Coupled equations for Kähler metrics and Yang–Mills connections

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Joint work with:

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A moduli problem

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G compact Lie group,
$\mathfrak{g}$ Lie algebra of G,
E smooth principal G-bundle over X.

A moduli problem: **Construct a moduli space with a Kähler structure**

$$(1) \left\{ \text{ pairs } (g, A) \textbf{ satisfying suitable PDE } \right\} \Big/ \sim$$

A connection on E, g Kähler metric on X.

Problem 1 of this talk: **Find a well suited PDE for (1)**

Relation with physics: interaction between gauge fields and gravity.

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Strategy

• **Look for a PDE with symplectic interpretation:** its solutions are points in the symplectic reduction

$$\mu_\alpha^{-1}(0)/\widetilde{\mathcal{G}}$$

of a suitable space $\mathcal{P} \supset \mu_\alpha^{-1}(0)$ parameterizing Kähler structures on X and holomorphic structures on a bundle associated to the G-bundle E.

• **We rely on the symplectic interpretation of two fundamental equations in Kähler geometry:**

1) the Hermite–Yang–Mills (HYM) equations for a connection and
2) the constant scalar curvature equation for a Kähler metric (cscK).

Once we have our nice PDE ...

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Moment maps

(X, ω) = symplectic manifold,
G = Lie group with Lie algebra $\mathfrak{g}$,
$G \times X \to X$, left G-action preserving ω.

Suppose that $\exists$ a G-equivariant **moment map** i.e. $\exists\ \mu \colon X \to \mathfrak{g}^*$ such that

$$d\langle \mu, \zeta \rangle = \omega(Y_\zeta, \cdot) \quad \text{and} \quad \mu(g \cdot x) = \mathrm{Ad}(g)^{-1} \cdot \mu(x),$$

for all $g \in G$ and $\zeta \in \mathfrak{g}$, where $Y_{\zeta|x} = \frac{d}{dt}_{t=0} \exp(t\zeta) \cdot x \in T_x X$.

Symplectic quotient (Marsden & Weinstein '74): If we have a "good" action then $\mu^{-1}(0)/G$ inherits a natural symplectic structure.

Kähler quotient (Guillemin & Stenberg '82): If (X, J, ω) is Kähler and we have a "good" action of $G \curvearrowright (X, \omega, J)$ then $\mu^{-1}(0)/G$ inherits a natural Kähler structure.

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Example 1: The Hermite–Yang–Mills equations

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The infinite-dimensional manifold $\mathcal{A}$ has a Kähler structure $(\omega_{\mathcal{A}}, I_{\mathcal{A}}, g_{\mathcal{A}})$ preserved by $\mathcal{G}$.

$$\omega_{\mathcal{A}}(a_0, a_1) = \int_X (a_0 \wedge a_1) \wedge \omega^{n-1}, \; I_{\mathcal{A}} a_0 = -a_0(J\cdot) \quad \text{with } a_j \in \Omega^1(\mathrm{ad}E).$$

Moment map(Atiyah–Bott ('83) & Donaldson): $\mu_{\mathcal{A}} \colon \mathcal{A} \to (\mathrm{Lie}\,\mathcal{G})^*$

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$\mathcal{G} \curvearrowright \mathcal{A}^{1,1} = \{A \in \mathcal{A} \colon F_A^{0,2} = 0\} \equiv$ holomorphic struct. on $E^c = E \times_G G^c$

HYM equations:

$$\Lambda_\omega F_A = z, \qquad F_A^{0,2} = 0, \qquad z \in \mathfrak{z} \; (\textit{centre of } \mathfrak{g}).$$

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Example 2: The constant scalar curvature equation

(X,ω) smooth compact symplectic manifold of Kähler type.

$\mathcal{J}$=\{complex structures on X compatible with ω\}

$\mathcal{H}$=\{ Hamiltonian symplectomorphisms of (X,ω)\} $\curvearrowright \mathcal{J}$

The infinite-dimensional (singular) manifold $\mathcal{J}$ has a Kähler structure $(\omega_{\mathcal{J}}, I_{\mathcal{J}}, g_{\mathcal{J}})$ preserved by $\mathcal{H}$. Given $b_j \in T_J\mathcal{J} \subset \Omega^0(\mathrm{End}\, TX)$,

$$\omega_{\mathcal{J}|J}(b_0,b_1) = \int_X \mathrm{tr}(J\cdot b_0 \cdot b_1)\frac{\omega^n}{n!}, \qquad I_{\mathcal{J}} b_0 = J b_0.$$

Moment map (Fujiki(1992)–Donaldson(1997)): $\mu_{\mathcal{J}}\colon \mathcal{J} \to (\mathrm{Lie}\,\mathcal{H})^*$

$$\langle \mu_{\mathcal{J}}(J), \phi\rangle = -\int_X \phi(S_J - \hat{S})\frac{\omega^n}{n!}$$

$$\phi \in C^\infty(X)/\mathbb{R} \cong \mathrm{Lie}\mathcal{H} \qquad \hat{S} = \frac{1}{\mathrm{Vol}(X)}\int_X S_J \frac{\omega^n}{n!}$$

CscK equation: $\qquad S_J = \hat{S}, \qquad J \in \mathcal{J}.$

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Example 2: The constant scalar curvature equation

(X, ω) smooth compact symplectic manifold of Kähler type.
$\mathcal{J}=\{$complex structures on X compatible with $\omega\}$
$\mathcal{H}=\{$ Hamiltonian symplectomorphisms of $(X,\omega)\} \curvearrowright \mathcal{J}$

The infinite-dimensional (singular) manifold $\mathcal{J}$ has a Kähler structure $(\omega_{\mathcal{J}}, I_{\mathcal{J}}, g_{\mathcal{J}})$ preserved by $\mathcal{H}$. Given $b_j \in T_J\mathcal{J} \subset \Omega^0(\mathrm{End}\, TX)$,

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Group of symmetries: $1\to\mathcal{G}\to\widetilde{\mathcal{G}}\to\mathcal{H}\to 1$, with $\widetilde{\mathcal{G}}\curvearrowright\mathcal{J}\times\mathcal{A}$.

Symplectic structure: $\omega_\alpha=\alpha_0\omega_{\mathcal{J}}+\frac{4\alpha_1}{(n-1)!}\omega_{\mathcal{A}}$, $0\neq\alpha_0,\alpha_1\in\mathbb{R}$.

Remarks:

- $\mathcal{J}\times\mathcal{A}$ has an integrable complex structure that fibers over $(\mathcal{J},I_{\mathcal{J}})$, given by $\mathbf{I}_{(J,A)}(b,a)=(Jb,-a(J\cdot))$ and ω_α is Kähler if $\frac{\alpha_1}{\alpha_0}>0$!!!
- Why $\widetilde{\mathcal{G}}$? Geometry: It preserves $\mathbf{I}$, ω_α and the complex submanifold $\mathcal{P}=\{(J,A)\in\mathcal{J}\times\mathcal{A}\colon A\in\mathcal{A}_J^{1,1}\}\equiv$ Kähler structure on X with fixed ω + holomorphic structure on E^c over X.
Physics: Natural group of symmetries for (J,A) (grav. field + gauge field) $\Rightarrow\mathrm{Diff}(E)^G$. $\widetilde{\mathcal{G}}\subset\mathrm{Diff}(E)^G$ "biggest" subgroup preserving ω_α and $\mathbf{I}$.
- Why ω_α? For simplicity (following cscK & HYM).

Problem 1: **We find a solution if $\widetilde{\mathcal{G}}\curvearrowright\mathcal{J}\times\mathcal{A}$ is Hamiltonian.**

Lie group extensions and Hamiltonian actions

Question: **Is $\widetilde{\mathcal{G}} \curvearrowright (\mathcal{J} \times \mathcal{A}, \omega_\alpha)$ Hamiltonian?**

Recall: $1 \to \mathcal{G} \to \widetilde{\mathcal{G}} \to \mathcal{H} \to 1$ and the $\widetilde{\mathcal{G}}$-action is symplectic.

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Coupled equations for Kähler metrics and connections

This proves that ...

Proposition [—, L. Álvarez Cónsul, O. García Prada]

For any α_0 and α_1 there exists a $\widetilde{\mathcal{G}}$-equivariant moment map $\mu_\alpha \colon \mathcal{J} \times \mathcal{A} \to \operatorname{Lie} \widetilde{\mathcal{G}}^*$ for the $\widetilde{\mathcal{G}}$-action. If $\zeta \in \operatorname{Lie} \widetilde{\mathcal{G}}$, covering $\phi \in C^\infty(X)/\mathbb{R} \cong \operatorname{Lie} \mathcal{H}$ then,

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The $\widetilde{\mathcal{G}}$-action preserves the complex submanifold $\mathcal{P} = \{(J,A) \in \mathcal{J} \times \mathcal{A} \colon A \in \mathcal{A}_J^{1,1}\}$. $\Rightarrow \mu_\alpha : \mathcal{P} \to \operatorname{Lie} \widetilde{\mathcal{G}}^*$ and the conditions

$$\mu_\alpha(J,A) = 0, \qquad (J,A) \in \mathcal{P}$$

defines (**completely!**) coupled equations for (ω, J, g, A) that can be written as follows (after a suitable shift by $z \in \mathfrak{z}$, the center of $\mathfrak{g}$):

Definition:

$$\left.\begin{aligned} &\Lambda_\omega F_A = z, \\ &F_A^{0,2_J} = 0, \\ &\alpha_0 S_g \ + \ \alpha_1 \Lambda_\omega^2 (F_A \wedge F_A) = c. \end{aligned}\right\} \qquad (1)$$

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For any α_0 and α_1 there exists a $\widetilde{\mathcal{G}}$-equivariant moment map $\mu_\alpha \colon \mathcal{J} \times \mathcal{A} \to \operatorname{Lie} \widetilde{\mathcal{G}}^*$ for the $\widetilde{\mathcal{G}}$-action. If $\zeta \in \operatorname{Lie} \widetilde{\mathcal{G}}$, covering $\phi \in C^\infty(X)/\mathbb{R} \cong \operatorname{Lie} \mathcal{H}$ then,

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HYM: *1. Construction of moduli spaces with Kähler structure* ⇒
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First examples of solutions

We fix a compact complex manifold (X, J) and a G-bundle over X. Consider the equations for (ω, A), with $\omega \in [\omega]$ and $A \in \mathcal{A}^{1,1}$.

Trivial examples:

- The system of equations (1) decouples when $\dim_{\mathbb{C}} X = 1$ since $(F_A \wedge F_A) = 0$. Solutions = stable holomorphic bundles over (X, J).
- If $E = L$, or if E es projectively flat, with $c_1(E) = \lambda[\omega]$ then the coupled equations admit decoupled solutions: cscK + HYM.

Remark: In both cases $\exists$ a solution to $F_A = \lambda\omega$, which implies $\mathrm{Lie}\,\widetilde{\mathcal{G}} = \mathrm{Lie}\,\mathcal{G} \ltimes \mathrm{Lie}\,\mathcal{H}$.

Less trivial examples:

- The coupled equations (1) have solutions on Homogenous holomorphic bundles E^c over homogeneous Kähler manifolds if the bundle comes from an irreducible representation ($\equiv \exists$ HYM connection). Proof: invariant structures and representation theory.
- Solutions are given by simultaneous solutions for the cases $\alpha_1 = 0, \alpha_0 \neq 0$ and $\alpha_0 = 0, \alpha_1 \neq 0$.

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An existence criterion

In the previous examples the Kähler metric on (X, J) is always cscK. Are there any examples of solutions (ω, A) with ω non cscK?

Theorem [—, L. Álvarez Cónsul, O. García Prada]

Let (X, L) be a compact polarised manifold, G^c be a complex reductive Lie group and E^c be a holomorphic G^c-bundle over X. If there exists a cscK metric $\omega \in c_1(L)$, X has finite automorphism group and E^c is stable with respect to L then, given a pair of positive real constants $\alpha_0, \alpha_1 > 0$ with small ratio $0 < \frac{\alpha_1}{\alpha_0} << 1$, there exists a solution $(\omega_\alpha, A_\alpha)$ to (1) with these coupling constants and $\omega_\alpha \in c_1(L)$.

Proof: Deformation argument using the Implicit Function Theorem in Banach spaces (either fixing ω and moving J or viceversa). Idea (fixing ω): suppose $\widetilde{\mathcal{G}}$ has a complexification $\widetilde{\mathcal{G}}^c$ that extends the $\widetilde{\mathcal{G}}$-action on $\mathcal{P}$. Consider the map $L\colon \operatorname{Lie}\widetilde{\mathcal{G}} \to \operatorname{Lie}\widetilde{\mathcal{G}}^*\colon \zeta \to \mu_\alpha(e^{\mathbf{i}\zeta})$. Then,

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Examples

Example: Let X be a high degree hypersurface of $\mathbb{P}^3$. Then, $\exists$ KE metric $\omega \in c_1(X)$ (in particular cscK) (Aubin & Yau). Moreover, $c_1(X) < 0 \Rightarrow \mathrm{Aut}(X)$ finite.

Let E be a smooth $SU(2)$-bundle over X with second Chern number $k = \frac{1}{8\pi^2}\int_X \operatorname{tr} F_A \wedge F_A \in \mathbb{Z}$, where A is a connection on E. If $k \gg 0$, the moduli space M_k of Anti-Self-Dual (ASD) connections A on E with respect to ω is non-empty, non-compact but admits a compactification.Let A be a connection that determines a point in M_k. Then, A is irreducible and so we can apply our Theorem obtaining solutions $(\omega_\alpha, A_\alpha)$ to (1) for small $0 < \alpha = \frac{\alpha_1}{\alpha_0}$.

How can we assure that ω_α is not cscK? Recall that the scalar equation in (1) is equivalent to $S_{\omega_\alpha} - \alpha|F_{A_\alpha}|^2 = const$. Since $(\omega_\alpha, A_\alpha) \to (\omega, A)$ uniformly as $\alpha \to 0$ it is enough to take A such that $|F_A|^2$ is not a constant function on X.Take A near to the boundary of the moduli space (bubbling). Can we make this argument explicit?Locally yes.

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Consider $\mathbb{C}^2 \times SU(2)$, the trivial bundle over $\mathbb{C}^2$. Let ω be the euclidean metric on $\mathbb{C}^2$ (Kähler) and consider the basic 1-instanton (in quaternionic notation $\mathbb{C}^2 \equiv \mathbb{H}$)

$$A = \mathrm{Im}\frac{\overline{x}dx}{1+|x|^2} = \frac{1}{2}\cdot\frac{\overline{x}dx - d\overline{x}x}{1+|x|^2},$$

where $x = x_1 + x_2 \cdot \mathbf{i} + x_3 \cdot \mathbf{j} + x_4 \cdot \mathbf{k}$, with curvature

$$F_A = \frac{d\overline{x}\wedge dx}{(1+|x|^2)^2}.$$

Then $|F_A|^2 = \frac{24}{(1+|x|^2)^4}$.

Theorem

Let $k \in \mathbb{Z}$. For each $\alpha \in \mathbb{R}$ there exists a solution $(\omega_\alpha, A_\alpha)$ of the coupled equations with coupling constant α and fixed topological invariant $k = \frac{1}{8\pi^2}\int_{\mathbb{C}^2} \mathrm{tr}\, F_A \wedge F_A \in \mathbb{Z}$. The metric ω_α is an assymptotically euclidean Kähler metric and for each α there exists a k-instanton A'_α, such that A_α converges assymptotically to A' at infinity.

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An algebro-geometric problem: **Construct a moduli space with a structure of variety or separated scheme**

$$(3)\left\{\begin{array}{c}\textbf{semiestable} \text{ pairs with `fixed invariants':} \\ \text{projective variety} + \text{bundle} \\ (\text{projective scheme} + \text{coherent sheaf})\end{array}\right\}\Big/\sim$$

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Strategy: the Kempf–Ness Theorem

G^c = complexification of a compact Lie group G,
V = representation of G^c,
$X \subset \mathbb{P}(V)$, projective variety, G^c-invariant.

$\exists$ a G-equivariant **moment map** $\mu\colon X \to (\mathrm{Lie}\, G)^*$
$\exists$ **linearization** of the G^c-action, i.e. $L = \mathcal{O}_X(1)$ is a G^c-bundle over X.

The Kempf-Ness Theorem tell us that for every $x \in X$:

x is GIT-stable $\iff$ *$\exists\, g \in G^c$ such that $\mu(g \cdot x) = 0$ and the G^c-stabilizer of x is finite.*

The stability of a point can be checked (Hilbert–Mumford) computing, for any $\lambda\colon \mathbb{C}^* \to G^c$,

$$\text{weight of the } \mathbb{C}^* - \text{action on} L_{|x_0} = \langle \mu(x_0), \zeta \rangle,$$

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α-K-stability

To apply the previous picture we have a problem : there exists no $\widetilde{\mathcal{G}}^c$.
Idea: consider finite dimensional 'approximations' of $\widetilde{\mathcal{G}}$, that can be always complexified (adapt Donaldson's arguments for the cscK problem to our problem).
Let (X, L) = smooth compact (complex) polarised manifold and E = vector bundle over X. Taking $k >> 0$, we can consider $X \subset \mathbb{P}(V_k)$, $V_k = H^0(X, L^k)^*$. Hence, X defines a point on Hilb^P, $P(k) = \chi(X, L^k)$.
There exists a proper scheme

$$\mathrm{Quot}^{P_E} \to \mathrm{Hilb}^P$$

which parametrises sheaves over the corresponding point on Hilb, with Hilbert polynomial $P_E(k) = \chi(X, E \otimes L^k)$. Let $W_k = H^0(X, E \times L^k)$. The group $G_k = GL(V_k) \times GL(W_k) \curvearrowright \mathrm{Quot}^{P_E}$ and for any $\lambda \colon \mathbb{C}^* \to G_k$

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We take (X_0, L_0, E_0) representing ϵ_0, endowed with a natural $\mathbb{C}^*$-action and measure a weight F_α.

α-K-stability

To apply the previous picture we have a problem : there exists no $\widetilde{\mathcal{G}}^c$.
Idea: consider finite dimensional ‘approximations’ of $\widetilde{\mathcal{G}}$, that can be always complexified (adapt Donaldson’s arguments for the cscK problem to our problem).
Let $(X, L) =$ smooth compact (complex) polarised manifold and $E =$ vector bundle over X. Taking $k >> 0$, we can consider $X \subset \mathbb{P}(V_k)$, $V_k = H^0(X, L^k)^*$. Hence, X defines a point on Hilb^P, $P(k) = \chi(X, L^k)$.
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$\mathbb{C}^* \curvearrowright (X_0, L_0, E_0)$:

$P_{L_0}(E_0) =$ Hilbert polynomial of E_0 with respect to L_0,
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α-**invariant** of the $\mathbb{C}^*$-action on (X_0, L_0, E_0):

$$F_\alpha(X_0, L_0, E_0) = F_1(L_0, \mathcal{O}_{X_0}) + \alpha\left(F_2(L_0, E_0) - F_2(L_0, \mathcal{O}_{X_0})\right)$$

Proposition [—, L. Álvarez Cónsul, O. García Prada]

If (X_0, L_0, E_0) is smooth then

$$F_\alpha(X_0, L_0, E_0) \sim \mu_\alpha(\zeta),$$

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Conjecture [—, L. Álvarez Cónsul, O. García Prada]

If there exists a solution (ω, A) to the coupled equations (1) with $\omega \in c_1(L)$ and positive coupling constants α_0 and α_1, then

$$F_\alpha(X_0, L_0, E_0) \geq 0,$$

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