Monopoles, Periods and Problems

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Bath 2010

Monopole Results in collaboration with V.Z. Enolskii, A.D'Avanzo.
Spectral curve programs with T.Northover.

Overview

Equations	$\overset{Zero\ \ Curvature/Lax}{\longrightarrow}$	Spectral Curve $\mathcal{C} \subset \mathcal{S}$
$\uparrow$		$\downarrow$
Reconstruction	$\longleftarrow$	Baker-Akhiezer Function

- BPS Monopoles
- Sigma Model reductions in AdS/CFT
- KP, KdV solitons
- Harmonic Maps
- SW Theory/Integrable Systems

Setting
BPS Monopoles

- Reduction of $F = *F$

$$L = -\frac{1}{2}\mathrm{Tr}\, F_{ij}F^{ij} + \mathrm{Tr}\, D_i\Phi\, D^i\Phi.$$

- $B_i = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk} F^{jk} = D_i\Phi$
- A *monopole* of charge n

$$\sqrt{-\frac{1}{2}\mathrm{Tr}\,\Phi(r)^2}\bigg|_{r\to\infty} \sim 1 - \frac{n}{2r} + O(r^{-2}), \quad r = \sqrt{x_1^2 + x_2^2 + x_3^2}$$

- Monopoles $\leftrightarrow$ Nahm Data $\leftrightarrow$ Hitchin Data

Setting

BPS Monopoles: Nahm Data for charge n $SU(2)$ monopoles

Three $n \times n$ matrices $T_i(s)$ with $s \in [0, 2]$ satisfying the following:

- N1 Nahm's equation $$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k].$$
- N2 $T_i(s)$ is regular for $s \in (0, 2)$ and has simple poles at $s = 0, 2$. Residues form $su(2)$ irreducible n-dimensional representation.
- N3 $T_i(s) = -T_i^{\dagger}(s), \qquad T_i(s) = T_i^{t}(2 - s).$

$$A(\zeta) = T_1 + iT_2 - 2iT_3\zeta + (T_1 - iT_2)\zeta^2$$
$$M(\zeta) = -iT_3 + (T_1 - iT_2)\zeta$$

Nahm's eqn. $$\frac{dT_i}{ds} = \frac{1}{2}\sum_{j,k=1}^{3} \epsilon_{ijk}[T_j, T_k] \Longleftrightarrow [\frac{d}{ds} + M, A] = 0.$$

Setting
Spectral Curve

- $[\frac{d}{ds} + M, A] = 0, \qquad \mathcal{C}:\ 0 = \det(\eta 1_n + A(\zeta)) := P(\eta, \zeta)$

  $$P(\eta,\zeta) = \eta^n + a_1(\zeta)\eta^{n-1} + \ldots + a_n(\zeta), \qquad \deg a_r(\zeta) \leq 2r$$

- Where does $\mathcal{C}$ lie? $\quad \mathcal{C} \subset \mathcal{S}$
  - $\mathcal{C}_{\text{monopole}} \subset T\mathbb{P}^1 := \mathcal{S} \qquad (\eta, \zeta) \to \eta \frac{d}{d\zeta} \in T\mathbb{P}^1$

    Minitwistor description
  - $\mathcal{C}_{\sigma-\text{model}} \subset \mathbb{P}^2 := \mathcal{S}$
  - $\mathcal{S} = T^*\Sigma$ Hitchin Systems on a Riemann surface Σ
  - $\mathcal{S} = K3$
  - $\mathcal{S}$ a Poisson surface
  - separation of variables $\leftrightarrow \mathrm{Hilb}^{[N]}(\mathcal{S})$
  - X the total space of an appropriate line bundle $\mathcal{L}$ over $\mathcal{S} \leftrightarrow$ noncompact CY
- genus given by Riemann Hurwitz formula $g_{\text{monopole}} = (n-1)^2$

- H1 Reality conditions $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})}$
- H2 L^λ denote the holomorphic line bundle on $T\mathbb{P}^1$ defined by the transition function $g_{01} = \exp(-\lambda\eta/\zeta)$
  $L^\lambda(m) \equiv L^\lambda \otimes \pi^* \mathcal{O}(m)$ be similarly defined in terms of the transition function $g_{01} = \zeta^m \exp\left(-\lambda\eta/\zeta\right)$.
  L^2 is trivial on $\mathcal{C}$ and $L^1(n-1)$ is real.
  L^2 is trivial $\implies$ $\exists$ nowhere-vanishing holomorphic section.
- H3 $H^0(\mathcal{C}, L^\lambda(n-2)) = 0$ for $\lambda \in (0, 2)$

$\mathcal{C}$ often comes with an antiholomorphic involution or real structure

- Reverse orientation of lines $(\eta, \zeta) \rightarrow (-\bar{\eta}/\bar{\zeta}^2, -1/\bar{\zeta})$
  $a_r(\zeta) = (-1)^r \zeta^{2r} \overline{a_r(-\frac{1}{\bar{\zeta}})} \Longrightarrow$
  $a_r(\zeta) = \chi_r \left[\prod_{l=1}^r \left(\frac{\overline{\alpha}_{r,l}}{\alpha_{r,l}}\right)^{1/2}\right] \prod_{k=1}^r (\zeta - \alpha_{r,k})(\zeta + \frac{1}{\overline{\alpha}_{r,k}})$
  $\alpha_{r,k} \in \mathbb{C}, \ \chi_r \in \mathbb{R}$, $a_r(\zeta)$ given by $2r+1$ (real) parameters
- reality constrains the form of the period matrix.
- there may be between 0 and $g+1$ ovals of fixed points of the antiholomorphic involution.
- Imposing reality can be one of the hardest steps.

- $SO(3)$ induces an action on $T\mathbb{P}^1$ via $PSU(2)$

$$\begin{pmatrix} p & q \\ -\bar{q} & \bar{p} \end{pmatrix} \in PSU(2), \qquad |p|^2 + |q|^2 = 1,$$

$$\zeta \to \frac{\bar{p}\,\zeta - \bar{q}}{q\,\zeta + p}, \qquad \eta \to \frac{\eta}{(q\,\zeta + p)^2}$$

- corresponds to a rotation by θ around $\mathbf{n} \in S^2$

$$n_1 \sin(\theta/2) = \mathrm{Im}\, q, \qquad n_2 \sin(\theta/2) = -\mathrm{Re}\, q,$$
$$n_3 \sin(\theta/2) = \mathrm{Im}\, p, \qquad \cos(\theta/2) = -\mathrm{Re}\, p.$$

- Invariant curves yield symmetric monopoles.

Spectral Curves

Extrinsic Properties: Example of Cyclically Symmetric Monopoles

- $\omega = \exp(2\pi i/n)$, $\bar{p} = \omega^{1/2}$ $q = 0$ $\quad (\eta, \zeta) \to (\omega\eta, \omega\zeta)$
  $\eta^i \zeta^j$ invariant for $i + j \equiv 0 \mod n$

$$\eta^n + a_1 \eta^{n-1}\zeta + a_2 \eta^{n-2}\zeta^2 + \ldots + a_n \zeta^n + \beta\zeta^{2n} + \gamma = 0$$

- Impose reality conditions and centre $a_1 = 0$

$$\eta^n + a_2 \eta^{n-2}\zeta^2 + \ldots + a_n \zeta^n + \beta\zeta^{2n} + (-1)^n \bar{\beta} = 0, \qquad a_i \in \mathbf{R}$$

  By an overall rotation we may choose β real

- $x = \eta/\zeta$, $\nu = \zeta^n \beta$,

$$x^n + a_2 x^{n-2} + \ldots + a_n + \nu + \frac{(-1)^n |\beta|^2}{\nu} = 0$$

- Affine Toda Spectral Curve $\; y = \nu - \frac{(-1)^n |\beta|^2}{\nu}$

$$y^2 = (x^n + a_2 x^{n-2} + \ldots + a_n)^2 - 4(-1)^n |\beta|^2$$

- Meromorphic differentials describe flows
- L^2 trivial $\Longrightarrow f_0(\eta,\zeta) = \exp\left\{-2\frac{\eta}{\zeta}\right\} f_1(\eta,\zeta)$

$$\mathrm{dlog}\, f_0 = \mathrm{d}\left(-2\frac{\eta}{\zeta}\right) + \mathrm{dlog} f_1, \quad \exp \oint_\lambda \mathrm{dlog}\, f_0 = 1 \quad \forall \lambda \in H_1(\mathcal{C},\mathbb{Z})$$

- $\{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^g$ basis for $H_1(\mathcal{C},\mathbb{Z})$: $\mathfrak{a}_i \cap \mathfrak{b}_j = -\mathfrak{b}_j \cap \mathfrak{a}_i = \delta_{ij}$
- $\gamma_\infty(P) = \frac{1}{2}\,\mathrm{dlog}\, f_0(P) + \imath\pi \sum_{j=1}^{g} m_j\, v_j(P), \qquad \oint_{\mathfrak{a}_k} v_j = \delta_{jk}$

$$2\pi\imath\mathbf{U} = \oint_{\mathfrak{b}_k} \gamma_\infty = \imath\pi n_k + \imath\pi \sum_{l=1}^{g} m_l \tau_{lk}, \qquad 2\mathbf{U} \in \Lambda$$

- **H3**
  $H^0\left(\mathcal{C}, \mathcal{O}(L^s(n-2))\right) = 0 \Rightarrow H^0\left(\mathcal{C}, \mathcal{O}(L^s)\right) = 0,\ s \in (0,2).$
  $(\mathcal{O}(L^s) \hookrightarrow \mathcal{O}(L^s(n-2))\ \times$ a section of $\pi^*\mathcal{O}(n-2)|_{\mathcal{C}})$
  L^s trivial $\Longleftrightarrow s\,\mathbf{U} \in \Lambda, \qquad 2\mathbf{U}$ is a primitive vector in Λ

- Ercolani-Sinha Constraints: The following are equivalent:
  1. L^2 is trivial on $\mathcal{C}$.
  2. $2\mathbf{U} \in \Lambda \Longleftrightarrow \mathbf{U} = \frac{1}{2\pi\imath}\left(\oint_{\mathfrak{b}_1}\gamma_\infty, \ldots, \oint_{\mathfrak{b}_g}\gamma_\infty\right)^T = \frac{1}{2}\mathbf{n} + \frac{1}{2}\tau\mathbf{m}.$
  3. There exists a 1-cycle $\mathfrak{c} = \mathbf{n}\cdot\mathfrak{a} + \mathbf{m}\cdot\mathfrak{b}$ such that for every holomorphic differential
     $$\Omega = \frac{\beta_0\eta^{n-2} + \beta_1(\zeta)\eta^{n-3} + \ldots + \beta_{n-2}(\zeta)}{\frac{\partial\mathcal{P}}{\partial\eta}}\, d\zeta, \ \oint_c \Omega = -2\beta_0$$
- ES constraints impose g *transcendental constraints* on curve

$$\sum_{j=2}^{n}(2j+1) - g = (n+3)(n-1) - (n-1)^2 = 4(n-1)$$

- $H^0(\mathcal{C}, L^\lambda(n-2)) \neq 0 \Longleftrightarrow \theta(\lambda\mathbf{U} - \widetilde{\mathbf{K}}\,|\,\tau) = 0$ where $\widetilde{\mathbf{K}} = \mathbf{K} + \phi\left((n-2)\sum_{k=1}^{n}\infty_k\right)$, $\mathbf{K}$ vector of Riemann constants

Cyclic Monopoles and Toda

New Results

- $\mathcal{C}_{\rm monopole}$ is an unbranched $n:1$ cover $\mathcal{C}_{\rm Toda}$
  $g_{\rm monopole} = (n-1)^2$, $g_{\rm Toda} = (n-1)$
- Sutcliffe: Ansatz for Nahm's equations for cyclic monopoles in terms of Affine Toda equation.
  Cyclic Nahm eqns. $\supset$ Affine Toda eqns.
- Cyclic Nahm eqns. $\equiv$ Affine Toda eqns.
- Cyclic monopoles $\equiv$ (particular) Affine Toda solns.
- Implementation in terms of curves, period matrices, theta functions etc.

$$T_1 + iT_2 = (T_1 - iT_2)^T$$
$$= \begin{pmatrix} 0 & e^{(q_1-q_2)/2} & 0 & \dots & 0 \\ 0 & 0 & e^{(q_2-q_3)/2} & \dots & 0 \\ \vdots & & & \ddots & \vdots \\ 0 & 0 & 0 & \dots & e^{(q_{n-1}-q_n)/2} \\ e^{(q_n-q_1)/2} & 0 & 0 & \dots & 0 \end{pmatrix}$$
$$T_3 = -\frac{i}{2}\,\mathrm{Diag}(p_1, p_2, \dots, p_n)$$

$$\frac{d}{ds}(T_1 + iT_2) = i[T_3, T_1 + iT_2] \Rightarrow p_i - p_{i+1} = \dot{q}_i - \dot{q}_{i+1}$$
$$\frac{d}{ds}T_3 = [T_1, T_2] = \frac{i}{2}[T_1 + iT_2, T_1 - iT_2] \Rightarrow \dot{p}_i = -e^{q_i - q_{i+1}} + e^{q_{i-1}-q_i}$$

- p_i, q_i real

$$H = \frac{1}{2}\left(p_1^2 + \ldots + p_n^2\right) - \left[e^{q_1 - q_2} + e^{q_2 - q_3} + \ldots + e^{q_n - q_1}\right].$$

Toda $\Rightarrow$ Nahm   Affine Toda eqns. $\subset$ Cyclic Nahm eqns.

- $G \subset SO(3)$ acts on triples $\mathbf{t} = (T_1, T_2, T_3) \in \mathbb{R}^3 \otimes SL(n, \mathbb{C})$ via natural action on $\mathbb{R}^3$ and conjugation on $SL(n, \mathbb{C})$
- $g' \in SO(3)$ and $g = \rho(g') \in SL(n, \mathbb{C})$. Invariance of curve $\Rightarrow$

$$g(T_1 + iT_2)g^{-1} = \omega(T_1 + iT_2),$$
$$gT_3g^{-1} = T_3,$$
$$g(T_1 - iT_2)g^{-1} = \omega^{-1}(T_1 - iT_2).$$

Cyclic Monopoles and Toda

Cyclic Nahm eqns. $\equiv$ Affine Toda eqns: New Result I

- $SL(n,\mathbb{C}) \sim \underline{2n-1} \oplus \underline{2n-3} \oplus \ldots \oplus \underline{5} \oplus \underline{3}$
- Kostant $\Rightarrow$ $\rho(SO(3))$ principal three dimensional subgroup.
- $g = \rho(g') = \exp\left[\frac{2\pi}{n} H\right]$, H semi-simple, generator Cartan TDS
- $g \equiv \mathrm{Diag}(\omega^{n-1}, \ldots, \omega, 1)$, $gE_{ij}g^{-1} = \omega^{j-i}\, E_{ij}$.
- For a cyclically invariant monopole

$$T_1 + iT_2 = \sum_{\alpha \in \hat{\Delta}} e^{(\alpha,\tilde{q})/2}\, E_\alpha, \qquad T_3 = -\frac{i}{2} \sum_j \tilde{p}_j\, H_j$$

- Sutcliffe follows if $\tilde{q}_i$ and $\tilde{p}_i$ may be chosen real.
  $\tilde{q}_i \in \mathbb{R}$ with $SU(n)$ conjug. $+$ overall $SO(3)$ rotation.
  $\tilde{p}_i \in \mathbb{R}$ from $T_i(s) = -T_i^\dagger(s)$ which also fixes $T_1 - iT_2$.
- Any cyclically symmetric monopole is gauge equivalent to Nahm data given by Sutcliffe's ansatz, and so obtained from the affine Toda equations.

Theorem

The Ercolani-Sinha vector is invariant under the group of symmetries of the spectral curve arising from rotations.

- $\pi : \mathcal{C}_{\text{monopole}} \to \mathcal{C}_{\text{Toda}}$

$$\text{Jac}(\mathcal{C}_{\text{monopole}}) = \pi^* \, \text{Jac}(\mathcal{C}_{\text{Toda}}) + \text{Prym}$$

- $\mathbf{U} = \pi^* \mathbf{u}$
- $\widetilde{\mathbf{K}} \in \Theta_{\text{singular}} \subset \text{Jac}(\mathcal{C}_{\text{monopole}})$, $\quad 2\widetilde{\mathbf{K}} \in \Lambda$, $\quad \widetilde{\mathbf{K}} = \pi^* \mathbf{e}_1$
- Fay-Accola

$$\theta[\widetilde{\mathbf{K}}](\pi^* z;\, \tau_{\text{monopole}}) = c \prod_{i=1}^{n} \theta[\mathbf{e}_i](z;\, \tau_{\text{Toda}})$$

"*θ-functions are still far from being a spectator sport.*"(L.V. Ahlfors)

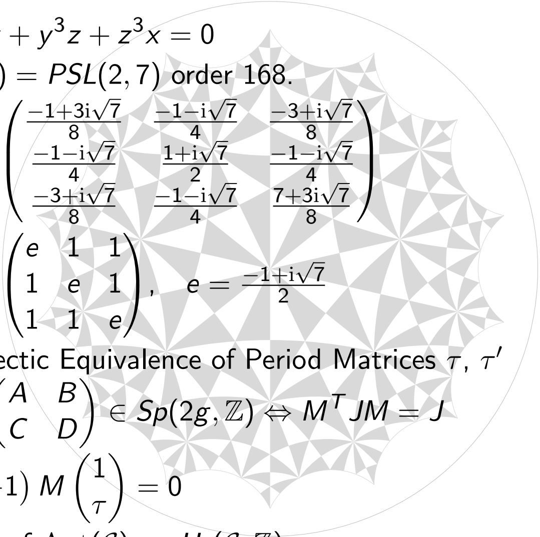
Curves

Fundamental Ingredients

- Homology basis $\{\mathfrak{a}_i, \mathfrak{b}_i\}_{i=1}^{g}$
  - algorithm for branched covers of $\mathbb{P}^1$ (Tretkoff & Tretkoff)
  - poor if curve has symmetries
- Holomorphic differentials du_i $(i = 1, \ldots, g)$
- Period Matrix $\tau = BA^{-1}$ where $\begin{pmatrix} A \\ B \end{pmatrix} = \begin{pmatrix} \oint_{\mathfrak{a}_i} du_j \\ \oint_{\mathfrak{b}_i} du_j \end{pmatrix}$
  - normalized holomorphic differentials ω_i, $\oint_{\mathfrak{a}_i} \omega_j = \delta_{ij}$ $\oint_{\mathfrak{b}_i} \omega_j = \tau_{ij}$
  - curves with lots of symmetries: evaluate τ via character theory

$$w^2 = z^{2g+2} - 1 \ (D_{2g+2}),\ \ w^2 = z(z^{2g+1} - 1) \ (C_{2g+1})$$

  - Principle (Kontsevich, Zagier): *Whenever you meet a new number, and have decided (or convinced yourself) that it is transcendental, try to figure out whether it is a period*



- $\mathcal{C}$: $x^3y + y^3z + z^3x = 0$
- $\mathrm{Aut}(\mathcal{C}) = PSL(2,7)$ order 168.
- $\tau_{RL} = \begin{pmatrix} \frac{-1+3\mathrm{i}\sqrt{7}}{8} & \frac{-1-\mathrm{i}\sqrt{7}}{4} & \frac{-3+\mathrm{i}\sqrt{7}}{8} \\ \frac{-1-\mathrm{i}\sqrt{7}}{4} & \frac{1+\mathrm{i}\sqrt{7}}{2} & \frac{-1-\mathrm{i}\sqrt{7}}{4} \\ \frac{-3+\mathrm{i}\sqrt{7}}{8} & \frac{-1-\mathrm{i}\sqrt{7}}{4} & \frac{7+3\mathrm{i}\sqrt{7}}{8} \end{pmatrix}$
- $\tau = \frac{1}{2}\begin{pmatrix} e & 1 & 1 \\ 1 & e & 1 \\ 1 & 1 & e \end{pmatrix}, \quad e = \frac{-1+\mathrm{i}\sqrt{7}}{2}$
- Symplectic Equivalence of Period Matrices τ, τ'

  $M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2g,\mathbb{Z}) \Leftrightarrow M^T JM = J$

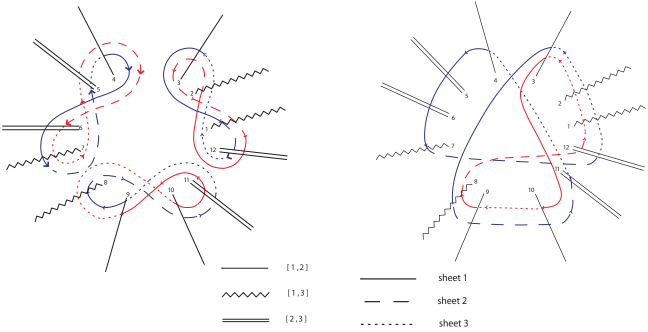
  $\begin{pmatrix} \tau' & -1 \end{pmatrix} M \begin{pmatrix} 1 \\ \tau \end{pmatrix} = 0$
- Action of $\mathrm{Aut}(\mathcal{C})$ on $H_1(\mathcal{C},\mathbb{Z})$

The spectral curve of genus 4

$$w^3 + \alpha w z^2 + \beta z^6 + \gamma z^3 - \beta = 0$$

$$\tau_{\hat{\mathcal{C}}_{\text{monopole}}} = \begin{pmatrix} a & b & b & b \\ b & c & d & d \\ b & d & c & d \\ b & d & d & c \end{pmatrix}$$

$$\begin{aligned} \sigma_*^k(\mathfrak{a}_i) &= \mathfrak{a}_{i+k} \\ \sigma_*^k(\mathfrak{b}_i) &= \mathfrak{b}_{i+k} \\ \sigma_*^k(\mathfrak{a}_0) &= \mathfrak{a}_0 \\ \sigma_*^k(\mathfrak{b}_0) &\sim \mathfrak{b}_0 \end{aligned}$$



The spectral curve of genus 2

$$y^2 = (x^3 + \alpha x - 2\imath\beta + \gamma)(x^3 + \alpha x + 2\imath\beta + \gamma)$$

$$\tau = \begin{pmatrix} \frac{a}{3} & b \\ b & c + 2d \end{pmatrix}$$

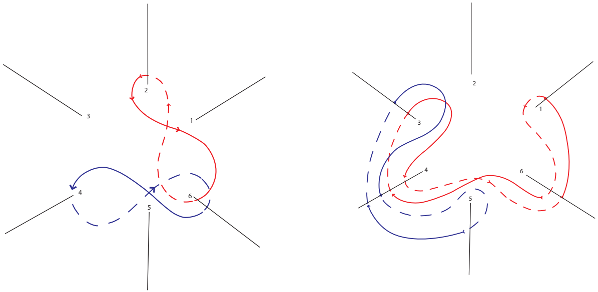


Figure: Projection of the previous basis

The Humbert Variety

τ the period matrix of a genus 2 curve $\mathcal{C}$.

- $\tau \in \mathcal{H}_\Delta$ if there exist $q_i \in \mathbb{Z}$
  $q_1 + q_2\tau_{11} + q_3\tau_{12} + q_4\tau_{22} + q_5(\tau_{12}^2 - \tau_{11}\tau_{22}) = 0$
  $q_3^2 - 4(q_1 q_5 + q_2 q_4) = \Delta$
- $\mathcal{C}$ covers elliptic curves $\mathcal{E}_\pm \Leftrightarrow \Delta = h^2 \geq 1,\ h \in \mathbb{N}$.
- Bierman-Humbert: $\tau \in \mathcal{H}_{h^2} \Rightarrow \exists\, \mathfrak{S} \in \mathrm{Sp}(4,\mathbb{Z})$, such that
  $\mathfrak{S} \circ \tau = \widetilde{\tau} = \begin{pmatrix} \widetilde{\tau}_{11} & \frac{1}{h} \\ \frac{1}{h} & \widetilde{\tau}_{22} \end{pmatrix}$
- $$\theta(z_1, z_2 \,|\, \widetilde{\tau}) = \sum_{k=0}^{h-1} \vartheta_3\left(z_1 + \frac{k}{h} \,\Big|\, \widetilde{\tau}_{1,1}\right) \theta\begin{bmatrix} \frac{k}{h} \\ 0 \end{bmatrix}\left(hz_2 \,|\, h^2\widetilde{\tau}_{2,2}\right)$$
- $$\theta(z_1, z_2 \,|\, \widetilde{\tau}) = \vartheta_3\left(z_1 \,|\, \widetilde{\tau}_{11}\right)\vartheta_3\left(2z_2 \,|\, 4\widetilde{\tau}_{22}\right) + \vartheta_3\left(z_1 + 1/2 \,|\, \widetilde{\tau}_{11}\right)\vartheta_2\left(2z_2 \,|\, 4\widetilde{\tau}_{22}\right)$$

The Symmetric Monopole

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \qquad b \in \mathbb{R}$$

Theorem

To each pair of relatively prime integers $(n, m) = 1$ for which $(m+n)(m-2n) < 0$ we obtain a solution to the Ercolani-Sinha constraints for the symmetric curve as follows. First we solve for t, where

$$\frac{2n-m}{m+n} = \frac{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, t)}{{}_2F_1(\frac{1}{3}, \frac{2}{3}; 1, 1-t)}.$$

Then $b = \dfrac{1-2t}{\sqrt{t(1-t)}}$, $t = \dfrac{-b+\sqrt{b^2+4}}{2\sqrt{b^2+4}}$. *With* $\alpha^6 = t/(1-t)$ *then* $\chi^{\frac{1}{3}} = -(n+m)\,\dfrac{2\pi}{3\sqrt{3}}\,\dfrac{\alpha}{(1+\alpha^6)^{\frac{1}{3}}}\,{}_2F_1(\dfrac{1}{3}, \dfrac{2}{3}; 1, t)$.

The Symmetric Monopole

$$\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \qquad b \in \mathbb{R}$$

satisfies **H1** and **H2** $\Leftrightarrow \exists\ n, m\ (n, m) = 1\ (m+n)(m-2n) < 0$

$$b = b(m,n) = -\frac{\sqrt{3}(p(m,n)^6 - 45p(m,n)^4 + 135p(m,n)^2 - 27)}{9p(m,n)(p(m,n)^4 - 10p(m,n)^2 + 9)}$$

$$p(m,n) = \frac{3\vartheta_3^2\left(0|\frac{\mathcal{T}(m,n)}{2}\right)}{\vartheta_3^2\left(0|\frac{\mathcal{T}(m,n)}{6}\right)}, \qquad \mathcal{T}(m,n) = 2\imath\sqrt{3}\frac{n+m}{2n-m}$$

Expression for $\chi = \chi(m,n)$ can be given.

The Symmetric Monopole and H3

$\mathcal{C}_{\rm monopole} : \eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0, \quad \mathcal{C}_{\rm Toda} : y^2 = (x^3 + b)^2 + 4$

H3 $H^0(\mathcal{C}_{\rm monopole}, L^\lambda(n-2)) = 0$ for $\lambda \in (0,2)$

- $\theta(\lambda \mathbf{U} - \widetilde{\mathbf{K}};\, \tau_{\rm monopole}) \neq 0$ for $\lambda \in (0,2)$
  $\theta[\mathbf{e}_i](\lambda \mathbf{u};\, \tau_{\rm Toda}) \neq 0$ for $\lambda \in (0,2)$
- Bierman-Humbert+Weierstrass-Poincáre+Martens
  $\theta(\lambda \mathbf{U} - \widetilde{\mathbf{K}};\, \tau_{\rm monopole}) = 0$ for $\lambda \in [0,2] \Leftrightarrow$ at least one of the functions $(k = -1, 0, 1 \mod 3)$

$$h_k(y) := \frac{\vartheta_3}{\vartheta_2}\left(i\sqrt{3}\,y + \frac{k\,\mathcal{T}}{3}\,\Big|\,\mathcal{T}\right) + (-1)^k\,\frac{\vartheta_2}{\vartheta_3}\left(y + \frac{k}{3}\,\Big|\,\frac{\mathcal{T}}{3}\right)$$

also vanishes. $y := y(\lambda) = \lambda\,(n+m)\rho/3, \quad \mathcal{T} = 2i\sqrt{3}\dfrac{n+m}{2n-m}$
$\rho = \exp(2\pi\imath/3)$

An Elliptic function Conjecture and the Tetrahedral Monopole

$$h_k(y) := \frac{\vartheta_3}{\vartheta_2}\left(i\sqrt{3}\,y + \frac{k\,\mathcal{T}}{3}\,\Big|\,\mathcal{T}\right) + (-1)^k\,\frac{\vartheta_2}{\vartheta_3}\left(y + \frac{k}{3}\,|\,\frac{\mathcal{T}}{3}\right)$$

$$y := y(\lambda) = \lambda\,(n+m)\rho/3, \quad \mathcal{T} = 2i\sqrt{3}\frac{n+m}{2n-m}$$

- **Conjecture** $h_{-1}(y)h_0(y)h_1(y)$ vanishes $2(|n|-1)$ times on the interval $\lambda \in (0,2)$
- $\dfrac{\vartheta_3\left(\frac{\tau}{3}\mid\tau\right)}{\vartheta_2\left(\frac{\tau}{3}\mid\tau\right)} = \dfrac{\vartheta_2\left(\frac{1}{3}\mid\frac{\tau}{3}\right)}{\vartheta_3\left(\frac{1}{3}\mid\frac{\tau}{3}\right)}$
- **Theorem** Only $(m,n) = (1,1)$ and $(0,1)$ have no zeros within the range.
- **Theorem** The only curves $\eta^3 + \chi(\zeta^6 + b\zeta^3 - 1) = 0$ that yield BPS monopoles have $b = \pm 5\sqrt{2}$, $\chi^{\frac{1}{3}} = -\frac{1}{6}\,\frac{\Gamma(\frac{1}{6})\Gamma(\frac{1}{3})}{2^{\frac{1}{6}}\,\pi^{\frac{1}{2}}}$. These correspond to tetrahedrally symmetric monopoles.