, D)$$



e.g., $mult(p) = 1$ limiting procedure

$$\lim_{r \to 0} [\gamma_r] = \lim_{r \to 0} \log \frac{\gamma_r(1)}{\gamma_r(0)} \in \mathbb{C}$$

giving "loops" at D.

Parallel transport

- Meromorphic connection $\nabla : \mathcal{E} \to \Omega^1_X(D) \otimes \mathcal{E}$
- Usual parallel transport defined on $\Pi_1(X \setminus D)$ extends to

$$\Psi : s^*\mathcal{E} \to t^*\mathcal{E}$$

  globally defined and holomorphic on $\Pi_1(X, D)$.
- **Caveat:** $\Pi_1(X, D)$ was constructed as an infinite-dimensional quotient—not very explicit.

Our paper

With M. Gualtieri and S. Li we give

- Explicit local normal forms, the **Stokes groupoids**
- Finite-dimensional global construction using the uniformization theorem
  - analytic open embedding in a $\mathbb{P}^1$-bundle

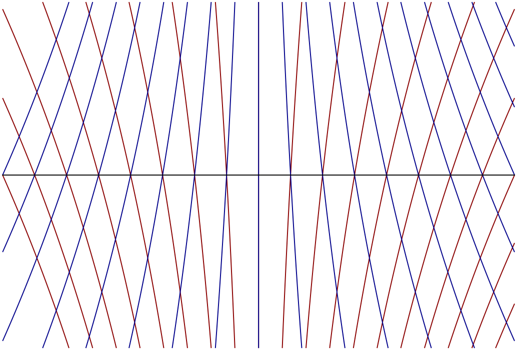
$$\Pi_1(X, D) \hookrightarrow \mathbb{P}(J^1_{X,D}\Omega_X^{1/2})$$

  - groupoid structure maps given by solving the uniformizing ODE
  - e.g. groupoid for $X = \mathbb{P}^1$ and $D = 0 + 1 + \infty$ involves hypergeometric functions and the elliptic modular function $\lambda(\tau)$
- Constructions of $Pair(X, D)$ by iterated blowups
- Application to divergent series

Local normal form: the Stokes groupoids

The case $X = \mathbb{C}$ and $D = k \cdot 0$ is the Stokes groupoid $\mathsf{Sto}_k = \Pi_1(\mathbb{C}, k \cdot 0)$

$$\mathsf{Sto}_k = \mathbb{C} \times \mathbb{C} \rightrightarrows \mathbb{C}$$
$$s(x, y) = \exp(-x^{k-1}y) \cdot x$$
$$t(x, y) = \exp(x^{k-1}y) \cdot x$$
$$i(z) = (z, 0)$$



$k = 1$

Or $s(z, \lambda) = z$ and $t(z, \lambda) = \exp(\lambda^{k-1} z) \cdot z$, in which case

$$(z_1, \lambda_1)(z_2, \lambda_2) = (z_1, u_2 \exp((k-1)u_1 z_1^{k-1}) + u_1)$$

Demonstration on my web site

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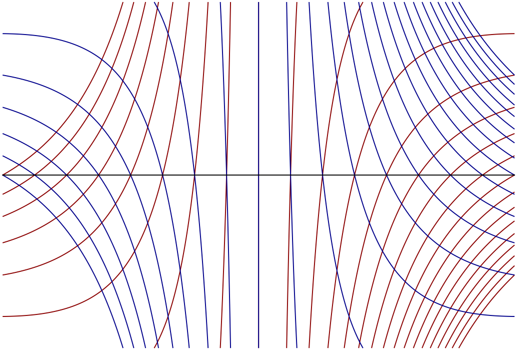

$k = 2$

Or $s(z, \lambda) = z$ and $t(z, \lambda) = \exp(\lambda^{k-1} z) \cdot z$, in which case

$$(z_1, \lambda_1)(z_2, \lambda_2) = (z_1, u_2 \exp((k-1)u_1 z_1^{k-1}) + u_1)$$

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Local normal form: the Stokes groupoids

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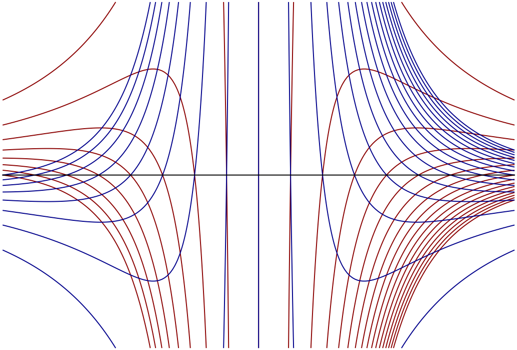
$$i(z) = (z, 0)$$



$k = 3$

Or $s(z, \lambda) = z$ and $t(z, \lambda) = \exp(\lambda^{k-1}z) \cdot z$, in which case

$$(z_1, \lambda_1)(z_2, \lambda_2) = (z_1, u_2 \exp((k-1)u_1 z_1^{k-1}) + u_1)$$

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Local normal form: the Stokes groupoids

The case $X = \mathbb{C}$ and $D = k \cdot 0$ is the Stokes groupoid $\mathsf{Sto}_k = \Pi_1(\mathbb{C}, k \cdot 0)$

$$\mathsf{Sto}_k = \mathbb{C} \times \mathbb{C} \rightrightarrows \mathbb{C}$$
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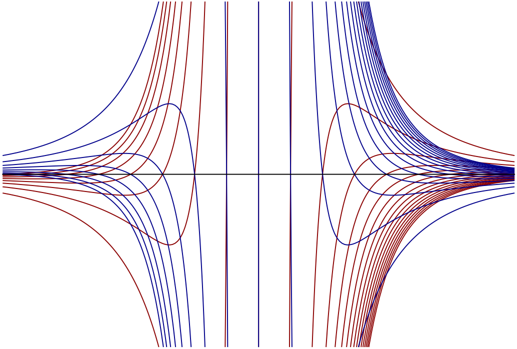
$$i(z) = (z, 0)$$



$k = 4$

Or $s(z, \lambda) = z$ and $t(z, \lambda) = \exp(\lambda^{k-1}z) \cdot z$, in which case

$$(z_1, \lambda_1)(z_2, \lambda_2) = (z_1, u_2 \exp((k-1)u_1 z_1^{k-1}) + u_1)$$

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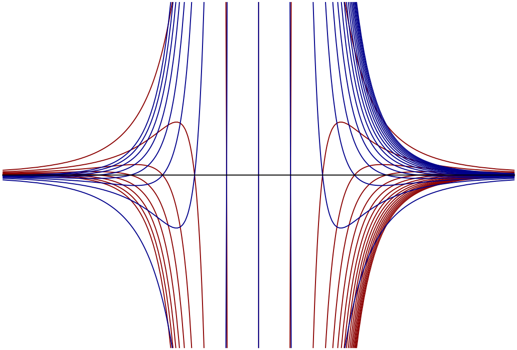
Local normal form: the Stokes groupoids

The case $X = \mathbb{C}$ and $D = k \cdot 0$ is the Stokes groupoid $\mathsf{Sto}_k = \Pi_1(\mathbb{C}, k \cdot 0)$

$$\mathsf{Sto}_k = \mathbb{C} \times \mathbb{C} \rightrightarrows \mathbb{C}$$
$$s(x, y) = \exp(-x^{k-1}y) \cdot x$$
$$t(x, y) = \exp(x^{k-1}y) \cdot x$$
$$i(z) = (z, 0)$$



$k = 5$

Or $s(z, \lambda) = z$ and $t(z, \lambda) = \exp(\lambda^{k-1}z) \cdot z$, in which case

$$(z_1, \lambda_1)(z_2, \lambda_2) = (z_1, u_2 \exp((k-1)u_1 z_1^{k-1}) + u_1)$$

Demonstration on my web site

Resummation, redux

Suppose given the following data:

- Two connections $\nabla, \nabla_0 : \mathcal{E} \to \Omega^1_X(D) \otimes \mathcal{E}$
- A point $p \in D$
- An isomorphism on the formal completion $\hat{X} \subset X$ at p:

$$\hat{\phi} : \nabla_0|_{\hat{X}} \to \nabla|_{\hat{X}}.$$

Integrating ∇, ∇_0 and $\hat{\phi}$, we get the parallel transports on $\Pi_1(X, D)$:

$$\Psi, \Psi_0 : s^*\mathcal{E} \to t^*\mathcal{E},$$

and their Taylor expansions $\hat{\Psi}, \hat{\Psi}_0$ on $\Pi_1\widehat{(X, D)} \subset \Pi_1(X, D)$.

Resummation, redux

$$\hat{\phi} : (\mathcal{E}, \nabla_0)|_{\hat{X}} \to (\mathcal{E}, \nabla)|_{\hat{X}} \qquad \Psi, \Psi_0 : s^*\mathcal{E} \to t^*\mathcal{E}$$

Theorem (Gualtieri–Li–P.)

The formal power series

$$t^*\hat{\phi} \cdot \widehat{\Psi_0} \cdot s^*\hat{\phi}^{-1}$$

converges *to* Ψ *in a neighbourhood of* $\mathrm{id}(p) \in \Pi_1(X, D)$.

Proof.

Because $\hat{\phi}$ is an isomorphism, we have the identity of formal power series:

$$\widehat{\Psi} = t^*\hat{\phi} \cdot \widehat{\Psi_0} \cdot s^*\hat{\phi}^{-1}.$$

But Ψ is holomorphic a priori, so its Taylor expansion converges. □

Resummation example

In our example:

$$\nabla = d - \begin{pmatrix} 1 & -z \\ 0 & 0 \end{pmatrix} \frac{dz}{z^2} \qquad \nabla_0 = d - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \frac{dz}{z^2}$$

$$\hat{\phi} = \begin{pmatrix} 1 & f(z) \\ 0 & 1 \end{pmatrix} : \hat{\nabla}_0 \to \hat{\nabla}_1$$

with $f(z) = \sum_{n \geq 0} n! z^{n+1}$.

Fundamental solutions:

$$\psi_0 = \begin{pmatrix} e^{-1/z} & 0 \\ 0 & 1 \end{pmatrix} \qquad \psi = \begin{pmatrix} e^{-1/z} & f \\ 0 & 1 \end{pmatrix}$$

Resummation example

Choose local coordinates (μ, z) on Sto_2 in which $s = z$ and $t = \frac{z}{1-\mu z}$.

We compute

$$\begin{aligned}\Psi_0 &= t^*\psi_0 \cdot s^*\psi_0^{-1} \\ &= \begin{pmatrix} e^{-1/t} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-1/s} & 0 \\ 0 & 1 \end{pmatrix}^{-1} \\ &= \begin{pmatrix} e^{-\frac{1-\mu z}{z}} & \hat{0} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{1/z} & 0 \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\mu} & 0 \\ 0 & 1 \end{pmatrix}\end{aligned}$$

$$\begin{aligned}\Psi &= t^*\phi \cdot \Psi_0 \cdot s^*\phi^{-1} \\ &= \begin{pmatrix} 1 & f(t) \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{\mu} & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & -f(s) \\ 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} e^{\mu} & f(\frac{z}{1-\mu z}) - e^{\mu} f(z) \\ 0 & 1 \end{pmatrix}\end{aligned}$$

which **must be holomorphic**.

Resummation example

Using $f = \sum_{n=0}^{\infty} n! z^{n+1}$, we find

$$f\left(\tfrac{z}{1-\mu z}\right) - e^{\mu} f(z) = -\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{z^{i+1}\mu^{i+j+1}}{(i+1)(i+2)\cdots(i+j+1)}$$

$$= -\sum_{n=0}^{\infty} \frac{\mu^n}{n!} \sum_{k=0}^{n} k!\, z^{k+1}$$

which is **holomorphic on the groupoid** Sto_2.

Result: We have taken the divergent series, and the solutions of the "simple" connection ∇_0, and obtained a convergent series for the parallel transport of ∇ by **elementary algebraic manipulations**

Recovering the Borel sum

So far we have the parallel transport Ψ between points in $X \setminus D = \mathbb{C}^*$.

To see the Borel sum: look for gauge transformations $\tilde{\phi}(z)$ such that

$$\lim_{z \to 0} \tilde{\phi}(z) = 1$$

Recall that we have

$$\tilde{\phi}(t) = \Psi \tilde{\phi}(s) \Psi_0^{-1} \in Aut(\mathcal{E}|_t)$$

for any gauge transformation.

Using the previous formula, we easily find

$$\tilde{\phi}(z) = \begin{pmatrix} 1 & \lim_{\mu \to \infty} e^{-\mu} \sum_{n=0}^{\infty} \frac{\mu^n}{n!} \sum_{k=0}^{n} k! z^k \\ 0 & 1 \end{pmatrix} = BorelSum(\phi)$$

in the appropriate sector.

Conclusion

Moral: The formula for **Borel resummation is a consequence of the geometry** of the groupoid $\Pi_1(X, D)$.

Future directions:

- Recover the Riemann–Hilbert correspondence/Stokes data
- Extend this method to other situations, e.g. WKB approximation in quantum mechanics, other types of singular DEs
- Isomonodromic deformations via Morita equivalences (in prep. with Gualtieri)