Instantons and Holomorphic Bundles on the Blown-Up Plane)

by Alastair D. King, Worcester College, Oxford.

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Contents

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Introduction

The primary aim of this thesis is to describe a small piece of an expanding story about geometry in four dimensions, which brings together ideas from both mathematical physics and algebraic geometry. From physics comes a special class of soliton-type solutions to the Euclidean Yang-Mills equations. These 'instantons' are the vacuum (i.e. minimum energy) solutions of the equations. In algebraic geometry, one studies holomorphic vector bundles over algebraic varieties. It is natural to try to construct moduli spaces of such bundles and, in doing so, one is lead to the consideration of bundles which are stable, in a suitable sense.

The physics is related to differential geometry because the Yang-Mills equations and the instanton condition can be interpreted, over any Riemannian 4-manifold, as curvature conditions on a unitary connection on a vector bundle. Indeed, they can be so interpreted in all dimensions, but they are particularly natural in four dimensions, where they depend only on the conformal class of the Riemannian metric and the instanton condition is equivalent to the self-duality or anti-self-duality condition. It is by studying the moduli spaces of such instantons that Donaldson has proved some striking new results on the differential topology of 4-manifolds (see **[D4]**).

The link with algebraic geometry is provided by the fact that, while a connection is determined a covariant derivative ∇, a holomorphic structure is determined by a $\overline{\partial}$ operator, which is, in a suitable sense, half the covariant derivative. Over a 4-manifold carrying compatible complex and Riemannian structures, the anti-self-duality condition on ∇ includes the condition that the associated $\overline{\partial}$ operator defines a holomorphic structure. It is then natural to ask which holomorphic structures are associated to anti-self-dual instantons. More precisely, since the notion of equivalence for instantons is stronger than that for holomorphic bundles, we can ask about the map $\mathsf{h} : \mathsf{MI} \to \mathsf{MH}$ from the instanton moduli space (on a fixed topological vector bundle) to the corresponding holomorphic bundle moduli space, induced by the association $\nabla \mapsto \overline{\partial}$. It is a special case of a conjecture of Hitchin and Kobayashi that, over a compact complex surface, h should be a bijection, when MH is taken to be the moduli space of (Mumford-Takemoto) stable holomorphic bundles. This was proved by Donaldson **[D3]** for projective algebraic surfaces, by Uhlenbeck & Yau **[UY]** for general Kähler manifolds and by Buchdahl **[Bu3]** for arbitrary surfaces, equipped with a Gauduchon metric ($\partial\overline{\partial}\omega = 0$). It is well known that, in the presence of a hermitian structure on the vector bundle, $\overline{\partial}$ determines ∇ by the requirement that it be compatible (i.e. has $\overline{\partial}$ as one half) and unitary. Thus, the Hitchin-Kobayashi conjecture can be interpreted as saying that a holomorphic structure is stable if and only if it admits a hermitian structure which determines an anti-self-dual instanton connection. Furthermore, when such a hermitian structure exists, it is unique.

Another version of the Hitchin-Kobayashi conjecture, also proved by Donaldson **[D1]**, is for the (non-compact) affine plane $\mathbb{C}^2$ with, as a Kähler metric, the flat Euclidean metric. Here the result is basically the same as the one just stated for a compact surface, but with two differences. Firstly, the condition that the holomorphic bundle be stable is replaced by the requirement that it has associated to it an extension to $\mathbb{C}\mathrm{P}^2$, which is trivial when restricted to the line added in compactifing $\mathbb{C}^2$. Secondly, the hermitian structure that determines an instanton connection is only unique once it is specified at infinity. These two differences are handled simultaneously by taking MI to be the moduli space of instantons on $S^4(=\mathbb{R}^4\cup\{\infty\})$, with a unitary framing at ∞, while MH is the moduli space of holomorphic bundles on $\mathbb{C}\mathrm{P}^2(=\mathbb{C}^2\cup\ell_\infty)$, with a holomorphic framing along ℓ_∞. Then, $\mathfrak{h}:\mathsf{MI}\to\mathsf{MH}$ is a bijection, as before.

In this thesis, we shall prove a second example of this type of non-compact Hitchin-Kobayashi correspondence. In this case it will be over $\widetilde{\mathbb{C}}^2$, the affine plane blown up at the origin. This carries a complete conformally anti-self-dual Kähler metric, which has, as a conformal compactification, the Fubini-Study metric on the oppositely oriented projective plane $\overline{\mathbb{C}\mathrm{P}}^2=\widetilde{\mathbb{C}}^2\cup\{\infty\}$. The smallest complex compactification is the first Hirzebruch surface $\Sigma^1=\widetilde{\mathbb{C}}^2\cup\ell_\infty$. Thus, the main result that we shall prove is

Theorem. If MI is the moduli space of anti-self-dual instantons, framed at ∞, on a fixed hermitian vector bundle $\mathcal{E}$ over $\overline{\mathbb{C}\mathrm{P}}^2$ (with $c_1(\mathcal{E})=0$) and MH is the moduli space of holomorphic structures on the pullback of $\mathcal{E}$ to Σ^1, framed along ℓ_∞, then the canonical map $\mathfrak{h}:\mathsf{MI}\to\mathsf{MH}$ is a bijection.

To describe MI, we use the Euclidean version of Penrose's twistor theory, which, in general, converts solutions of differential equations into holomorphic objects. In our case, this relies on the fact that the Fubini-Study metric on $\overline{\mathbb{C}\mathrm{P}}^2$ is conformally anti-self-dual and, further, has a complex algebraic twistor space, the flag manifold $\mathsf{F}=\mathsf{F}(\mathbb{C}^3)$. The Ward correspondence **[Wa]** is used to describe instantons on $\overline{\mathbb{C}\mathrm{P}}^2$ in terms of holomorphic bundles on F and the map $\mathfrak{h}:\mathsf{MI}\to\mathsf{MH}$ is induced by restriction to a copy of Σ^1 sitting as a hypersurface in F. The relevant moduli spaces of holomorphic bundles over F and Σ^1 are described using monads in special canonical forms determined by certain linear algebra data. The equivalence of the two spaces follows from a general equivalence between symplectic and algebraic quotients.

The fact that this version of the Hitchin-Kobayashi correspondence holds for a second non-compact Kähler surface gives us the hope of generalising it much further. Two particular families of non-compact surfaces which could also be studied, using some of the techiniques employed in this work, are the total spaces of $\mathcal{O}_{\mathbb{P}^1}(-n)$ and the blow-ups of $\mathbb{C}^2$ in n points, both of which have $\widetilde{\mathbb{C}}^2$ as the $n=1$ case. LeBrun has recently shown that the former family admits conformally anti-self-dual Kähler metrics **[LeB1]** and that the latter family does also for $n=2,3$ **[LeB2]**. An intriguing feature of the above proof is that the method of proving the compact version of the correspondence is essentially an infinite dimensional version of the same symplectic/algebraic equivalence. The suggestion is that the non-compact version of the Hitchin-Kobayashi correspondence should be able to be formulated and proved, using analytic techniques, over a large class of non-compact surfaces and that the tranformation into linear algebra data is a sort of Nahm transform which works for a special class of surfaces and preserves the geometric form of the correspondence.

A second interesting feature of the Hitchin-Kobayashi correspondence over, say, a projective algebraic (hence Kähler) surface is that the two moduli spaces have, a priori, quite different features. In particular, MI has a Riemannian metric (induced from the L^2 metric on the space of all connections) and a natural compactification in a space of 'ideal instantons'. On the other hand, **MH** has a complex algebraic structure and a natural compactification in the space of torsion-free sheaves. The relationship between these features is, firstly, that the Riemannian and complex structures are compatible and actually give the moduli space a Kähler structure (as is true over non-algebraic Kähler surfaces as well [It]). Secondly, Maruyama [Ma] has shown, at least for rational surfaces, that one can find a smaller algebraic compactification by taking universal extensions of sheaves and that this gives the same space as the ideal instanton compactification.

In terms of the explicit linear algebra data descriptions of the moduli spaces, we shall show that similar results hold for the moduli spaces **MI** and **MH** over $\mathbb{C}^2$ and $\widetilde{\mathbb{C}}^2$. Thus, we shall see that the moduli spaces are complex algebraic Kähler manifolds and that they have completions — not compactifications, since the underlying manifold is not compact — in both the metric and algebraic sense, which are identical and can be interpreted in terms of ideal instantons. We note that the Kähler metric we construct is not necessarily the same as the L^2 metric, but that there are indications that it in fact is.

Finally, we shall show that there is an explicit algebraic map from the completed moduli space over $\widetilde{\mathbb{C}}^2$ to the completed moduli space over $\mathbb{C}^2$, which should be interpreted as the direct image map. We formulate the conjecture that this map is a blow-up and verify this for the simplest moduli spaces (for bundles with $c_2(\mathcal{E}) = 1$). This conjecture is a local paradigm for a general conjecture, due to Peter Kronheimer, that the completed moduli space of instantons/holomorphic bundles over a blown-up compact complex surface should be the blow-up of the corresponding moduli space over the original surface.

The layout of the material is as follows. In §1, we set the scene and introduce the two compactifications of $\widetilde{\mathbb{C}}^2$: the complex one, Σ^1, and the conformal one, $\overline{\mathbb{CP}}^2$, along with its twistor space F. We also discuss the use of moment maps and the notion of analytic stability, which provide the key ingredients in proving the main correspondence theorem. In §2, we describe the method, introduced by Horrocks [Ho], of using monads to classify holomorphic bundles and we show how they reflect the jumping behaviour of these bundles. In §3, we quote Buchdahl's monad description [Bu1] of the holomorphic bundles on F which correspond to instantons on $\overline{\mathbb{CP}}^2$, and derive a similar description of holomorphic bundles on Σ^1. We show that both these monad descriptions can be reduced to give a description of the moduli spaces of instantons and holomorphic bundles on $\widetilde{\mathbb{C}}^2$ as quotients of two closely related spaces of linear maps. We also show that this linear algebra data, describing the holomorphic bundles, has a natural cohomological interpretation. In §4, we show how these descriptions of the moduli spaces relate to the general theory of analytic stability and moment maps introduced earlier. We are thus able to prove the Hitchin-Kobayashi correspondence over $\widetilde{\mathbb{C}}^2$ as a particular case of the equivalence of algebraic and symplectic quotients and also provide the moduli space with the structure of a Kähler manifold. In §5, we describe the metric and algebraic completions of the moduli spaces over $\widetilde{\mathbb{C}}^2$ and $\mathbb{C}^2$, showing that they are the same and have an 'ideal instanton' interpretation. In §6, we describe the 'direct image' map between the moduli spaces and verify the conjecture that it is a blow-up in the case of the index one moduli spaces.

1 Basic Material

1.1 Geometry in Four Dimensions

In two real dimensions Riemannian geometry and complex geometry are very closely related. In fact, having a complex structure on a (real) surface is the same as having a conformal structure and an orientation. In higher dimensions this is no longer the case, but there are still some strong links, which give four dimensional geometry a special flavour, as we shall now describe. For a more detailed exposition, see [**AHS**],[**Sa**].

Let M, g be an oriented Riemannian 4-manifold. One important feature of four dimensions is that the Hodge star operator acts on $\Lambda^2(M)$, the middle bundle in the de Rham complex, as an involution, i.e. $*^2 = 1$, and thus defines a splitting $\Lambda^2(M) = \Lambda^2_+(M) \oplus \Lambda^2_-(M)$. As $\Lambda^2(M)$ is the bundle associated to the adjoint representation of $SO(4)$, this fact can also be understood in terms of the splitting $so(4) \cong su(2) \oplus su(2)$, which is a phenomenon unique to four dimensions, i.e. $so(n)$ is indecomposable for all other n.

One immediate consequence of this splitting is that the Riemannian curvature of M, which is a section of $S^2\left(\Lambda^2(M)\right)$, can be split up as

$$R = \begin{pmatrix} R_{++} & R_{+-} \\ R_{+-} & R_{--} \end{pmatrix}$$

In this decomposition, R_{+-} can be identified with the trace-free Ricci curvature and both $\operatorname{tr} R_{++}$ and $\operatorname{tr} R_{--}$ with the scalar curvature. The trace free parts of R_{++} and R_{--} make up the Weyl curvature W. In particular, this shows how the Weyl curvature splits ($W = W_+ + W_-$) in four dimensions, which does not happen in higher dimensions.

Now consider the space of possible complex structures J on TM which preserve the metric and are compatible with the orientation. We associate to each J a two-form $\omega(\bullet,\bullet) = g(\bullet, J\bullet)$ and this identifies the bundle of compatible complex structures $Z(M)$ with the sphere bundle in $\Lambda^2_+(M)$. The fibres of $Z(M)$ are naturally copies of the Riemann sphere and, at a point in a fibre, the horizontal subspace, defined using the Levi-Civita connection on $\Lambda^2_+(M)$, has a tautological complex structure. Thus we get an almost complex structure on $Z(M)$, the *twistor space* of M.

If one is now allowed to conformally rescale the metric g, then the above story does not change very much. The Hodge star operator is conformally invariant in the middle dimension and so the decomposition $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ remains the same. The compatible complex structures remain the same, but we now see that $Z(M)$ should be more naturally identified with the set of rays from the origin in Λ^2_+. The almost complex structure on $Z(M)$ is also conformally invariant and the obstruction to its integrability is, in fact, given simply by W_+. Thus a Riemannian manifold which is *conformally anti-self-dual*, i.e. has $W_+ = 0$, has a twistor space which is a three dimensional complex manifold. The twistor space also has a real structure, i.e. an anti-holomorphic involution, which is given in each fibre by taking any complex structure J to its conjugate structure $-J$. Geometrically, this is the antipodal map on the Riemann sphere.

The twistor space is of most use when it is actually an algebraic variety. However, Hitchin [**Hi1**] has shown that the only compact manifolds with projective algebraic (or, even, Kähler) twistor spaces are S^4 and $\overline{\mathbb{CP}}^2$, whose twistor spaces are $\mathbb{CP}^3$ and the flag manifold $\mathbb{F}(\mathbb{C}^3)$ respectively.

1.2 Instantons and the Ward Correspondence

The special features mentioned in the previous section also give Yang-Mills theory a special flavour in four dimensions (see [**At**] for a fuller discussion).

Suppose that $\mathcal{E}$ is a hermitian vector bundle over a compact Riemannian (or conformal) four-manifold M and that $\mathcal{E}$ has a unitary connection (or gauge potential) ∇. The curvature of the connection (or gauge field) F is a two-form with values in the endomorphism bundle of $\mathcal{E}$, i.e. $F \in \Lambda^2(M) \otimes \mathrm{End}(\mathcal{E})$, and thus we can split it into two components F_+ and F_-, determined by the splitting of $\Lambda^2(M)$. The total energy of the field F is given by the Yang-Mills action

$$\mathrm{YM}(F) = \int_M ||F||^2 d\mu = \int_M (||F_+||^2 + ||F_-||^2) d\mu,$$

where we can also write the Lagrangian $||F||^2 d\mu = -\operatorname{tr}(F \wedge *F)$. The Euler-Lagrange equations for this system are the Yang-Mills equations: $\nabla \wedge *F = 0$. The conformal invariance of the Hodge star on Λ^2 shows that the Yang-Mills equations are conformally invariant in four dimensions.

The quantity

$$\int_M \operatorname{tr}(F \wedge F) = \int_M (||F_-||^2 - ||F_+||^2) d\mu$$

is a topological invariant of the bundle $\mathcal{E}$ and so it is constant on the space of all connections on $\mathcal{E}$. This constant is $8\pi^2 k$, where k is the characteristic number $c_2 - \frac{1}{2}c_1^2$. Thus we see that the Yang-Mills action is minimised on the set of connections for which $F_+ = 0$ or $F_- = 0$, depending on whether $k \geqslant 0$ or $k \leqslant 0$. Such connections are called *instantons*. The constant k is the *index* (or *topological charge*) of the instanton and the total energy of the instanton is then $8\pi^2|k|$. When the distinction needs to be made, *self-dual* instantons are those with $*F = F$, i.e. $F_- = 0$, and *anti-self-dual* instantons are those with $*F = -F$, i.e. $F_+ = 0$. The Bianchi identity $\nabla \wedge F = 0$ shows directly that instantons satisfy the Yang-Mills equations, though of course we know they must, being minima of the Yang-Mills action.

Notice that, as defined above, it is the anti-self-dual instantons that have positive index. This may be in conflict with some definitions elsewhere. In this thesis, we shall be considering mostly anti-self-dual instantons and therefore the term 'instanton' will be taken to refer to these, if not otherwise qualified. We shall also restrict our attention to bundles with vanishing first Chern class and thus the index k will be equal to the second Chern number c_2. The notions of self-duality and anti-self-duality can be interchanged by reversing the orientation of the base manifold. However, this is often not an 'allowed symmetry', because the base manifold may have a preferred orientation.

This is the case for a complex manifold and over such a base manifold it is specifically the anti-self-dual instantons which are of interest, because of their relationship to holomorphic bundles, as we now describe.

If M is a complex manifold, then the algebra of complex differential forms is bigraded — the subspace of forms of bitype (p, q) being generated by simple forms $f dz_{i_1} \dots dz_{i_p} d\overline{z}_{j_1} \dots d\overline{z}_{j_q}$. Thus,

$$\Lambda^r(M; \mathbb{C}) = \bigoplus_{p+q=r} \Lambda^{p,q}(M).$$

Writing $\Omega^r(\mathcal{E})$ for the space of smooth sections of the bundle $\Lambda^r \otimes \mathcal{E}$, a connection on $\mathcal{E}$ is identified with a covariant derivative $\nabla : \Omega^0(\mathcal{E}) \to \Omega^1(\mathcal{E})$, and from this we can define an anti-holomorphic derivative $\overline{\partial} = \nabla^{(0,1)} : \Omega^0(\mathcal{E}) \to \Omega^{0,1}(\mathcal{E})$. Just as the covariant derivative extends to all $\Omega^r(\mathcal{E})$ and its curvature is given by ∇^2, so $\overline{\partial}$ extends to all $\Omega^{p,q}(\mathcal{E})$ and is integrable if and only if $\overline{\partial}^2 = 0$. Under the complex structure, $\Lambda^2(M; \mathbb{C})$ splits as $\Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \Lambda^{1,1}$ and, if M is a hermitian manifold, i.e. has a compatible Riemannian structure, we get a further splitting $\Lambda^{1,1} = \langle\omega\rangle \oplus \Lambda_0^{1,1}$, where $\Lambda_0^{1,1}$ is the bundle of 1,1-forms orthogonal to ω. This decomposition is related to the self-dual/anti-self-dual decomposition by

$$\Lambda^2_+(M; \mathbb{C}) = \Lambda^{2,0} \oplus \Lambda^{0,2} \oplus \langle\omega\rangle \quad \text{and} \quad \Lambda^2_-(M; \mathbb{C}) = \Lambda_0^{1,1}.$$

Thus we see that, if ∇ is an (anti-self-dual) instanton, then $\overline{\partial} = \nabla^{(0,1)}$ will be integrable. Indeed, ∇ is anti-self-dual if and only if it defines an integrable $\overline{\partial}$ operator with respect to any local almost complex structure on M compatible with the underlying Riemannian metric.

Now, suppose that M is not complex, but is conformally anti-self-dual, so that the twistor space $Z(M)$ is a complex manifold. Then, since the fibres of $\pi : Z(M) \to M$ precisely give the compatible complex structures on TM, we see that a connection ∇ on $\mathcal{E}$ is anti-self-dual if and only if the pulled-back connection determines a holomorphic structure on $\tilde{\mathcal{E}} = \pi^*(\mathcal{E})$. This procedure of assigning a holomorphic bundle on $Z(M)$ to an instanton on M is called the Ward transform. The bundle $\tilde{\mathcal{E}}$ is trivial on all fibres and the unitary structure on $\mathcal{E}$ induces a real structure on $\tilde{\mathcal{E}}$, i.e. an anti-holomorphic involution on $\tilde{\mathcal{E}}$ covering the real structure on $Z(M)$. In fact **[At;Thm 2.9]**, the Ward transform sets up a one-one correspondence between instantons on M and holomorphic bundles on $Z(M)$ with 'positive' real structures. It is this correspondence that makes it possible to use algebraic geometry techniques to classify instantons on S^4 **[ADHM]** and also on $\overline{\mathbb{CP}}^2$ **[Bu1]** (the latter paper actually classifies self-dual instantons on $\mathbb{CP}^2$, which are the same objects).

1.3 The Hitchin-Kobayashi Correspondence

The study of instantons overlaps more directly with algebraic geometry on a much wider class of four-manifolds, namely, projective algebraic surfaces through the Hitchin-Kobayashi correspondence:

THEOREM 1.3.1. [**D3**] *Let M be a projective algebraic surface. An indecomposable holomorphic bundle $\mathcal{E}$ over M with $c_1(\mathcal{E}) = 0$ is stable if and only if it admits a compatible anti-self-dual unitary connection.*

The stability condition occuring here is the one introduced by Mumford and Takemoto [**Ta**], namely that, if $\mathcal{F} \subseteq \mathcal{E}$ is any subsheaf of $\mathcal{E}$ with torsion-free quotient, then the quantity $\mu(\mathcal{F}) = c_1(\mathcal{F}) \cdot [\omega]/\operatorname{rk}\mathcal{F}$ should be negative, where $[\omega]$ is the hyperplane class with respect to the projective embedding, represented in de Rham cohomology by the Kähler form ω of the restriction of the Fubini-Study metric on projective space. It is also with respect to this restricted metric that the notion of anti-self-duality is defined. The compatibility condition between a holomorphic structure $\overline{\partial}$ and a connection ∇ is simply $\nabla^{(0,1)} = \overline{\partial}$. In fact (e.g. [**We;III.2**]), $\overline{\partial}$ and the unitary structure uniquely determine ∇. This means that, if we fix a unitary structure, then we can make the full complex gauge group, which acts naturally on the space of $\overline{\partial}$ operators, act on the space of all holomorphic connections, i.e. those with curvature $F \in \Omega^{1,1}(\operatorname{End}\mathcal{E})$. The extra condition for ∇ to be an instanton is simply $F \wedge \omega = 0$, which can be interpreted as a 'moment map' for the action of the unitary gauge group. Indeed, Donaldson's proof of Theorem 1.3.1 (see also [**D5**]) gives an infinite dimensional analogue of the equivalence of symplectic and complex quotients which we shall discuss in §1.5.

The first non-compact version of the Hitchin-Kobayashi correspondence was proved over $\mathbb{C}^2$ with the flat Euclidean metric — also by Donaldson [**D1**]. An instanton on $\mathbb{C}^2$ (or $\mathbb{R}^4$) is an anti-self-dual connection which has finite total energy. The appropriate notion of equivalence is given by the action of the group of gauge transformations (sections of $GL(\mathcal{E})$) which tend to the identity at infinity. The moduli space $\mathsf{MI}(\mathbb{C}^2; r, k)$ of instantons on $\mathbb{C}^2$, of rank r and index k (which is defined to be the total energy divided by $8\pi^2$) is the quotient of the space of all such instantons by the gauge group just described. The finite action condition means that we can extend an instanton on $\mathbb{C}^2$ to one on the conformal compactification S^4, and, furthermore, the gauge transformations extend to ones on S^4 which are the identity at ∞ (the point added in compactifying). Thus we see that $\mathsf{MI}(\mathbb{C}^2; r, k)$ is canonically identified with the framed moduli space $\mathsf{MI}(S^4, \infty; r, k)$ of instantons on S^4 together with an identification $\mathcal{E}_\infty \cong \mathbb{C}^r$. The index k is the second Chern number of the bundle over S^4 and is therefore an integer.

It is not immediately obvious how one should obtain a moduli space of holomorphic bundles over $\mathbb{C}^2$. From the naive point of view, such bundles have no topological invariant one could call the index, no notion of stability and, in fact, no moduli. However, by analogy with the instantons, which extend to a conformal compactification of $\mathbb{C}^2$, the solution to all these problems is to ask that the holomorphic bundles extend to the canonical complex compactification $\mathbb{CP}^2$ and be holomorphically trivial on the added line ℓ_∞. Indeed, we essentially ask that the extension be part of the information carried by the bundle, by taking as the gauge group only those complex gauge transformations over $\mathbb{C}^2$ which tend to the identity at infinity. The second Chern number of the extension provides the bundles with an index and gives a moduli space which is an open subset of the moduli space of stable holomorphic bundles on $\mathbb{C}P^2$ (see [**Ba**]). Thus, the moduli space $\mathsf{MH}(\mathbb{C}^2; r, k)$ of "stable" holomorphic bundles on $\mathbb{C}^2$ with rank r and $c_2 = k$ is identified with the framed moduli space $\mathsf{MH}(\mathbb{C}P^2, \ell_\infty; r, k)$ of holomorphic bundles on $\mathbb{C}P^2$ with rank r and $c_2 = k$, with a given holomorphic identification of $\mathcal{E}|_{\ell_\infty}$ with the trivial bundle with fibre $\mathbb{C}^r$.

Now, the Hitchin-Kobayashi correspondence in this case is proved via the Ward correspondence. If we consider the twistor space $Z(S^4) = \mathbb{C}P^3$, the complex structure on $\mathbb{C}^2 = S^4 \setminus \{\infty\}$ will give a section of the twistor fibration away from ∞. The closure of the graph of this section is a hyperplane in $\mathbb{C}P^3$, i.e. a copy of $\mathbb{C}P^2$, and contains the whole of the fibre over ∞, which will be ℓ_∞. The Ward transform of an instanton on S^4, when restricted to this hyperplane, gives a holomorphic bundle on $\mathbb{C}P^2$ trivial on ℓ_∞. Furthermore, the restriction of this bundle to $\mathbb{C}^2$ is just the holomorphic part of the instanton restricted to $\mathbb{C}^2$. The framing of the instanton at ∞ lifts to a trivialisation of the holomorphic bundle along ℓ_∞. Since the unitary gauge group sits inside the complex gauge group, we get an induced map

$$\mathsf{h} : \mathsf{MI}(\mathbb{C}^2; r, k) \to \mathsf{MH}(\mathbb{C}^2; r, k)$$

which can also be interpreted as being induced by restricting holomorphic bundles on $\mathbb{C}P^3$ to a hyperplane. By constructing both moduli spaces explicitly and using a simple direct proof of the equivalence of symplectic and algebraic quotients, Donaldson shows that h is a bijection, thus demonstrating the required correspondence.

The method used to prove the correspondence on $\mathbb{C}^2$ should also be applicable to $\widetilde{\mathbb{C}}^2$, since its conformal compactification is $\overline{\mathbb{C}P}^2$, the other compact four-manifold with a projective algebraic twistor space. The fact that this method does work and yield a Hitchin-Kobayashi correspondence for $\widetilde{\mathbb{C}}^2$ is the main result that we prove in what follows, the formal statement being given in §4.3. The precise definitions of the moduli spaces $\mathsf{MI}(\widetilde{\mathbb{C}}^2; r, k)$ and $\mathsf{MH}(\widetilde{\mathbb{C}}^2; r, k)$ will be given in the next section. They are the direct analogues of the instanton and holomorphic bundle moduli spaces on $\mathbb{C}^2$ with the added restriction that the underlying vector bundle should have vanishing first Chern class.

1.4 The Geometric Environment: $\widetilde{\mathbb{C}}^2$, $\overline{\mathbb{C}P}^2$, Σ^1 and F

We now describe the various spaces which are associated to the problem we are going to study. The blow-up of the affine plane at the origin, $\widetilde{\mathbb{C}}^2$, has a natural complex description as the set

$$\{((x_1, x_2), [z_1, z_2]) \in \mathbb{C}^2 \times \mathbb{C}P^1 \mid z_1 x_2 = z_2 x_1\}$$

We thus get two projections $\pi_1 : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$, which is the blowing-down map onto the first factor, i.e. is a bijection except that it maps the whole of one line, the exceptional line E, to 0, and $\pi_2 : \widetilde{\mathbb{C}}^2 \to \mathbb{C}P^1$, which is a fiberation with fibre $\mathbb{C}$ and identifies $\widetilde{\mathbb{C}}^2$ with the total space of the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$. The exceptional line E is the zero section, with self-intersection -1.

The above description also provides a Kähler metric on $\widetilde{\mathbb{C}}^2$ by restriction of the product of the flat metric on $\mathbb{C}^2$ and the Fubini-Study metric on $\mathbb{CP}^1$. If we write $\zeta = z_1/z_2$, then this metric is

$$ds^2 = \frac{R^2}{4}\frac{d\zeta d\bar{\zeta}}{(1+||\zeta||^2)^2} + dx_1 d\bar{x}_1 + dx_2\bar{x}_2,$$

where R is a parameter giving the 'radius' of the exceptional line.

By analogy with the $\mathbb{C}^2$ case, we must now find two compactifications of $\widetilde{\mathbb{C}}^2$: one conformal and one complex. We first describe the conformal one, using the orientation reversing map

$$\widetilde{\mathbb{C}}^2 \to \mathbb{CP}^2 : \big((x_1,x_2),[z_1,z_2]\big) \mapsto \begin{cases} [||x||^2, x_1, x_2] & \text{if } x \neq 0 \\ [0, z_1, z_2] & \text{if } x = 0 \end{cases}$$

This map is clearly well defined and injective and its image is $\mathbb{CP}^2 \setminus \{[1,0,0]\}$. Furthermore, it is an isometry with respect to the Fubini-Study metric on $\mathbb{CP}^2$ and the conformally rescaled metric $\left(1+||x||^2\right)^{-2}ds^2$ on $\widetilde{\mathbb{C}}^2$. Thus we see that, taking into account the orientation, $\widetilde{\mathbb{C}}^2$ has a conformal compactification $\overline{\mathbb{CP}}^2$, equipped with the Fubini-Study metric. Now the Fubini-Study metric on $\mathbb{CP}^2$ is conformally self-dual ([**AHS**]) and so $\overline{\mathbb{CP}}^2$ is conformally anti-self-dual. Thus $\widetilde{\mathbb{C}}^2$ is conformally anti-self-dual, which we could have seen also by calculating that its scalar curvature is zero and using the well-known result that a Kähler metric is anti-self-dual if and only if it has zero scalar curvature. Thus we can identify the moduli space $\mathsf{MI}(\widetilde{\mathbb{C}}^2; r, k)$ with $\mathsf{MI}(\overline{\mathbb{CP}}^2, \infty; r, k)$, the moduli space of instantons on $\overline{\mathbb{CP}}^2$, framed at ∞, whose underlying vector bundle has rank r, $c_1 = 0$ and $c_2 = k$.

Secondly, we describe the canonical complex compactification. Since $\widetilde{\mathbb{C}}^2$ is the total space of a line bundle over $\mathbb{P}^1$, it is natural to take the associated projective line bundle obtained by adding a point at infinity in each fibre, i.e. a line ℓ_∞. More explicitly, the 'projectivisation' of a general vector bundle $\mathcal{E}$ over X is the projective bundle $\mathbb{P}(\mathcal{O}_X \oplus \mathcal{E})$. Thus we see that the complex compactification of $\widetilde{\mathbb{C}}^2$ is the Hirzebruch surface Σ^1, where the general Hirzebruch surface Σ^n is the total space of the projective line bundle $\mathbb{P}\big(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n)\big)$. Therefore, essentially by definition, we identify the holomorphic moduli space $\mathsf{MH}(\widetilde{\mathbb{C}}^2; r, k)$ with $\mathsf{MH}(\Sigma^1, \ell_\infty; r, k)$, the moduli space of holomorphic bundles on Σ^1, with a trivialisation along ℓ_∞ and whose underlying vector bundle has rank r, $c_1 = 0$ and $c_2 = k$.

Finally, we describe the twistor space of $\overline{\mathbb{CP}}^2$ and identify a copy of Σ^1 sitting inside it. This description comes originally from [**D2**]. The twistor space is the flag manifold $\mathsf{F} = \mathsf{F}(\mathbb{C}^3)$ and is constructed as follows. Let $\mathbb{P}_2 = \mathbb{P}(\mathbb{C}^3)$ and $\mathbb{P}_2^* = \mathbb{P}(\mathbb{C}^{3*})$. Then

$$\mathsf{F} = \left\{ \big([x],[y]\big) \in \mathbb{P}_2 \times \mathbb{P}_2^* \mid y(x) = \textstyle\sum x_i y_i = 0 \right\}$$

and thus we have two canonical holomorphic projection maps $\pi_1 : \mathsf{F} \to \mathbb{P}_2$ and $\pi_2 : \mathsf{F} \to \mathbb{P}_2^*$. If we consider $\mathbb{C}^3$ to be equipped with the standard hermitian inner product (given by an anti-linear isomorphism $\mathbb{C}^3 \to \mathbb{C}^{3*} : w \mapsto \hat{w}$), then we can define a further 'projection', which we shall write as $\pi_Z : \mathsf{F} \to \overline{\mathbb{CP}}^2$ to emphasize the fact that it is not holomorphic,

$$\pi_Z : \big([x],[y]\big) \mapsto [x]^\perp \cap \ker y$$

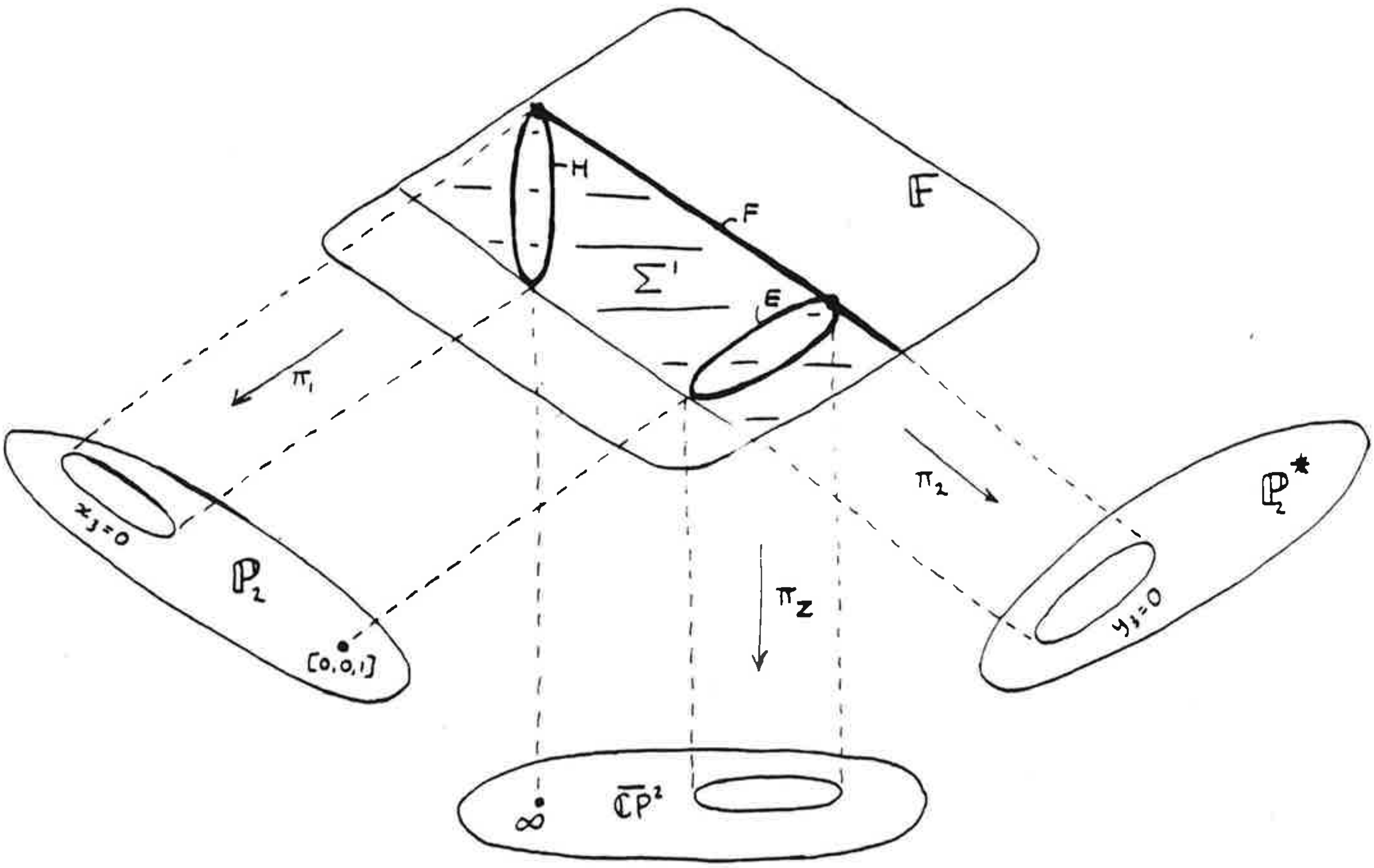
The fibres of this 'projection' are the lines

$$\ell_w = \{([x],[y]) \in \mathsf{F} \mid \hat{w}(x) = y(w) = 0\},$$

which are clearly preserved by the real structure $\sigma : ([x],[y]) \mapsto ([\hat{y}],[\hat{x}])$.

If we now choose $\infty = [0,0,1] \in \overline{\mathbb{CP}}^2$, then the hypersurface $\Sigma^1 \subseteq \mathsf{F}$ with equation $y_3 = 0$ contains the whole of ℓ_∞, but otherwise π_Z gives a one-one correspondence between $\Sigma^1 \setminus \ell_\infty$ and $\overline{\mathbb{CP}}^2 \setminus \{\infty\} = \tilde{\mathbb{C}}^2$. The restriction of π_2 realises Σ^1 as a ruled surface over the line $y_3 = 0$ in $\mathbb{P}_2^*$. The restriction of π_1 is the blowing down map to $\mathbb{P}_2$, with the exceptional divisor E given by the equations $x_1 = 0$ and $x_2 = 0$. We also identify two more divisors which are projective lines: $H = \ell_\infty$ and F, which is a typical fibre of the projection π_2.

The above information is summarised in the following diagram:



The line bundles on F are given by

$$\mathcal{O}_{\mathsf{F}}(p,q) = \pi_1^*\mathcal{O}_{\mathbb{P}_2}(p) \otimes \pi_2^*\mathcal{O}_{\mathbb{P}_2^*}(q).$$

The coordinate functions x_1,x_2,x_3 can be interpreted as a basis for the space of sections $H^0(\mathcal{O}_{\mathbf{P}_2}(1))$ and thus of $H^0(\mathcal{O}_{\mathsf{F}}(1,0))$. Similarly, the dual coordinate functions y_1,y_2,y_3 can be thought of as a basis for $H^0(\mathcal{O}_{\mathsf{F}}(0,1))$.

The line bundles on Σ^1 are given by the restrictions of $\mathcal{O}_{\mathsf{F}}(p,q)$, and thus $H^0(\mathcal{O}_{\Sigma^1}(1,0))$ also has basis x_1,x_2,x_3, while $H^0(\mathcal{O}_{\Sigma^1}(0,1))$ has basis y_1,y_2. The linear system $|\mathcal{O}_{\Sigma^1}(0,1)|$ consists of the fibres of π_2, e.g. F, and the linear system $|\mathcal{O}_{\Sigma^1}(1,0)|$ consists of the inverse images of lines in $\mathbf{P}_2$, e.g. H. When the line in $\mathbf{P}_2$ passes through $[0,0,1]$, its inverse image consists of the exceptional fibre and one fibre of π_2, demonstrating that, up to linear equivalence, $H = F + E$. We can also easily write down the various intersection relationships of these divisors:

$$\begin{array}{ccc} H^2 = 1 & F^2 = 0 & E^2 = -1 \\ H \cdot F = 1 & H \cdot E = 0 & E \cdot F = 1 \end{array}$$

We observe that $E \in |\mathcal{O}_{\Sigma^1}(1,-1)|$ and that there is indeed a section $s = x_2/y_1 = -x_1/y_2$ (well defined because $x_1y_1 + x_2y_2 = 0$ on Σ^1) which vanishes on E. Since E is exceptional, we must have $H^0(\mathcal{O}_{\Sigma^1}(1,-1)) = \langle s\rangle$. Finally, we note that the canonical bundle on $\mathbf{P}_2 \times \mathbf{P}_2^*$ is $\mathcal{O}(-3,-3)$ and so the adjunction formula tells us that the canonical bundles of F and Σ^1 are $\mathcal{O}_{\mathsf{F}}(-2,-2)$ and $\mathcal{O}_{\Sigma^1}(-2,-1)$ respectively.

1.5 Moment Maps and Analytic Stability

We now present a short discussion of holomorphic actions of reductive groups on Hodge manifolds. We describe the rôle played by moment maps for such actions and the relationship with the analytic notion of stability. In what follows, X will denote a Hodge manifold. That is to say that X is a Kähler manifold (with Kähler form ω) over which there is a hermitian line bundle L equipped with a unitary connection whose curvature is $-i\omega$. Since $-i\omega$ is a 1,1-form, L is a holomorphic bundle and, as usual, the holomorphic and hermitian structures together determine the connection. Thus, given X (as just a complex manifold) and the holomorphic line bundle L, the Kähler metric is determined by the hermitian structure on L, which we can think of as a global Kähler potential. Locally, we can pick a non-zero section f of L and define the familiar local Kähler potential as the function $\phi = \log ||f||^2$ which gives $\omega = i\partial\overline{\partial}\phi$.

Now let G be a compact Lie group with complexification $G_{\mathbf{C}}$. The complex Lie group $G_{\mathbf{C}}$ is then reductive and has Lie algebra $\mathbf{g}_{\mathbf{C}} = \mathbf{g} \oplus i\mathbf{g}$, where $\mathbf{g}$ is the Lie algebra of G. As a real manifold, $G_{\mathbf{C}} \cong TG \cong G \times \mathbf{g}$, via the map $G \times \mathbf{g} \to G_{\mathbf{C}} : (g, A) \mapsto g\,e^{iA}$. Thus we can identify $P = G_{\mathbf{C}}/G$ with $\mathbf{g}$. In fact, since P is a symmetric space, while $\mathbf{g}$ is a vector space, it would be more natural to identify $\mathbf{g}$ with the tangent space T_0P at the point $0 \in P$ corresponding to the coset G itself. The map $A \mapsto Ge^{iA}$ is then the metric exponential map.

In this section, we shall be considering specifically actions of $G_{\mathbf{C}}$ on X by holomorphic transformations, for which G acts by isometries. Thus, G acts symplectically with respect to ω. A *moment map* for such an action of G is a map $\mu : X \to \mathbf{g}^*$, which is equivariant with respect to the co-adjoint action and satisfies

$$d\mu(A) = i_{\hat{A}}\omega \tag{1.5.1}$$

where $i_{\tilde{A}}$ denotes interior multiplication by the vector field $\tilde{A}$ which gives the infinitesimal action of $A \in \mathbf{g}$. If we have a moment map μ, we can lift each vector field $\tilde{A}$ on X to a vector field on the total space of L

$$\hat{A}_{(\zeta,x)} = h(\tilde{A}) - i\mu_x(A)\zeta,$$

where $\zeta \in L_x$ and $h(\tilde{A})$ is the horizontal lift of $\tilde{A}$ determined by the connection on L. Under mild topological assumptions, this infinitesimal action of $\mathbf{g}$ will generate a linear unitary action of G on L, covering the original G-action on X. This can then be extended to a linear $G_{\mathbf{C}}$-action which covers the $G_{\mathbf{C}}$-action on X. Such an action of $G_{\mathbf{C}}$ on L, for which G acts unitarily, will be called a *linearisation* of the $G_{\mathbf{C}}$-action on X. Note that this is slightly stronger than in the general holomorphic or algebraic case, where the extra condition on the G-action is inappropriate.

As a converse to the above, if we start with a linearisation of the $G_{\mathbf{C}}$-action on X, then we can define a moment map from it as follows. For each point $x \in X$, define the function

$$M_x : G_{\mathbf{C}} \to \mathbf{R} : g \mapsto \log \frac{||g \cdot \zeta||}{||\zeta||},$$

where ζ is a non-zero point in the fibre L_x and the norm is induced by the hermitian inner product on L. This function is clearly independent of ζ and constant on cosets of G, so it induces a function $m_x : P \to \mathbf{R}$. Given the earlier identification, we can define a map $\mu : X \to \mathbf{g}^*$ by $\mu_x = dm_x(0)$. The map μ is then a moment map for the G-action and, by the earlier procedure, gives back the initial linearisation. We can further consider the second derivative $D^2 m_x(0)$, which we take to be the quadratic form on $\mathbf{g}$ which assigns to an element $A \in \mathbf{g} = T_0 P$ the second derivative of m_x along the geodesic through 0 with tangent vector A. This is given by $A \mapsto 2||\tilde{A}_x||^2$ and, hence, if x has no infinitesimal stabiliser (i.e. $\{A \in \mathbf{g}_{\mathbf{C}} \mid \tilde{A}_x = 0\} = \{0\}$), then m_x will be strictly convex at $0 \in P$. Conversely, if m_x is strictly convex at 0, it cannot be constant on any geodesic through 0 and so x can have no infinitesimal stabiliser. Note that, if m_x is strictly convex at 0, then m_{hx} will also be, for any $h \in G_{\mathbf{C}}$, and hence m_x will be strictly convex on the whole of P.

Now, when one has a moment map μ for a free symplectic G-action on any symplectic manifold X, one can define the Marsden-Weinstein reduction $\widehat{X} = \mu^{-1}(0)/G$, which is again a symplectic manifold — also called the symplectic quotient of X by G. We shall now describe how, on a Hodge manifold with a holomorphic $G_{\mathbf{C}}$-action as above, there is a sufficiently nice subset for which the symplectic quotient is equal to the set-theoretic quotient under the $G_{\mathbf{C}}$-action and is (almost) a smooth Kähler manifold. The precise notion of "sufficiently nice" is given by analytic stability.

DEFINITION 1.5.1. A point $x \in X$ is *analytically stable*, with respect to a linearised $G_{\mathbf{C}}$-action, if, given any non-zero $\zeta \in L_x$, the function $g \mapsto ||g \cdot \zeta||$ is proper, i.e.

$$||g \cdot \zeta|| \to \infty \quad \text{as} \quad g \to \infty \quad \text{in } G_{\mathbf{C}}.$$

Clearly, if x is analytically stable, then so is $g \cdot x$ for any $g \in G_{\mathbf{C}}$, so that analytic stability is a property of $G_{\mathbf{C}}$ orbits in X.

Proposition 1.5.2. *An orbit $G_{\mathbf{C}} \cdot x \subseteq X$ is analytically stable if and only if it has no infinitesimal stabilisers and it contains a point on which $\mu = 0$. Furthermore, if it does contain such a point, then the set $G_{\mathbf{C}} \cdot x \cap \mu^{-1}(0)$ will consist of a single G orbit.*

Proof. Consider the function $m_x : P \to \mathbf{R}$ as defined above. Since G is compact, x is analytically stable if and only if m_x is proper. But, if m_x is proper, then there can be no geodesic through 0 along which it is constant, so x can have no infinitesimal stabiliser. Hence, m_x is a strictly convex proper function and has precisely one critical point, which is a minimum. Now, the critical points of m_x correspond to the G orbits on which $\mu = 0$, and thus most of the proposition is proved.

To show the converse, note that if m_x is strictly convex and has a critical point, then this must be a minimum and a strictly convex function with a minimum is necessarily proper. □

Theorem 1.5.3. *Let X_{as} be the set of analytically stable points of X with respect to a linearised $G_{\mathbf{C}}$-action with moment map μ. Then*

$$X_{as}/G_{\mathbf{C}} = X_{as} \cap \mu^{-1}(0)/G$$

and, if the $G_{\mathbf{C}}$-action on X_{as} is actually free, the quotient is a smooth Kähler manifold.

Proof. The two quotients are equivalent by Proposition 1.5.2 and, in general, smooth when the action is free. The compatibility of the complex and Riemannian structures on the quotient follows from the moment map condition (1.5.1), which ensures that the subspace of $T_x X$, tangent to $\mu^{-1}(0)$ and orthogonal to $G \cdot x$, is a complex subspace complementary to $G_{\mathbf{C}} \cdot x$. This is the subspace on which the tangent space to the quotient is modelled. For the existence of a Hodge structure on the Marsden-Weinstein reduction in this case see e.g. [**HKLR**, §3]). □

In order to be able to identify analytically stable points, it is useful to have the following analytic version of Hilbert's criterion (cf. [**Mu,Thm2.1**]).

Lemma 1.5.4. *A point $x \in X$ is analytically stable with respect to a linearised $G_{\mathbf{C}}$-action if and only if it is analytically stable for the restricted action of all real one parameter subgroups $\lambda : \mathbf{R} \to G_{\mathbf{C}} : t \mapsto e^{tA}$ for $A \in i\mathbf{g}$.*

Proof. This lemma is equivalent to the statement that $m_x : P \to \mathbf{R}$ is proper if and only if it is proper when restricted to all geodesics through 0. The "only if" part is immediate. The "if" part follows from the fact that a strictly convex function on P is proper if its values on some sphere, centered at 0, are all greater than its value at 0. The compactness of the set of geodesics through 0, together with the fact that m_x is proper when restricted to all of these, implies that we can find such a sphere. □

Remark 1.5.5. Kempf & Ness [**KN**] have shown that the notion of analytic stability coincides with the usual notion of stability in geometric invariant theory [**Mu**] when X is a projective variety, equipped with the standard Kähler metric coming from the restriction of the Fubini-Study metric on $\mathbf{CP}^N$.

2 Monads

In this chapter we shall introduce monads and explain their use in describing families of holomorphic vector bundles. Monads were first introduced by Horrocks [**Ho**] and further developed by Barth and Hulek [**BH,Ba**] as tools in the classification of vector bundles on projective space (in particular on $\mathbb{CP}^2$). They were combined with twistor methods by Atiyah et al [**ADHM**] to give a classification of instantons on $\mathbb{R}^4$. For a more detailed discussion of monads, see [**OSS**].

2.1 Introduction to Monads

A *monad* is a complex of holomorphic vector bundles

$$0 \longrightarrow \mathcal{U} \xrightarrow{\mathcal{A}} \mathcal{V} \xrightarrow{\mathcal{B}} \mathcal{W} \longrightarrow 0$$

which is exact at $\mathcal{U}$ and $\mathcal{W}$, so that it has cohomology only in the middle position and thus defines a vector bundle $\mathcal{E} = \ker \mathcal{B} / \operatorname{im} \mathcal{A}$. Put another way, a monad is a short two-sided resolution of a vector bundle.

The power of monads lies in the fact that we can describe whole families of vector bundles (e.g. all stable bundles on $\mathbb{CP}^2$ with fixed rank and Chern classes) using monads in which the terms $\mathcal{U}$,$\mathcal{V}$ and $\mathcal{W}$ remain fixed and only the maps $\mathcal{A}$ and $\mathcal{B}$ vary. Furthermore, we can hope to find $\mathcal{U}$,$\mathcal{V}$ and $\mathcal{W}$ of a particularly simple form, e.g. direct sums of line bundles. We will then have a fairly concrete description of the spaces $\operatorname{Hom}(\mathcal{U},\mathcal{V})$ and $\operatorname{Hom}(\mathcal{V},\mathcal{W})$. The family of vector bundles can then be described as a subset of

$$\big\{(\mathcal{A},\mathcal{B}) \in \operatorname{Hom}(\mathcal{U},\mathcal{V}) \times \operatorname{Hom}(\mathcal{V},\mathcal{W}) \mid \mathcal{B}\circ\mathcal{A} = 0,\ \mathcal{A} \text{ is injective and } \mathcal{B} \text{ is surjective}\big\}$$

modulo a notion of equivalence which we now describe.

A map from one monad $M : \mathcal{U} \to \mathcal{V} \to \mathcal{W}$ to another $M' : \mathcal{U}' \to \mathcal{V}' \to \mathcal{W}'$ is simply a map of complexes, i.e. a triple (ϕ, ψ, χ) making the following diagram commute

$$\begin{array}{ccccc} \mathcal{U} & \xrightarrow{\mathcal{A}} & \mathcal{V} & \xrightarrow{\mathcal{B}} & \mathcal{W} \\ \downarrow{\scriptstyle\phi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\chi} \\ \mathcal{U}' & \xrightarrow{\mathcal{A}'} & \mathcal{V}' & \xrightarrow{\mathcal{B}'} & \mathcal{W}' \end{array}$$

Thus the category of monads is a full subcategory of the category of all (bounded) complexes. Since $\mathcal{A}$ and $\mathcal{A}'$ are injective and $\mathcal{B}$ and $\mathcal{B}'$ are surjective, ϕ and χ are uniquely determined by ψ, as long as it in turn satisfies

$$\psi(\operatorname{im}\mathcal{A}) \subseteq \operatorname{im}\mathcal{A}' \quad \text{and} \quad \psi(\ker\mathcal{B}) \subseteq \ker\mathcal{B}'.$$

Then ψ induces a map $H(\psi) : \mathcal{E} \to \mathcal{E}'$. The process of taking cohomology of a complex is, in general, functorial, so that two monads which are isomorphic (in the usual categorical sense) define isomorphic vector bundles. Therefore, the natural notion of equivalence on the set of monads with fixed terms $\mathcal{U}$,$\mathcal{V}$ and $\mathcal{W}$ is given by the action of the group $\mathrm{Aut}(M) = \mathrm{Aut}(\mathcal{U}) \times \mathrm{Aut}(\mathcal{V}) \times \mathrm{Aut}(\mathcal{W})$, which will also have a fairly concrete description when $\mathcal{U}$,$\mathcal{V}$ and $\mathcal{W}$ are direct sums of line bundles.

It is clearly important to know whether all automorphisms of a bundle $\mathcal{E}$ are induced by automorphisms of the monad M defining it, i.e. whether the quotient by $\mathrm{Aut}(M)$ necessarily determines an effective parametrisation. To this end we can ask the slightly more general question: when is the map $H : \mathrm{Hom}(M, M') \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ surjective? We should also like to know what its kernel is, at least in the good case when the map is surjective.

2.2 Some General Properties

In this section, we shall use the following shorthand notation, which was introduced in [Ru]. We shall write $\langle X \mid Y \rangle$ for $\mathrm{Hom}(X,Y)$ and ${}^i\langle X \mid Y \rangle$ for $\mathrm{Ext}^i(X,Y)$, so that ${}^0\langle X \mid Y \rangle = \langle X \mid Y \rangle$. Further, we shall write $| \theta \rangle : {}^i\langle X \mid Y \rangle \to {}^i\langle X \mid Z \rangle$ for the natural map induced by $\theta : Y \to Z$, $\langle \phi \,| : {}^i\langle X \mid Y \rangle \to {}^i\langle W \mid Y \rangle$ for the map induced by $\phi : W \to X$ and $\langle \phi \mid \theta \rangle : {}^i\langle X \mid Y \rangle \to {}^i\langle W \mid Z \rangle$ for the map induced by θ and ϕ simultaneously. Recall that ${}^i\langle \bullet \mid \bullet \rangle$ is a bifunctor, contravariant in the first place and covariant in the second place.

The relationship between a monad and the bundle it defines is conveniently represented by the *display* of the monad, which is a commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{U} & \xrightarrow{i_1} & \mathcal{K} & \xrightarrow{j_1} & \mathcal{E} & \to & 0 \\
 & & \| & & \downarrow{\scriptstyle i_3} & & \downarrow{\scriptstyle i_4} & & \\
0 & \to & \mathcal{U} & \xrightarrow{\mathcal{A}} & \mathcal{V} & \xrightarrow{j_2} & \mathcal{Q} & \to & 0 \\
 & & & & \downarrow{\scriptstyle \mathcal{B}} & & \downarrow{\scriptstyle j_4} & & \\
 & & & & \mathcal{W} & = & \mathcal{W} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array} \qquad (2.2.1)$$

in which $\mathcal{K} = \ker \mathcal{B}$ and $\mathcal{Q} = \operatorname{coker} \mathcal{A}$.

Suppose we have two monads $M : \mathcal{U} \to \mathcal{V} \to \mathcal{W}$ and $M' : \mathcal{U}' \to \mathcal{V}' \to \mathcal{W}'$. We can use the extra spaces and maps provided by their displays to give a simple description of $\mathrm{Hom}(M, M')$ as a subset of $\langle \mathcal{V} \mid \mathcal{V}' \rangle$. Indeed, for $\psi \in \langle \mathcal{V} \mid \mathcal{V}' \rangle$ the condition $\psi(\mathrm{im}\,\mathcal{A}) \subseteq \mathrm{im}\,\mathcal{A}'$ is simply $\psi \in \ker \langle \mathcal{A} \mid j_2' \rangle$, while $\psi(\ker \mathcal{B}) \subseteq \ker \mathcal{B}'$ if and only if $\psi \in \ker \langle i_3 \mid \mathcal{B}' \rangle$. Thus,

$$\mathrm{Hom}(M, M') = \ker \langle \mathcal{A} \mid j_2' \rangle \cap \ker \langle i_3 \mid \mathcal{B}' \rangle \subseteq \langle \mathcal{V} \mid \mathcal{V}' \rangle.$$

In addition, by considering the diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \langle \mathcal{E} \mid \mathcal{E}' \rangle & \to & \langle \mathcal{K} \mid \mathcal{E}' \rangle & \longrightarrow & \langle \mathcal{U} \mid \mathcal{E}' \rangle & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \langle \mathcal{E} \mid \mathcal{Q}' \rangle & \to & \langle \mathcal{K} \mid \mathcal{Q}' \rangle & \xrightarrow{\langle i_1 \mid} & \langle \mathcal{U} \mid \mathcal{Q}' \rangle & \to & \cdots \\
 & & \downarrow & & \downarrow{\scriptstyle \mid j_4' \rangle} & & \downarrow & & \\
0 & \to & \langle \mathcal{E} \mid \mathcal{W}' \rangle & \to & \langle \mathcal{K} \mid \mathcal{W}' \rangle & \longrightarrow & \langle \mathcal{U} \mid \mathcal{W}' \rangle & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & \vdots & & \vdots & & \vdots & &
\end{array}$$

we can identify $\mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ with $\ker(\langle i_1 \mid) \cap \ker(\mid j_4' \rangle) \subseteq \langle K \mid Q' \rangle$. Under the above identifications the map $H : \mathrm{Hom}(M, M') \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ is the restriction of the map $\langle i_3 \mid j_2' \rangle : \langle V \mid V' \rangle \to \langle K \mid Q' \rangle$ and, furthermore,

$$\langle i_3 \mid j_2' \rangle^{-1}(\mathrm{Hom}(\mathcal{E}, \mathcal{E}')) = \mathrm{Hom}(M, M'). \qquad (2.2.2)$$

We are now in a position to prove

PROPOSITION 2.2.1. *Suppose $M : \mathcal{U} \to \mathcal{V} \to \mathcal{W}$ and $M' : \mathcal{U}' \to \mathcal{V}' \to \mathcal{W}'$ are two monads defining vector bundles $\mathcal{E}$ and $\mathcal{E}'$ respectively. If*

$$\mathrm{Ext}^1(\mathcal{W}, \mathcal{V}') = \mathrm{Ext}^1(\mathcal{V}, \mathcal{U}') = \mathrm{Ext}^2(\mathcal{W}, \mathcal{U}') = 0,$$

then the map $H : \mathrm{Hom}(M, M') \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ is onto. If further

$$\mathcal{H}om(\mathcal{W}, \mathcal{V}') = \mathrm{Hom}(\mathcal{V}, \mathcal{U}') = 0,$$

then $\ker(H)$ can be identified with $\mathrm{Ext}^1(\mathcal{W}, \mathcal{U}')$.

PROOF. The pair of short exact sequences

$$0 \to \mathcal{K} \to \mathcal{V} \to \mathcal{W} \to 0 \quad \text{and} \quad 0 \to \mathcal{U}' \to \mathcal{V}' \to \mathcal{Q}' \to 0$$

induce the following exact diagram of Hom/Ext groups:

$$\begin{array}{ccccccccccc}
 & & 0 & & 0 & & 0 & & \vdots & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \langle \mathcal{W} \mid \mathcal{U}' \rangle & \to & \langle \mathcal{W} \mid \mathcal{V}' \rangle & \longrightarrow & \langle \mathcal{W} \mid \mathcal{Q}' \rangle & \to & {}^1\langle \mathcal{W} \mid \mathcal{U}' \rangle & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \langle \mathcal{V} \mid \mathcal{U}' \rangle & \to & \langle \mathcal{V} \mid \mathcal{V}' \rangle & \xrightarrow{|j_2'\rangle} & \langle \mathcal{V} \mid \mathcal{Q}' \rangle & \to & {}^1\langle \mathcal{V} \mid \mathcal{U}' \rangle & \to & \cdots \\
 & & \Big\downarrow & & \Big\downarrow{\scriptstyle \langle i_3|} & & \Big\downarrow{\scriptstyle \langle i_3|} & & \Big\downarrow & & \\
0 & \to & \langle \mathcal{K} \mid \mathcal{U}' \rangle & \to & \langle \mathcal{K} \mid \mathcal{V}' \rangle & \xrightarrow{|j_2'\rangle} & \langle \mathcal{K} \mid \mathcal{Q}' \rangle & \to & {}^1\langle \mathcal{K} \mid \mathcal{U}' \rangle & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & {}^1\langle \mathcal{W} \mid \mathcal{U}' \rangle & \to & {}^1\langle \mathcal{W} \mid \mathcal{V}' \rangle & \longrightarrow & {}^1\langle \mathcal{W} \mid \mathcal{Q}' \rangle & \to & {}^2\langle \mathcal{W} \mid \mathcal{U}' \rangle & \to & \cdots \\
 & & \downarrow & & \downarrow & & \downarrow & & \downarrow & & \\
\cdots & \to & {}^1\langle \mathcal{V} \mid \mathcal{U}' \rangle & \to & \vdots & & \vdots & & \vdots & & \\
 & & \downarrow & & & & & & & & \\
 & & \vdots & & & & & & & &
\end{array}$$

The first vanishing conditions in the proposition show that

$$\langle i_3 \mid j_2' \rangle : \langle \mathcal{V} \mid \mathcal{V}' \rangle \to \langle \mathcal{K} \mid \mathcal{Q}' \rangle$$

is onto, in which case (2.2.2) means that $H : \mathrm{Hom}(M, M') \to \mathrm{Hom}(\mathcal{E}, \mathcal{E}')$ is onto and has the same kernel as $\langle i_3 \mid j_2' \rangle$. Given the additional vanishing conditions in the proposition, the above diagram allows us to identify this kernel with ${}^1\langle \mathcal{W} \mid \mathcal{U}' \rangle$. □

COROLLARY 2.2.2. *Let $M : \mathcal{U} \to \mathcal{V} \to \mathcal{W}$ be a monad defining a vector bundle $\mathcal{E}$ and suppose that*

$$\mathrm{Ext}^1(\mathcal{W}, \mathcal{V}) = \mathrm{Ext}^1(\mathcal{V}, \mathcal{U}) = \mathrm{Ext}^2(\mathcal{W}, \mathcal{U}) = 0.$$

Then another monad $M' : \mathcal{U} \to \mathcal{V} \to \mathcal{W}$ defines a bundle isomorphic to $\mathcal{E}$ if and only if M' is isomorphic (as a monad) to M.

The natural problem to consider next is that of reversing the above procedure, i.e. of finding monads of a particular form which define vector bundles with given properties. Although not so much can be said about this in general, we can make a few simple observations arising from the monad display (2.2.1).

Firstly, the additivity of the Chern character *ch* on short exact sequences gives the following relationship between the topological invariants of $\mathcal{U}$,$\mathcal{V}$,$\mathcal{W}$ and $\mathcal{E}$:

$$ch(\mathcal{E}) = \mathrm{ch}(\mathcal{V}) - \mathrm{ch}(\mathcal{U}) - \mathrm{ch}(\mathcal{W}).$$

In particular, this means that

$$\mathrm{rk}(\mathcal{E}) = \mathrm{rk}(\mathcal{V}) - \mathrm{rk}(\mathcal{U}) - \mathrm{rk}(\mathcal{W}) \quad \text{and} \quad c_1(\mathcal{E}) = c_1(\mathcal{V}) - c_1(\mathcal{U}) - c_1(\mathcal{W}).$$

Secondly, we can see that the double extension

$$0 \to \mathcal{U} \xrightarrow{i_1} \mathcal{K} \xrightarrow{i_4 \circ j_1} \mathcal{Q} \xrightarrow{j_4} \mathcal{W} \to 0 \tag{2.2.3}$$

represents the trivial class in $\mathrm{Ext}^2(\mathcal{W}, \mathcal{U})$. This is a corollary of the following proposition

PROPOSITION 2.2.3. *Suppose we are given two extensions*

$$0 \to \mathcal{U} \xrightarrow{i_1} \mathcal{K} \xrightarrow{j_1} \mathcal{E} \to 0 \tag{2.2.4}$$

$$0 \to \mathcal{E} \xrightarrow{i_4} \mathcal{Q} \xrightarrow{j_4} \mathcal{W} \to 0. \tag{2.2.5}$$

Then we can fit them into a completed monad display (2.2.1) *if and only if the double extension* (2.2.3), *i.e. their Ext-product, is trivial in* $\mathrm{Ext}^2(\mathcal{W}, \mathcal{U})$. *Furthermore, any two ways of completing the display differ by a natural action of* $\mathrm{Ext}^1(\mathcal{W}, \mathcal{U})$.

PROOF. The second sequence (2.2.5) induces a long exact sequence of Ext groups

$$\cdots \to {}^1\langle \mathcal{W} \mid \mathcal{U} \rangle \xrightarrow{\langle j_4 |} {}^1\langle \mathcal{Q} \mid \mathcal{U} \rangle \xrightarrow{\langle i_4 |} {}^1\langle \mathcal{E} \mid \mathcal{U} \rangle \xrightarrow{\vartheta} {}^2\langle \mathcal{W} \mid \mathcal{U} \rangle \to \cdots$$

and ϑ maps the first extension (2.2.4) to the double extension (2.2.3). Thus, it is precisely when the double extension is trivial that we can find an extension

$$0 \to \mathcal{U} \xrightarrow{\mathcal{A}} \mathcal{V} \xrightarrow{j_2} \mathcal{Q} \to 0$$

in ${}^1\langle \mathcal{Q} \mid \mathcal{U} \rangle$ which is mapped to (2.2.4) by $\langle i_4 |$. This will entail there being a (uniquely determined) map $i_3 : \mathcal{K} \to \mathcal{V}$ such that

$$\begin{array}{ccccccc} 0 \to & \mathcal{U} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E} & \to 0 \\ & \| & & \downarrow i_3 & & \downarrow i_4 & \\ 0 \to & \mathcal{U} & \longrightarrow & \mathcal{V} & \longrightarrow & \mathcal{Q} & \to 0 \end{array}$$

commutes. Finally, setting $\mathcal{B} = j_4 \circ j_2$ we get the sequence required to complete the display:

$$0 \to \mathcal{K} \xrightarrow{i_3} \mathcal{V} \xrightarrow{\mathcal{B}} \mathcal{W} \to 0.$$

By this procedure, the completions of the display are identified with a coset of $\langle j_4 \mid \left({}^1\langle \mathcal{W} \mid \mathcal{U} \rangle\right)$ in ${}^1\langle \mathcal{Q} \mid \mathcal{U} \rangle$, i.e. an affine space carrying a transitive action of ${}^1\langle \mathcal{W} \mid \mathcal{U} \rangle$. In fact, ${}^1\langle \mathcal{W} \mid \mathcal{U} \rangle$ acts naturally on the space of all monads with ends $\mathcal{U}$ and $\mathcal{W}$, and the monads which occur in this particular completed display form one orbit under this action.

Notice that, as one would expect from the symmetry of the proposition, there is a "mirror image" proof which starts by considering the long exact sequence induced by the first extension

(2.2.4), but which encounters the same obstruction and the same ambiguity as in the proof given above. □

Observe that the obstruction to completing the display automatically vanishes if $\mathrm{Ext}^2(\mathcal{W},\mathcal{U}) = 0$, which will be the case in practice, because we will be looking for monads which satisfy the hypotheses of Proposition 2.2.1.

With the help of the two propositions proved above, we have the following general method of constructing monads for describing families of vector bundles (as used by **[At]** and **[Bu1]**):

1) Choose suitable bundles $\mathcal{U}$ and $\mathcal{W}$ for the ends of the monad, which satisfy $\mathrm{Ext}^2(\mathcal{W},\mathcal{U}) = 0$.
2) Choose suitable extensions $0 \to \mathcal{U} \to \mathcal{K} \to \mathcal{E} \to 0$ and $0 \to \mathcal{E} \to \mathcal{Q} \to \mathcal{W} \to 0$.
3) Observe, by Proposition 2.2.3, that the display can be completed, yielding a monad $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$.
4) Identify the middle term by cohomological considerations. This is where one requires the "suitability" in (1) and (2) and the special properties of the bundles in the family.
5) Check that $\mathrm{Ext}^1(\mathcal{V},\mathcal{U}) = \mathrm{Ext}^1(\mathcal{W},\mathcal{V}) = 0$, so that Corollary 2.2.2 holds.

The monads that we shall construct to classify holomorphic bundles on the Hirzebruch surface Σ^1 will have $\mathcal{U} = \bigoplus U_i \otimes \mathcal{L}_i$ and $\mathcal{W} = \bigoplus W_i \otimes \mathcal{L}_i^*$, for vector spaces U_i and W_i and line bundles $\mathcal{L}_i$, and we shall actually be able to deduce that $\mathcal{V}$ is trivial. The construction will be dealt with in detail in the next chapter.

2.3 Monads and Jumping Behaviour

One important phenomenon that helps in studying holomorphic vector bundles on some algebraic surfaces — particularly rational surfaces and ruled surfaces (c.f. **[Br1,Br2]**) — is that of jumping. Suppose we are considering a vector bundle $\mathcal{E}$ on a surface S, and that we have an algebraic family $\{L_p \mid p \in V\}$ of subvarieties of S, all of which are projective lines. Then $c_1(\mathcal{E})$ determines the topological type of $\mathcal{E}|_{L_p}$, which is independent of p, since all the lines are homologous. By Grotendieck's classification, we know that $\mathcal{E}|_{L_p}$ is, algebraically, a direct sum of line bundles

$$\mathcal{E}|_{L_p} \cong \bigoplus_{i=1}^{\mathrm{rk}\,\mathcal{E}} \mathcal{O}_{L_p}(d_i),$$

where the unordered sequence $\{d_i\}$ is the *splitting type* of $\mathcal{E}$ along L_p. The topological considerations only fix $\sum d_i = \chi(\mathcal{E}|_{L_p}(-1))$, but we can, in fact, typically expect the splitting type to be constant (and in some sense minimal) for generic $p \in V$. The lines along which the splitting type is not generic are called the *jumping lines* of the bundle $\mathcal{E}$. Now, isomorphic bundles clearly have the same jumping behaviour and bundles which are direct sums of line bundles clearly have no jumping lines. Hence, the jumping behaviour of $\mathcal{E}$ in some way measures the extent to which $\mathcal{E}$ is not a direct sum of line bundles.

We shall mostly be concerned with bundles with $c_1(\mathcal{E}) = 0$, which are trivial on generic lines. Triviality of a bundle on a line has the following simple cohomological characterisation.

Lemma 2.3.1. *A bundle $\mathcal{E}$ on $\mathbb{P}^1$ is trivial if and only if*

$$H^0(\mathcal{E}(-1)) = H^1(\mathcal{E}(-1)) = 0.$$

Proof. Immediate from the Grotendieck classification and the identity

$$H^0(\mathcal{E}(-1)) + H^1(\mathcal{E}(-1)) = \sum |d_i|,$$

where $\{d_i\}$ is the splitting type of $\mathcal{E}$. □

If we wish to study the jumping behaviour of a bundle defined by a monad, then it is very convenient for the terms of the monad to be direct sums of line bundles, because then the restrictions of the monad to the lines of an algebraic family will have essentially constant terms. Another convenient feature is for the monad to have a trivial middle term, because we then have the following result.

Proposition 2.3.2. *Suppose that $\mathcal{E}$ is a holomorphic vector bundle on $\mathbb{P}^1$ which is defined by a monad $\mathcal{U} \to \mathcal{V} \to \mathcal{W}$ with $\mathcal{V}$ trivial. Then there is a naturally induced map $\phi : H^1(\mathcal{U}(-1)) \to H^0(\mathcal{W}(-1))$, which is an isomorphism if and only if $\mathcal{E}$ is trivial.*

Proof. The monad display, twisted by the tautological bundle $\mathcal{O}_{\mathbb{P}^1}(-1)$, induces, after some rearrangement, the following exact commuting diagram

$$\begin{array}{ccccccc}
 & & 0 & \longrightarrow & H^0(\mathcal{K}(-1)) & \longrightarrow & H^0(\mathcal{V}(-1)) \\
 & & \downarrow & & \downarrow & & \\
 & & H^0(\mathcal{E}(-1)) & = & H^0(\mathcal{E}(-1)) & & \\
 & & \downarrow & & \downarrow & & \\
H^0(\mathcal{V}(-1)) & \longrightarrow & H^0(\mathcal{Q}(-1)) & \longrightarrow & H^1(\mathcal{U}(-1)) & \longrightarrow & H^1(\mathcal{V}(-1)) \\
\| & & \downarrow & & \downarrow & & \| \\
H^0(\mathcal{V}(-1)) & \longrightarrow & H^0(\mathcal{W}(-1)) & \longrightarrow & H^1(\mathcal{K}(-1)) & \longrightarrow & H^1(\mathcal{V}(-1)) \\
 & & \downarrow & & \downarrow & & \\
 & & H^1(\mathcal{E}(-1)) & = & H^1(\mathcal{E}(-1)) & & \\
 & & \downarrow & & \downarrow & & \\
H^1(\mathcal{V}(-1)) & \longrightarrow & H^1(\mathcal{Q}(-1)) & \longrightarrow & 0 & &
\end{array}$$

The triviality of $\mathcal{V}$ implies that $H^i(\mathcal{V}(-1)) = 0$, for $i = 0, 1$. Hence we can reduce the above diagram to the single exact sequence

$$0 \to H^0(\mathcal{E}(-1)) \to H^1(\mathcal{U}(-1)) \xrightarrow{\phi} H^0(\mathcal{W}(-1)) \to H^1(\mathcal{E}(-1)) \to 0$$

and the result follows from Lemma 2.3.1. □

Remark 2.3.3. If we call a bundle $\mathcal{E}$ over $\mathbb{P}^1$ *generic* when at least one of $H^0(\mathcal{E}(-1))$ and $H^1(\mathcal{E}(-1))$ vanishes, then Proposition 2.3.2 can be generalised to say that ϕ has maximal rank if and only if $\mathcal{E}$ is generic. The use of the term generic is justified by the Semicontinuity Theorem for cohomology **[Ha;III.12]**.

If we are in the special case when $\mathcal{U} = U \otimes \mathcal{O}(-1)$ and $\mathcal{W} = W \otimes \mathcal{O}(1)$ for vector spaces U and W then

$$\phi : U \otimes H^1(\mathcal{O}(-2)) \to W \otimes H^0(\mathcal{O}).$$

Now, $H^0(\mathcal{O}) = \mathbb{C}$ and $H^1(\mathcal{O}(-2))$ can be identified with $\Lambda^2(X)^*$, where X is the two dimensional vector space such that $\mathbb{P}^1 = \mathbb{P}(X)$. We can also identify $H^0(\mathcal{O}(1))$ with X^*, so that the monad maps $\mathcal{A} : \mathcal{U} \to \mathcal{V}$ and $\mathcal{B} : \mathcal{V} \to \mathcal{W}$ can be thought of as elements of $X^* \otimes \mathrm{Hom}(U, V)$ and $X^* \otimes \mathrm{Hom}(V, W)$ respectively. Combining these to get an element of $X^* \otimes X^* \otimes \mathrm{Hom}(U, W)$, we observe that the condition $\mathcal{B} \circ \mathcal{A} = 0$ means that this element is alternating in X, i.e. it is an element of $\Lambda^2(X)^* \otimes \mathrm{Hom}(U, W)$ and, in fact, it is just ϕ. Thus we get the following lemma (also proved in [OSS]):

Lemma 2.3.4. *If $[p]$ and $[q]$ are two (distinct) points on $\mathbb{P}^1$, then a bundle $\mathcal{E}$, defined by a monad of the form*

$$U \otimes \mathcal{O}(-1) \xrightarrow{\mathcal{A}} V \otimes \mathcal{O} \xrightarrow{\mathcal{B}} W \otimes \mathcal{O}(1),$$

is trivial if and only if $\mathcal{B}(p) \circ \mathcal{A}(q) : U \to W$ is an isomorphism.

Proof. Since p and q are independent, $p \wedge q$ spans $\Lambda^2(X)$ and so $\mathcal{B}(p) \circ \mathcal{A}(q)$ is essentially ϕ. The lemma then follows from Proposition 2.3.2. □

3 The Construction of the Moduli Spaces

3.1 Cohomology of Line Bundles over Σ^1

Before we proceed with the monad construction, it will be useful to have a 'cohomology map' of $\mathrm{Pic}(\Sigma^1)$, i.e. to know where various cohomology groups of line bundles on Σ^1 vanish and to know their dimensions in some cases where they do not vanish.

On an algebraic surface the primary tools are the Riemann-Roch formula

$$\chi(\mathcal{O}(D)) = \frac{1}{2}D \cdot (D - K) + 1 + p_a$$

and one instance of Serre duality

$$H^2(\mathcal{O}(D)) \cong H^0(\mathcal{O}(K - D)).$$

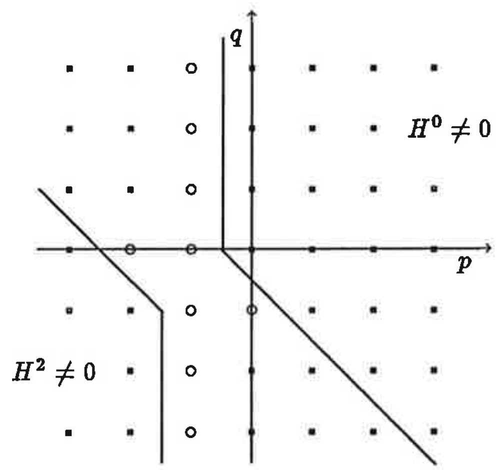
Here D is any divisor, $\mathcal{O}(D)$ is the associated line bundle, K is the canonical divisor and p_a is the arithmetic genus.

As we determined in §1.4, on Σ^1 we have $\mathcal{O}(1,0) = \mathcal{O}(H)$ and $\mathcal{O}(0,1) = \mathcal{O}(F)$, where $H^2 = 1$, $H \cdot F = 1$ and $F^2 = 0$. Furthermore, $K = -2H - F$ and, since Σ^1 is rational, $p_a = 0$. Hence we derive the following versions of Riemann-Roch and Serre duality on Σ^1:

$$\chi(\mathcal{O}(p,q)) = \frac{1}{2}(p+1)(p+2q+2) \qquad (3.1.1)$$

$$h^2(\mathcal{O}(p,q)) = h^0(\mathcal{O}(-2-p,-1-q)). \qquad (3.1.2)$$

Now we observe that, on any line (i.e. smooth rational curve) in the linear system $|\mathcal{O}(1,0)|$ (resp. $|\mathcal{O}(0,1)|$) the bundle $\mathcal{O}(p,q)$ restricts to $\mathcal{O}_{\mathbf{P}^1}(p+q)$ (resp. $\mathcal{O}_{\mathbf{P}^1}(p)$). But the set of all lines in either linear system sweeps out a dense open set in Σ^1 and so, when $p+q<0$ or $p<0$, $\mathcal{O}(p,q)$ has no non-zero sections, i.e. $H^0(\mathcal{O}(p,q)) = 0$. In all other cases, $H^0(\mathcal{O}(p,q)) \neq 0$, because $\mathcal{O}(0,1)$ and $\mathcal{O}(1,-1)$ are effective. Using Serre duality, we can also identify the region in $\mathrm{Pic}(\Sigma^1)$, where $H^2(\mathcal{O}(p,q)) \neq 0$ and, using the Riemann-Roch formula, we can identify a further region in which $H^i(\mathcal{O}(p,q)) = 0$, for all i. We summarise this information in the following diagram:

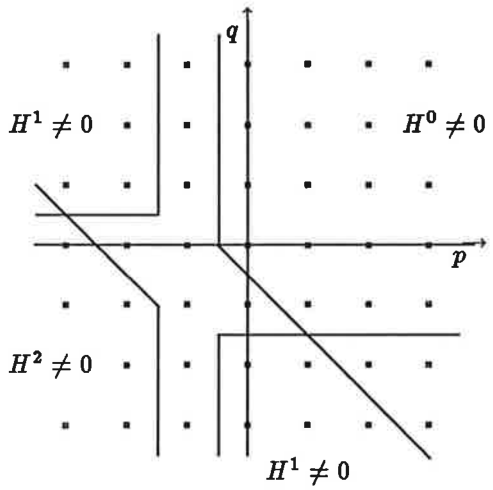


where o denotes a value of (p,q) for which $H^i(\mathcal{O}(p,q)) = 0$, for all i.

To identify the region '$H^1 \neq 0$', consider the long exact sequence

$$\cdots \to H^0(\mathcal{O}_H(p+q)) \to H^1(\mathcal{O}(p-1,q)) \to H^1(\mathcal{O}(p,q)) \to H^1(\mathcal{O}_H(p+q)) \to \cdots.$$

This allows us to transfer the condition '$H^1(\mathcal{O}(p,q)) = 0$' to the left if $p+q<0$ and to the right if $p+q+1>-2$. Thus we get the completed 'cohomology map'



To show that $H^1(\mathcal{O}(p,q))$ is indeed non-zero throughout the labelled regions, we need to use the structure sequence for E to show that

$$H^0(\mathcal{O}(p,q)) \cong H^0(\mathcal{O}(p+q,0)) \quad \text{for } q<0,$$

and then Riemann-Roch to calculate the dimension of H^1.

In fact the information we shall need from the above diagram is just the vanishing information and the fact that $h^1(\mathcal{O}(0,-2)) = 1$, which follows from (3.1.1).

3.2 Preliminary Results

In this section we prove two results which are important in determining the form of the monads which we shall construct in the following section. The first concerns the cohomology groups which are the basic building blocks of the monad; the second gives a criterion for determining the triviality of a holomorphic vector bundle over Σ^1.

PROPOSITION 3.2.1. *Let $\mathcal{E}$ be a holomorphic vector bundle over Σ^1 of rank r, with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$ and such that $\mathcal{E}|_{\ell_\infty}$ is trivial. Then*

$$h^i(\mathcal{E}(p,q)) = \begin{cases} 0 & \text{if } i = 0,2 \\ k & \text{if } i = 1 \end{cases}$$

for $(p,q) = (-2,0), (-1,0), (-1,-1)$ or $(0,-1)$.

Proof. First observe that $\mathcal{O}(1,0)$ and $\mathcal{O}(0,1)$ both restrict to $\mathcal{O}(1)$ on ℓ_∞. Therefore, the fact that $\mathcal{E}|_{\ell_\infty}$ is trivial implies that, when $p+q<0$, any section of $\mathcal{E}(p,q)$ vanishes along ℓ_∞. But the set $\{D \in |\mathcal{O}(1,0)| : \mathcal{E}|_D \text{ is trivial}\}$ is open and the set of points on some such D is open in Σ^1. Hence a section of $\mathcal{E}(p,q)$ must vanish on a non-empty open set and so must be identically zero. Thus

$$H^0(\mathcal{E}(p,q)) = 0 \qquad \text{for } p+q<0.$$

By Serre duality—because $\mathcal{E}^*$ is also trivial on ℓ_∞—we get

$$H^2(\mathcal{E}(p,q)) = 0 \qquad \text{for } p+q>-3.$$

Finally, the Hirzebruch-Riemann-Roch theorem **[Ha;A4]** combined with (3.1.1) gives

$$\chi(\mathcal{E}(p,q)) = \frac{1}{2}(p+1)(p+2q+2)r - k,$$

which implies that $h^1(\mathcal{E}(p,q)) = k$ for the values of (p,q) under consideration. □

Lemma 3.2.2. *Let $\mathcal{V}$ be a holomorphic vector bundle over Σ^1 with $c_1(\mathcal{V}) = 0$, $c_2(\mathcal{V}) = 0$ and such that $\mathcal{V}|_{\ell_\infty}$ is trivial. Then $\mathcal{V}$ is trivial.*

Proof. We extend the triviality of $\mathcal{V}$ on ℓ_∞ to Σ^1 by considering the sequence

$$0 \to \mathcal{V}(-1,0) \to \mathcal{V} \to \mathcal{V}|_{\ell_\infty} \to 0$$

and its cohomology sequence

$$H^0(\mathcal{V}(-1,0)) \to H^0(\mathcal{V}) \to H^0(\mathcal{V}|_{\ell_\infty}) \to H^1(\mathcal{V}(-1,0)) \to \cdots$$

Proposition 3.2.1 tells us that $H^0(\mathcal{V}(-1,0)) = H^1(\mathcal{V}(-1,0)) = 0$ and thus that the restriction map on sections induces an isomorphism $H^0(\mathcal{V}) \cong H^0(\mathcal{V}|_{\ell_\infty})$. Now $h^0(\mathcal{V}|_{\ell_\infty}) = \operatorname{rk}\mathcal{V}$, so picking a basis for $H^0(\mathcal{V})$ determines a map $\beta : (\mathcal{O}_{\Sigma^1})^{\operatorname{rk}\mathcal{V}} \to \mathcal{V}$. But $c_1(\mathcal{V}) = 0$ and Σ^1 is regular, so $\det(\mathcal{V}) = \mathcal{O}_{\Sigma^1}$ and therefore $\det(\beta)$ is constant. Now β is already an isomorphism over ℓ_∞, because $\mathcal{V}|_{\ell_\infty}$ is trivial, so β is actually an isomorphism over the whole of Σ^1, as required. □

Remark 3.2.3. In addition to being useful later on, this lemma shows us that the moduli space $\mathsf{MH}(\widetilde{\mathbb{C}}^2; 0, r)$ consists of a single point, representing the trivial bundle, and so is the same as the moduli space $\mathsf{MI}(\widetilde{\mathbb{C}}^2; 0, r)$ which consists of a single point, representing the trivial connection (the only flat connection).

3.3 Construction of Monads

We start this section by quoting Buchdahl's construction of monads on the flag manifold $\mathbb{F}$ which classify self-dual instantons on $\mathbb{C}\mathrm{P}^2$, or equivalently anti-self-dual instantons on $\overline{\mathbb{C}\mathrm{P}}^2$. We then give a compatible construction of monads on Σ^1 which is the key to proving our main correspondence theorem.

THEOREM 3.3.1. *Let $\mathcal{E}$ be a holomorphic vector bundle over $\mathbb{F}$ of rank r, which is the Ward transform of an (anti-self-dual) instanton on $\overline{\mathbb{C}\mathrm{P}}^2$ of index k. Then there exists a monad*

$$\mathcal{U} \xrightarrow{\mathcal{A}} \mathcal{V} \xrightarrow{\mathcal{B}} \mathcal{W},$$

with cohomology $\mathcal{E}$, in which $\mathcal{V}$ is a trivial bundle of rank $4k + r$,

$$\mathcal{U} = \bigoplus_{i=0}^{1} U_i \otimes \mathcal{L}_i \quad \text{and} \quad \mathcal{W} = \bigoplus_{i=0}^{1} W_i \otimes \mathcal{L}_i^*,$$

where U_i and W_i are complex vector spaces of dimension k and the line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ are $\mathcal{O}_{\mathbb{F}}(-1,0)$ and $\mathcal{O}_{\mathbb{F}}(0,-1)$ respectively.

PROOF. **[Bu1]**

THEOREM 3.3.2. *Let $\mathcal{E}$ be a holomorphic vector bundle over Σ^1 of rank r, with $c_1(\mathcal{E}) = 0$, $c_2(\mathcal{E}) = k$ and such that $\mathcal{E}|_{\ell_\infty}$ is trivial. Then there exists a monad*

$$\mathcal{U} \xrightarrow{\mathcal{A}} \mathcal{V} \xrightarrow{\mathcal{B}} \mathcal{W},$$

with cohomology $\mathcal{E}$, in which $\mathcal{V}$ is a trivial bundle of rank $4k + r$,

$$\mathcal{U} = \bigoplus_{i=0}^{1} U_i \otimes \mathcal{L}_i \quad \text{and} \quad \mathcal{W} = \bigoplus_{i=0}^{1} W_i \otimes \mathcal{L}_i^*,$$

where U_i and W_i are complex vector spaces of dimension k and the line bundles $\mathcal{L}_0$ and $\mathcal{L}_1$ are $\mathcal{O}_{\Sigma^1}(-1,0)$ and $\mathcal{O}_{\Sigma^1}(0,-1)$ respectively.

PROOF. We proceed in the standard manner described in §2.2 to construct the monad display:

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{U} & \longrightarrow & \mathcal{K} & \longrightarrow & \mathcal{E} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{U} & \xrightarrow{\mathcal{A}} & \mathcal{V} & \longrightarrow & \mathcal{Q} & \to & 0 \\
 & & & & \downarrow{\scriptstyle\mathcal{B}} & & \downarrow & & \\
 & & & & \mathcal{W} & = & \mathcal{W} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

The proof relies heavily on the results of §3.1 and §3.2 and we shall not always make specific reference to these results when they are used.

Stage 1

We start by choosing

$$U_i = \mathrm{Ext}^1(\mathcal{E}, \mathcal{L}_i)^* = H^1(\mathcal{E} \otimes \mathcal{L}_i^*(K)),$$
$$W_i = \mathrm{Ext}^1(\mathcal{L}_i^*, \mathcal{E}) \ = H^1(\mathcal{E} \otimes \mathcal{L}_i).$$

Notice that $\mathcal{L}_0^*(K) = \mathcal{O}(-1,-1)$ and $\mathcal{L}_1^*(K) = \mathcal{O}(-2,0)$, so, by Proposition 3.2.1, the U_i and W_i have dimension k as required.

Stage 2

Now choose $0 \to \mathcal{U} \to \mathcal{K} \to \mathcal{E} \to 0$ to be the 'universal' extension given by the canonical element $1 \oplus 1$ in

$$\mathrm{Ext}^1(\mathcal{E}, \mathcal{U}) = \bigoplus_{i=0}^{1} \mathrm{End}\big(\mathrm{Ext}^1(\mathcal{E}, \mathcal{L}_i)\big).$$

Similarly, choose $0 \to \mathcal{E} \to \mathcal{Q} \to \mathcal{W} \to 0$ to be the universal extension $1 \oplus 1$ in

$$\mathrm{Ext}^1(\mathcal{W}, \mathcal{E}) = \bigoplus_{i=0}^{1} \mathrm{End}\big(\mathrm{Ext}^1(\mathcal{L}_i^*, \mathcal{E})\big).$$

The effect of this choice of extension is that if we twist the extension by $\mathcal{L}_i$ then, in the long exact cohomology sequence, the coboundary map

$$\partial : H^0(\mathcal{W} \otimes \mathcal{L}_i) \longrightarrow H^1(\mathcal{E} \otimes \mathcal{L}_i)$$

is given by $\partial = \partial_0 + \partial_1$, where

$$\partial_j : H^1(\mathcal{E} \otimes \mathcal{L}_j) \otimes H^0(\mathcal{L}_j^* \otimes \mathcal{L}_i) \longrightarrow H^1(\mathcal{E} \otimes \mathcal{L}_i),$$

for $j = 0, 1$, are the natural multiplication maps. In particular, if $j = i$ then ∂_i is essentially the identity map.

Stage 3

We find that

$$\mathrm{Ext}^2(\mathcal{W}, \mathcal{U}) = H^2(\mathcal{W}^* \otimes \mathcal{U}) = \bigoplus_{i,j} W_i^* \otimes U_j \otimes H^2(\mathcal{L}_i \otimes \mathcal{L}_j) = 0$$

and so, by Proposition 2.2.3, we can automatically complete the monad display. Observe that the recipe described above is natural with respect to duality, in the sense that if we apply the same procedure to $\mathcal{E}^*$ we derive a display which is dual to a display derived for $\mathcal{E}$. To see this, first observe, by Serre duality, that the ends of the monad we would write down for $\mathcal{E}^*$, say $\mathcal{U}'$ and $\mathcal{W}'$, are simply $\mathcal{W}^*$ and $\mathcal{U}^*$, the duals of the ends of the monads for $\mathcal{E}$. The rest follows from the fact that the natural isomorphism $\mathrm{Ext}^i(A^*, B^*) \cong \mathrm{Ext}^i(B, A)$ is the one induced by dualising extensions, so that the universal extensions correspond correctly, as do the obstruction to and ambiguity in completing the display.

Stage 4

To complete the proof of Theorem 3.3.2 we need to show that the middle bundle in the monad can be chosen to be trivial. We can immediately see that $c_1(\mathcal{V}) = 0$, because the first Chern class is additive on short exact sequences. A more careful calculation shows that $c_2(\mathcal{V}) = 0$ as well, so that $\mathcal{V}$ is topologically trivial. The fact that it is holomorphically trivial will then follow from Lemma 3.2.2 provided we can show that $\mathcal{V}|_{\ell_\infty}$ is trivial. This, in turn, is reduced to a cohomology vanishing condition on $\mathcal{V}$ itself as follows.

LEMMA 3.3.3. *If $\mathcal{V}$ can be chosen so that $H^0(\mathcal{V}(0,-1)) = 0$, then $\mathcal{V}|_{\ell_\infty}$ is trivial.*

PROOF. Certainly, $\mathcal{V}|_{\ell_\infty}$ is topologically trivial, so it suffices to show that $H^0(\mathcal{V}|_{\ell_\infty}(-1)) = 0$, since then $H^1(\mathcal{V}|_{\ell_\infty}(-1)) = 0$, by Riemann-Roch, and thus $\mathcal{V}|_{\ell_\infty}$ is holomorphically trivial, by Lemma 2.3.1. This required cohomology vanishing can be deduced from the long exact sequence

$$\cdots \to H^0(\mathcal{V}(0,-1)) \to H^0(\mathcal{V}|_{\ell_\infty}(-1)) \to H^1(\mathcal{V}(-1,-1)) \to \cdots$$

induced by the structure sequence $0 \to \mathcal{O}(-1,-1) \to \mathcal{O}(0,-1) \to \mathcal{O}|_{\ell_\infty}(-1) \to 0$, provided that we can also show that $H^1(\mathcal{V}(-1,-1)) = 0$. But this holds for any choice of $\mathcal{V}$, as we can see from the diagram

$$\begin{array}{ccccccc} H^1(\mathcal{U}(-1,-1)) & \to & H^1(\mathcal{K}(-1,-1)) & \to & H^1(\mathcal{E}(-1,-1)) & \xrightarrow{\delta} & H^2(\mathcal{U}(-1,-1)) \\ & & \downarrow & & & & \\ & & H^1(\mathcal{V}(-1,-1)) & & & & \\ & & \downarrow & & & & \\ & & H^1(\mathcal{W}(-1,-1)) & & & & \end{array}$$

induced by the monad display, since $H^1(\mathcal{U}(-1,-1)) = H^1(\mathcal{W}(-1,-1)) = 0$ and we chose the extension $0 \to \mathcal{U} \to \mathcal{K} \to \mathcal{E} \to 0$ at Stage 2 so that the connecting homomorphism δ was an isomorphism. □

Since we know that $H^0(\mathcal{U}(0,-1)) = 0$, a cohomology sequence induced by the extension $0 \to \mathcal{U} \to \mathcal{V} \to \mathcal{Q} \to 0$ includes

$$0 \to H^0(\mathcal{V}(0,-1)) \to H^0(\mathcal{Q}(0,-1)) \xrightarrow{\partial} H^1(\mathcal{U}(0,-1)) \to \cdots$$

and we are lead to the requirement that we can choose this extension so that ∂ is injective. Now, $H^1(\mathcal{U}(0,-1)) = U_1 \otimes H^1(\mathcal{O}(0,-2))$ and so has dimension k. Furthermore, $H^0(\mathcal{Q}(0,-1))$ also has dimension k, because it is isomorphic to $W_0 \otimes H^0(\mathcal{O}(1,-1))$, as we see from the long exact sequence

$$H^0(\mathcal{E}(0,-1)) \to H^0(\mathcal{Q}(0,-1)) \to H^0(\mathcal{W}(0,-1)) \xrightarrow{\delta} H^1(\mathcal{E}(0,-1)) \to \cdots$$

in which $\delta : W_0 \otimes H^0(\mathcal{O}(1,-1)) \oplus W_1 \longrightarrow W_1$ has the identity map as its second component (by construction) and $H^0(\mathcal{E}(0,-1)) = 0$, by Proposition 3.2.1. Hence, we really want to show that ∂ can be chosen to have maximal possible rank, k.

As shown in Stage 3, the freedom we have in the choice of the extension $0 \to \mathcal{U} \to \mathcal{V} \to \mathcal{Q} \to 0$ is provided by an action of

$$\mathrm{Ext}^1(\mathcal{W},\mathcal{U}) = H^1(\mathcal{W}^* \otimes \mathcal{U}) = W_1^* \otimes U_1 \otimes H^1(\mathcal{O}(0,-2)).$$

This action can be described by saying that an element of $H^1(\mathcal{W}^* \otimes \mathcal{U})$ acts, by cohomology multiplication, as a map $\phi : H^0(\mathcal{W} \otimes \mathcal{L}) \to H^1(\mathcal{U} \otimes \mathcal{L})$ (for an arbitrary line bundle $\mathcal{L}$), which modifies the connecting homomorphism $\partial : H^0(\mathcal{Q} \otimes \mathcal{L}) \to H^1(\mathcal{U} \otimes \mathcal{L})$ by $\partial \mapsto \partial + \phi j_*$, where j_* is the naturally induced cohomology map

$$\begin{array}{ccc} H^0(\mathcal{Q} \otimes \mathcal{L}) & & \\ j_* \downarrow & \searrow & \\ H^0(\mathcal{W} \otimes \mathcal{L}) & \xrightarrow{\phi} & H^1(\mathcal{U} \otimes \mathcal{L}) \end{array}$$

In our case, where $\mathcal{L} = \mathcal{O}(0,-1)$, we can fix a non-zero section $s \in H^0(\mathcal{O}(1,-1))$ and then identify j_* with the inclusion $W_0 \hookrightarrow W_0 \oplus W_1$ of the graph of the cohomology multiplication map s_* : $H^1(\mathcal{E}(-1,0)) \to H^1(\mathcal{E}(0,-1))$. Since, the map ϕ can be any map which factors through the second projection, it follows that we can modify ∂ by adding on any map with rank $\leqslant \mathrm{rk}\, s_*$. Therefore, we can choose ∂ to be of maximal rank if and only if, for an arbitrary choice of ∂, $\mathrm{rk}\,\partial + \mathrm{rk}\, s_* \geqslant k$. This inequality is proved by means of the following sequence of lemmas.

Lemma 3.3.4. $H^0(\mathcal{K}|_E(-1)) = 0$

Proof. The cohomology sequences coming from the product of the sequences $0 \to \mathcal{U} \to \mathcal{K} \to \mathcal{E} \to 0$ and $0 \to \mathcal{O}(-2,0) \to \mathcal{O}(-1,-1) \to \mathcal{O}_E(-1) \to 0$ include

$$\begin{array}{ccccc} H^0(\mathcal{U}(-1,-1)) & & & & H^1(\mathcal{U}(-2,0)) \\ \downarrow & & & & \downarrow \\ H^0(\mathcal{K}(-1,-1)) & \to & H^0(\mathcal{K}|_E(-1)) & \to & H^1(\mathcal{K}(-2,0)) \\ \downarrow & & & & \downarrow \\ H^0(\mathcal{E}(-1,-1)) & & & & H^1(\mathcal{E}(-2,0)) \\ & & & & \downarrow \delta \\ & & & & H^2(\mathcal{U}(-2,0)) \end{array}$$

But the extension $0 \to \mathcal{U} \to \mathcal{K} \to \mathcal{E} \to 0$ was chosen so that the connecting homomorphism δ is injective. In addition, $H^0(\mathcal{U}(-1,-1)) = H^0(\mathcal{E}(-1,-1)) = H^1(\mathcal{U}(-2,0)) = 0$, from which the lemma follows. □

Lemma 3.3.5. *Let $\partial_E : H^0(\mathcal{Q}|_E(-1)) \to H^1(\mathcal{U}|_E(-1))$ be the boundary map induced by the extension $0 \to \mathcal{U} \to \mathcal{K} \to \mathcal{Q} \to 0$, when it is restricted to E and twisted by $\mathcal{O}_E(-1)$. Then $\mathrm{rk}\,\partial_E \geqslant h^0(\mathcal{E}|_E(-1))$.*

Proof. Use the previous Lemma and the following part of the exact grid of cohomology sequences induced by the monad display, restricted to E and twisted by $\mathcal{O}_E(-1)$.

$$\begin{array}{ccccc}
H^0(\mathcal{K}|_E(-1)) & \to & H^0(\mathcal{E}|_E(-1)) & \to & H^1(\mathcal{U}|_E(-1)) \\
\downarrow & & \downarrow & & \| \\
H^0(\mathcal{V}|_E(-1)) & \to & H^0(\mathcal{Q}|_E(-1)) & \xrightarrow{\partial_E} & H^1(\mathcal{U}|_E(-1))
\end{array}$$

□

Corollary 3.3.6. $\operatorname{rk} s_* + \operatorname{rk} \partial_E \geqslant k$.

Proof. Follows from the previous Lemma and the sequence

$$0 \to H^0(\mathcal{E}|_E(-1)) \to H^1(\mathcal{E}(-1,0)) \xrightarrow{s_*} H^1(\mathcal{E}(0,-1)) \to H^1(\mathcal{E}|_E(-1)) \to 0$$

where the zeroes at the ends are provided by Proposition 3.2.1. □

Lemma 3.3.7. $H^0(\mathcal{Q}(-1,0)) = H^1(\mathcal{Q}(-1,0)) = 0$

Proof. Since $H^0(\mathcal{E}(-1,0)) = H^1(\mathcal{W}(-1,0)) = 0$, the extension $0 \to \mathcal{E} \to \mathcal{Q} \to \mathcal{W} \to 0$ induces the following long exact sequence

$$0 \to H^0(\mathcal{Q}(-1,0)) \to H^0(\mathcal{W}(-1,0)) \xrightarrow{\delta} H^1(\mathcal{E}(-1,0)) \to H^1(\mathcal{Q}(-1,0)) \to 0$$

in which the connecting homomorphism δ is an isomorphism, by construction. □

Corollary 3.3.8. $\operatorname{rk} \partial = \operatorname{rk} \partial_E$

Proof. In fact, ∂ can be identified with ∂_E by the restriction map induced by the structure sequence $0 \to \mathcal{O}(-1,0) \to \mathcal{O}(0,-1) \to \mathcal{O}_E(-1) \to 0$, as we see from the diagram

$$\begin{array}{ccccccc}
H^0(\mathcal{Q}(-1,0)) & \to & H^0(\mathcal{Q}(0,-1)) & \to & H^0(\mathcal{Q}|_E(-1)) & \to & H^1(\mathcal{Q}(-1,0)) \\
\downarrow & & \downarrow \partial & & \downarrow \partial_E & & \downarrow \\
H^1(\mathcal{U}(-1,0)) & \to & H^1(\mathcal{U}(0,-1)) & \to & H^1(\mathcal{U}|_E(-1)) & \to & H^2(\mathcal{U}(-1,0))
\end{array}$$

because the four spaces at the corners of the diagram all vanish. □

The two Corollaries above provide the inequality that we had shown was necessary to complete the proof of Theorem 3.3.2.

Stage 5

Finally, we observe that $\operatorname{Ext}^1(\mathcal{V},\mathcal{U}) = \operatorname{Ext}^1(\mathcal{W},\mathcal{V}) = 0$, because $H^1(\mathcal{L}_i) = 0$ for $i = 0,1$. Therefore we can apply Corollary 2.2.2 to deduce that the monads of the form given in the theorem do effectively parametrise the bundles in which we are interested, once we have taken into account the monad automorphisms.

3.4 Reduction of Monads on Σ^1

In the previous section, we gave a monad construction for holomorphic vector bundles on Σ^1, trivial on ℓ_∞. However, this does not, as it stands, give a particularly convenient description of the moduli space of such bundles. One reason is that this description involves taking the quotient by a large symmetry group—the group of automorphisms of the monad. In this section, we find a canonical form for the monad which reduces the symmetry group, incorporates the triviality on ℓ_∞ and yields a much simpler description of the moduli space in terms of a system of linear maps.

Suppose we are given, as in Theorem 3.3.2, a monad of the form

$$\mathcal{U} \xrightarrow{\mathcal{A}} \mathcal{V} \xrightarrow{\mathcal{B}} \mathcal{W}$$

which describes a holomorphic bundle $\mathcal{E}$ on Σ^1 with $\mathcal{E}|_{\ell_\infty}$ trivial. Let V denote the vector space on which the trivial bundle $\mathcal{V}$ is modelled and choose bases for the folowing spaces, as in §1.4:

$$\begin{aligned} H^0(\mathcal{L}_0^*) &= H^0\big(\mathcal{O}_{\Sigma^1}(1,0)\big) = \langle x_1, x_2, x_3\rangle \\ H^0(\mathcal{L}_1^*) &= H^0\big(\mathcal{O}_{\Sigma^1}(0,1)\big) = \langle y_1, y_2\rangle \\ H^0(\mathcal{L}_0^* \otimes \mathcal{L}_1) &= H^0\big(\mathcal{O}_{\Sigma^1}(1,-1)\big) = \langle s\rangle \end{aligned}$$

where $s\, y_1 = x_2$ and $s\, y_2 = -x_1$. Note also that

$$H^0(\mathcal{L}_1^* \otimes \mathcal{L}_0) = H^0\big(\mathcal{O}_{\Sigma^1}(-1,1)\big) = 0$$

We now observe that

$$\begin{aligned} \mathcal{A} &\in \mathrm{Hom}(\mathcal{U},\mathcal{V}) = \bigoplus_{i=0}^{1} \mathrm{Hom}(U_i, V) \otimes H^0(\mathcal{L}_i^*) \\ \mathcal{B} &\in \mathrm{Hom}(\mathcal{V},\mathcal{W}) = \bigoplus_{i=0}^{1} \mathrm{Hom}(V, W_i) \otimes H^0(\mathcal{L}_i^*) \end{aligned}$$

and therefore we can write

$$\begin{aligned} \mathcal{A} &= \begin{pmatrix} A_0^1 x_1 + A_0^2 x_2 + A_0^3 x_3 & A_1^1 y_1 + A_1^2 y_2 \end{pmatrix} \\ \mathcal{B} &= \begin{pmatrix} B_0^1 x_1 + B_0^2 x_2 + B_0^3 x_3 \\ B_1^1 y_1 + B_1^2 y_2 \end{pmatrix} \end{aligned}$$

where $A_i^j : U_i \to V$ and $B_i^j : V \to W_i$. We also observe that the automorphisms of the monad are given by:

$$\begin{aligned} \mathrm{Aut}(\mathcal{U}) &= \begin{pmatrix} GL(U_0) & 0 \\ \mathrm{Hom}(U_0, U_1) \otimes \langle s\rangle & GL(U_1) \end{pmatrix} \\ \mathrm{Aut}(\mathcal{V}) &= GL(V) \\ \mathrm{Aut}(\mathcal{W}) &= \begin{pmatrix} GL(W_0) & \mathrm{Hom}(W_1, W_0) \otimes \langle s\rangle \\ 0 & GL(W_1) \end{pmatrix} \end{aligned}$$

We start the reduction process by considering the implication of $\mathcal{E}$ being trivial on ℓ_∞. If we choose two points p and q on ℓ_∞ e.g.

$$p = \big[(1,0,0)\,,\,(0,-1)\big] \quad \text{and} \quad q = \big[(0,1,0)\,,\,(1,0)\big]$$

then, by Lemma 2.3.4, the triviality condition can be expressed as

$$\mathcal{B}(q)\,\mathcal{A}(p) = -\mathcal{B}(p)\,\mathcal{A}(q) = D \tag{3.4.1}$$

where $D : U_0 \oplus U_1 \longrightarrow W_0 \oplus W_1$ is an isomorphism. Since, in addition, the monad conditions tell us that $\mathcal{B}(p)\,\mathcal{A}(p) = \mathcal{B}(q)\,\mathcal{A}(q) = 0$, that $\mathcal{A}$ is pointwise injective and that $\mathcal{B}$ is pointwise surjective, we can make the following direct sum decomposition:

$$V = V^{(p)} \oplus V^{(q)} \oplus V_\infty$$

where $V^{(p)} = \operatorname{im}\mathcal{A}(p)$, $V^{(q)} = \operatorname{im}\mathcal{A}(q)$ and $V_\infty = \ker\mathcal{B}(p) \cap \ker\mathcal{B}(q)$.

Note that we have a natural identification $V_\infty \cong H^0(\mathcal{E}|_{\ell_\infty})$. Thus a holomorphic framing of $\mathcal{E}$ along ℓ_∞ is equivalent to an isomorphism $V_\infty \cong \mathbf{C}^r$.

Now observe that, without loss of generality, we can assume

$$D = \begin{pmatrix} 0 & D_{01} \\ D_{10} & D_{11} \end{pmatrix},$$

because the off-diagonal parts of $\operatorname{Aut}(\mathcal{U})$ and $\operatorname{Aut}(\mathcal{W})$ effectively allow us to replace U_0 by a complement of both U_1 and $D^{-1}(W_0)$ in $U_0 \oplus U_1$, and then to replace W_1 by $D(U_0)$. Thus, if we write out condition (1) as

$$\begin{pmatrix} B_0^2A_0^1 & -B_0^2A_1^2 \\ B_1^1A_0^1 & -B_1^1A_1^2 \end{pmatrix} = \begin{pmatrix} -B_0^1A_0^2 & -B_0^1A_1^1 \\ B_1^2A_0^2 & B_1^2A_1^1 \end{pmatrix} = \begin{pmatrix} 0 & D_{01} \\ D_{10} & D_{11} \end{pmatrix},$$

and observe that $D_{01} : U_1 \longrightarrow W_0$ and $D_{10} : U_0 \longrightarrow W_1$ are isomorphisms, we see that we can make a further direct sum decomposition

$$V^{(p)} = \ker(\overline{B}_1^1) \oplus \operatorname{im}(A_0^1), \qquad V^{(q)} = \operatorname{im}(A_1^1) \oplus \ker(\overline{B}_0^1),$$

where $\overline{B}_1^1 = B_1^1|_{V^{(p)}}$ and $\overline{B}_0^1 = B_0^1|_{V^{(q)}}$. We then use the following diagrams, in which all the maps are isomorphisms, to identify all the spaces involved:

$$\begin{array}{ccccc} & & \operatorname{im}(A_1^1) & & \\ & {\scriptstyle A_1^1}\nearrow & & \searrow{\scriptstyle -B_0^1} & \\ U_1 & & \xrightarrow{\;D_{01}\;} & & W_0 \\ & {\scriptstyle -A_1^2}\searrow & & \nearrow{\scriptstyle B_0^2} & \\ & & \ker(\overline{B}_1^1) & & \end{array} \qquad\qquad \begin{array}{ccccc} & & \operatorname{im}(A_0^1) & & \\ & {\scriptstyle A_0^1}\nearrow & & \searrow{\scriptstyle B_1^1} & \\ U_0 & & \xrightarrow{\;D_{10}\;} & & W_1 \\ & {\scriptstyle A_0^2}\searrow & & \nearrow{\scriptstyle B_1^2} & \\ & & \ker(\overline{B}_0^1) & & \end{array}$$

Thus, we can make the identifications

$$U_0 = W_1, \qquad U_1 = W_0, \qquad V = W_0 \oplus W_1 \oplus W_0 \oplus W_1 \oplus V_\infty$$

and, writing D_{11} as $d : W_0 \longrightarrow W_1$, we arrive at the following canonical forms:

$$A_0^1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A_0^2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{pmatrix} \quad A_1^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad A_1^2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B_0^1 = (0 \quad 0 \quad -1 \quad 0 \quad 0) \qquad B_1^1 = (d \quad 1 \quad 0 \quad 0 \quad 0)$$

$$B_0^2 = (1 \quad 0 \quad 0 \quad 0 \quad 0) \qquad B_1^2 = (0 \quad 0 \quad d \quad 1 \quad 0)$$

The remaining variables

$$A_0^3 = \begin{pmatrix} a_1 \\ a_1' \\ a_2 \\ a_2' \\ c \end{pmatrix} \quad \text{and} \quad B_0^3 = (\beta_1 \quad \beta_1' \quad \beta_2 \quad \beta_2' \quad b)$$

can be reduced by using the vanishing of the coefficients of x_3 in the equation $\mathcal{B} \circ \mathcal{A} = 0$.

$$\begin{array}{lll} (x_1 x_3) & B_0^3 A_0^1 + B_0^1 A_0^3 = 0 & \Rightarrow \quad \beta_1' = a_2 \\ (x_2 x_3) & B_0^3 A_0^2 + B_0^2 A_0^3 = 0 & \Rightarrow \quad \beta_2' = -a_1 \\ (y_1 x_3) & B_1^1 A_0^3 = B_0^3 A_1^1 = 0 & \Rightarrow \quad d a_1 + a_1' = \beta_2 = 0 \\ (y_2 x_3) & B_1^2 A_0^3 = B_0^3 A_1^2 = 0 & \Rightarrow \quad d a_2 + a_2' = \beta_1 = 0 \\ (x_3{}^2) & B_0^3 A_0^3 = 0 \quad \Rightarrow & \sum_i \beta_i a_i + \sum_i \beta_i' a_i' + bc = 0 \end{array}$$

Thus we get the remaining forms

$$A_0^3 = \begin{pmatrix} a_1 \\ -da_1 \\ a_2 \\ -da_2 \\ c \end{pmatrix} \qquad B_0^3 = (0 \quad a_2 \quad 0 \quad -a_1 \quad b),$$

which means that the canonical form of the monad is given by:

$$\mathcal{A} = \begin{pmatrix} a_1 x_3 & -y_2 \\ x_1 - da_1 x_3 & 0 \\ a_2 x_3 & y_1 \\ x_2 - da_2 x_3 & 0 \\ c x_3 & 0 \end{pmatrix} \tag{C1}$$

$$\mathcal{B} = \begin{pmatrix} x_2 & a_2 x_3 & -x_1 & -a_1 x_3 & b x_3 \\ d y_1 & y_1 & d y_2 & y_2 & 0 \end{pmatrix}$$

where $a_i : W_1 \to W_0$, $b : V_\infty \to W_0$, $c : W_1 \to V_\infty$ and $d : W_0 \to W_1$. This 'linear algebra data' must satisfy the one remaining equation coming from $(x_3{}^2)$

$$a_1 d a_2 - a_2 d a_1 + bc = 0. \tag{I}$$

We shall call a configuration (a_1, a_2, b, c, d) *integrable* if satisfies the condition (I). This condition is equivalent to $\mathcal{B} \circ \mathcal{A} = 0$ for the monad in canonical form, which, in turn, is analogous to the integrability condition $\overline{\partial}^2 = 0$ for an almost holomorphic structure.

The residual symmetry group consists of those monad automorphisms which preserve the canonical form. Since we wish to describe the moduli space of framed holomorphic bundles, we further require that the automorphisms preserve an identification $V_\infty \cong \mathbb{C}^r$. Thus the correct notion of equivalence for linear configurations (a_1, a_2, b, c, d) is provided by the natural action of $GL(W_0) \times GL(W_1)$, in which a typical element (g_0, g_1) acts as follows:

$$(g_0, g_1) : \begin{cases} a_i \mapsto g_0 \, a_i \, g_1^{-1} & b \mapsto g_0 \, b \\ d \mapsto g_1 \, d \, g_0^{-1} & c \mapsto c \, g_1^{-1} \end{cases} \tag{A}$$

There now only remains to impose the non-degeneracy condition, namely that, at every point of Σ^1, the monad maps $\mathcal{A}$ and $\mathcal{B}$ should be injective and surjective respectively. This is already guaranteed on ℓ_∞ (i.e. $x_3 = 0$) so we need only consider points of the form $x = (\lambda_1, \lambda_2, 1)$, $y = (\mu_1, \mu_2) \neq (0,0)$. Hence, for the linear algebra data to give a non-degenerate monad, we require that

$\forall (\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2$ such that $\lambda_1\mu_1 + \lambda_2\mu_2 = 0$ and $(\mu_1, \mu_2) \neq (0,0)$,

$$\begin{pmatrix} a_1 & -\mu_2 \\ \lambda_1 - da_1 & 0 \\ a_2 & \mu_1 \\ \lambda_2 - da_2 & 0 \\ c & 0 \end{pmatrix} \quad \text{is injective, and}$$

$$\begin{pmatrix} \lambda_2 & a_2 & -\lambda_1 & -a_1 & b \\ \mu_1 d & \mu_1 & \mu_2 d & \mu_2 & 0 \end{pmatrix} \quad \text{is surjective.}$$

One easily checks that this condition is equivalent to the following:

$\forall (\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2$ such that $\lambda_1\mu_1 + \lambda_2\mu_2 = 0$ and $(\mu_1, \mu_2) \neq (0,0)$,

$$\nexists\, v \in W_1 \text{ such that } \begin{cases} da_1\, v = \lambda_1\, v & (\mu_1 a_1 + \mu_2 a_2)v = 0 \\ da_2\, v = \lambda_2\, v & c\, v = 0 \end{cases} \tag{N1}$$

$$\text{and } \nexists\, w \in W_0^* \text{ such that } \begin{cases} d^* a_1^*\, w = \lambda_1\, w & (\mu_1 a_1^* + \mu_2 a_2^*)w = 0 \\ d^* a_2^*\, w = \lambda_2\, w & b^*\, w = 0 \end{cases} \tag{N2}$$

We have shown above that we can realise any holomorphic bundle on Σ^1, with a framing along ℓ_∞, as the cohomology of a monad in the canonical form $(C1)$. Thus we have derived a description of the holomorphic moduli space over $\widetilde{\mathbb{C}}^2$ in terms of linear algebra data, as follows:

Theorem 3.4.1. *Given complex vector spaces W_0, W_1 of dimension k, we can construct the moduli space $\mathsf{MH}(\widetilde{\mathbb{C}}^2; r, k) = \mathsf{MH}(\Sigma^1, \ell_\infty; r, k)$ as the quotient of the set of configurations of linear maps*

$$(a, b, c, d) \in \mathrm{Hom}(W_1, W_0)^2 \oplus \mathrm{Hom}(\mathbb{C}^r, W_0) \oplus \mathrm{Hom}(W_1, \mathbb{C}^r) \oplus \mathrm{Hom}(W_0, W_1)$$

satisfying conditions (I), $(N1)$ and $(N2)$, by the action (A) of $GL(W_0) \times GL(W_1)$.

3.5 Reduction of Monads on F

The monads we have on F to describe anti-self-dual instantons on $\overline{\mathbb{CP}}^2$ are similar to those we have just dealt with on Σ^1. We can therefore hope that reducing them to an analogous canonical form will yield a description of the instanton moduli space over $\widetilde{\mathbb{C}}^2$, which is closely related to the description of the holomorphic moduli space in Theorem 3.4.1.

We start in precisely the same manner as before, but we observe that this time

$$H^0(\mathcal{L}_0^*) = H^0(\mathcal{O}_\mathsf{F}(1,0)) = \langle x_1, x_2, x_3 \rangle$$
$$H^0(\mathcal{L}_1^*) = H^0(\mathcal{O}_\mathsf{F}(0,1)) = \langle y_1, y_2, y_3 \rangle$$

$$H^0(\mathcal{L}_0^* \otimes \mathcal{L}_1) = H^0(\mathcal{O}_{\mathsf{F}}(1,-1)) = 0$$
$$H^0(\mathcal{L}_0 \otimes \mathcal{L}_1^*) = H^0(\mathcal{O}_{\mathsf{F}}(-1,1)) = 0$$

and therefore we have

$$\mathcal{A} = \begin{pmatrix} A_0^1 x_1 + A_0^2 x_2 + A_0^3 x_3 & A_1^1 y_1 + A_1^2 y_2 + A_1^3 y_3 \end{pmatrix}$$
$$\mathcal{B} = \begin{pmatrix} B_0^1 x_1 + B_0^2 x_2 + B_0^3 x_3 \\ B_1^1 y_1 + B_1^2 y_2 + B_1^3 y_3 \end{pmatrix}.$$

Furthermore, the automorphism group of the monad is smaller because $\mathrm{Aut}(\mathcal{U})$ and $\mathrm{Aut}(\mathcal{W})$ have no off-diagonal parts. This increase in degrees of freedom, due to the increase in variables and decrease in automorphisms, is compensated for by the imposition of a reality condition on the monad which is induced by the reality condition on the bundle it defines, namely that there is a positive-definite isomorphism $\sigma^*(\mathcal{E}) \cong \mathcal{E}^\star$, where $\mathcal{E}^\star$ is the hermitian adjoint of $\mathcal{E}$ and σ is the real structure induced by the map $(x, y) \mapsto (-\bar{y}, \bar{x})$.

Since an instanton bundle $\mathcal{E}$ on F is trivial on all real lines, in particular on ℓ_∞, we can consider the same points p and q as in §3.4 and equation (3.4.1) will still hold, giving the isomorphism $D : U_0 \oplus U_1 \longrightarrow W_0 \oplus W_1$ as before. Thus we can make the same direct sum decomposition

$$V = V^{(p)} \oplus V^{(q)} \oplus V_\infty.$$

In this case, V_∞ should be interpreted as the fibre at ∞ of the instanton bundle on $\overline{\mathbb{CP}}^2$. Hence, the 'framing at infinity' is again given by an identification $V_\infty \cong \mathbb{C}^r$. Imposing the reality condition on the monad

$$\begin{matrix} U_0 \otimes \mathcal{O}(-1,0) \\ \oplus \\ U_1 \otimes \mathcal{O}(0,-1) \end{matrix} \xrightarrow{\mathcal{A}} V \otimes \mathcal{O} \xrightarrow{\mathcal{B}} \begin{matrix} W_0 \otimes \mathcal{O}(1,0) \\ \oplus \\ W_1 \otimes \mathcal{O}(0,1) \end{matrix}$$

is equivalent to requiring the following:

1) $U_0 \cong W_1{}^\star$ and $U_1 \cong W_0{}^\star$ (since $\sigma^*(\mathcal{O}(1,0)) \cong \overline{\mathcal{O}(0,1)}$ etc).
2) There is a positive-definite isomorphism $V \cong V^\star$.
3) Given the identifications in (1) and (2), $B_0^j = -A_1^{j\star}$ and $B_1^j = A_0^{j\star}$.

The first observation to make is that, because of (3), the direct sum $V = V^{(p)} \oplus V^{(q)} \oplus V_\infty$ is an orthogonal direct sum with respect to the inner product of (2). Secondly, we observe that D is self-adjoint and, indeed,

$$D_{00} = A_1^{1\star} A_0^2 = D_{11}^\star, \quad D_{01} = A_1^{1\star} A_1^1 \quad \text{and} \quad D_{10} = A_0^{1\star} A_0^1,$$

so that D_{10} and D_{01} are actually self-adjoint isomorphisms, because A_0^1 and A_1^1 are injective. Therefore, we can make a further orthogonal direct sum decomposition

$$V^{(p)} = \ker(\overline{B}_1^1) \oplus \mathrm{im}(A_0^1) \qquad V^{(q)} = \mathrm{im}(A_1^1) \oplus \ker(\overline{B}_0^1).$$

where $\overline{B}_1^1 = B_1^1|_{V^{(p)}}$ and $\overline{B}_0^1 = B_0^1|_{V^{(q)}}$. Now we can use the following diagram, in which all the maps are isomorphisms, to identify the spaces involved:

$$\begin{array}{ccccc} & & \mathrm{im}(A_1^1) & & \\ & {\scriptstyle A_1^1}\nearrow & & \searrow{\scriptstyle -B_0^1} & \\ U_1 & & \xrightarrow{\quad D_{01}\quad} & & W_0 \\ & {\scriptstyle -A_1^2}\searrow & & \nearrow{\scriptstyle B_0^2} & \\ & & \ker(\overline{B}_1^1) & & \end{array}$$

Thus, making the identifications

$$U_1 = W_0, \qquad V = W_0 \oplus \mathrm{im}(A_0^1) \oplus W_0 \oplus \ker(\overline{B}_0^1) \oplus V_\infty$$

and writing D_{11} as $d : W_0 \to W_1$, we get the following partial canonical forms:

$$A_0^1 = \begin{pmatrix} D_{00} \\ \alpha_0^1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A_0^2 = \begin{pmatrix} 0 \\ 0 \\ D_{00} \\ \alpha_0^2 \\ 0 \end{pmatrix} \quad A_1^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad A_1^2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B_0^1 = (0 \quad 0 \quad -1 \quad 0 \quad 0) \qquad B_1^1 = (d \quad \beta_1^1 \quad 0 \quad 0 \quad 0)$$

$$B_0^2 = (1 \quad 0 \quad 0 \quad 0 \quad 0) \qquad B_1^2 = (0 \quad 0 \quad d \quad \beta_1^2 \quad 0)$$

where $\beta_1^j = \alpha_0^{j\star}$ and $D_{10} = dD_{00} + \beta_1^1\alpha_0^1 = dD_{00} + \beta_1^2\alpha_0^2$.

But, by the definition of $V^{(p)}$, $\begin{pmatrix} D_{00} & -1 \\ \alpha_0^1 & 0 \end{pmatrix}$ is an isomorphism, so α_0^1 and, similarly, α_0^2 are both isomorphisms. Therefore, we can use the following diagram to identify all the spaces involved:

$$\begin{array}{ccccc} & & \mathrm{im}(A_0^1) & & \\ & {\scriptstyle \alpha_0^1}\nearrow & & \searrow{\scriptstyle \beta_1^1} & \\ U_0 & & \xrightarrow{\quad \Delta\quad} & & W_1 \\ & {\scriptstyle \alpha_0^2}\searrow & & \nearrow{\scriptstyle \beta_1^2} & \\ & & \ker(\overline{B}_0^1) & & \end{array}$$

where $\Delta = D_{10} - D_{11}D_{01}^{-1}D_{00}$. Thus, with the additional identifications

$$U_0 = W_1, \qquad V = W_0 \oplus W_1 \oplus W_0 \oplus W_1 \oplus V_\infty$$

we arrive at the canonical forms

$$A_0^1 = \begin{pmatrix} d^\star \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad A_0^2 = \begin{pmatrix} 0 \\ 0 \\ d^\star \\ 1 \\ 0 \end{pmatrix} \quad A_1^1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \end{pmatrix} \quad A_1^2 = \begin{pmatrix} -1 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

$$B_0^1 = (0 \quad 0 \quad -1 \quad 0 \quad 0) \qquad B_1^1 = (d \quad 1 \quad 0 \quad 0 \quad 0)$$

$$B_0^2 = (1 \quad 0 \quad 0 \quad 0 \quad 0) \qquad B_1^2 = (0 \quad 0 \quad d \quad 1 \quad 0)$$

The remaining variables are reduced exactly as before, setting to zero the coefficients of the expression $\mathcal{BA}$ involving exactly one of x_3 and y_3 (half of these equations are actually the complex conjugates of the other half). This gives the remaining canonical forms

$$A_0^3 = \begin{pmatrix} a_1 \\ -da_1 \\ a_2 \\ -da_2 \\ c \end{pmatrix}, \quad B_0^3 = \begin{pmatrix} 0 & a_2 & 0 & -a_1 & b \end{pmatrix}, \quad A_1^3 = -B_0^{3\star}, \quad B_1^3 = A_0^{3\star},$$

and thus

$$\mathcal{A} = \begin{pmatrix} d^\star x_1 + a_1 x_3 & -y_2 \\ x_1 - da_1 x_3 & -a_2^\star y_3 \\ d^\star x_2 + a_2 x_3 & y_1 \\ x_2 - da_2 x_3 & a_1^\star y_3 \\ c\, x_3 & b^\star y_3 \end{pmatrix} \tag{C2}$$

$$\mathcal{B} = \begin{pmatrix} x_2 & a_2 x_3 & -x_1 & -a_1 x_3 & b\, x_3 \\ d\, y_1 + a_1^\star y_3 & y_1 - (da_1)^\star y_3 & d\, y_2 + a_2^\star y_3 & y_2 - (da_2)^\star y_3 & c^\star y_3 \end{pmatrix}$$

where $a_1 : W_1 \to W_0$, $b : V_\infty \to W_0$ and $c : W_1 \to V_\infty$ as before. The remaining conditions on this linear algebra data come from the coefficients of ${x_3}^2$ and $x_3 y_3$ in $\mathcal{B}\circ\mathcal{A} = 0$ (the ${y_3}^2$ equation is the conjugate of the ${x_3}^2$ one). These conditions are

$$a_1 da_2 - a_2 da_1 + bc = 0 \tag{I}$$

$$a_1 {a_1}^\star + a_2 {a_2}^\star + bb^\star = 1 \tag{$\mu 0$}$$

$${a_1}^\star(1 + d^\star d)a_1 + {a_2}^\star(1 + d^\star d)a_2 + c^\star c = 1 + dd^\star$$

Here (I) is the same as the integrability condition in §3.4. Using $(\mu 0)$ we write this last equation in the form

$$[da_1\,,\,(da_1)^\star] + [da_2\,,\,(da_2)^\star] - {a_1}^\star a_1 - {a_2}^\star a_2 + db(db)^\star - c^\star c = -1, \tag{$\mu 1$}$$

which is more suitable for the purposes of the next chapter.

Thus we see that the canonical form for monads on F which give instanton bundles is determined by the same linear algebra data (a_1, a_2, b, c, d) as for the monads on Σ^1, satisfying the same equation (I) and two further equations $(\mu 0)$ and $(\mu 1)$. One will notice that the restriction of a monad in canonical form $(C2)$ to Σ^1, given by setting $y_3 = 0$, is not a monad in canonical form $(C1)$ from §3.4. However, the restricted monad only differs from one of the required form by the automorphism (over Σ^1)

$$\left(\begin{pmatrix} 1 & 0 \\ d^\star s & 1 \end{pmatrix}, 1, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \right) \in \mathrm{Aut}(\mathcal{U}) \times \mathrm{Aut}(\mathcal{V}) \times \mathrm{Aut}(\mathcal{W})$$

and thus all is well at the bundle level. That is to say, the bundle on F determined by the data (a_1, a_2, b, c, d), satisfying the necessary conditions, restricts to a bundle isomorphic to the one determined on Σ^1 by the same data.

Now, for a monad with real structure, one only needs to check the non-degeneracy condition at one point on each fibre of the twistor space and so we immediately see that the non-degeneracy condition for the canonical monad on F is the same as the one for the canonical monad on Σ^1, because Σ^1 meets every fibre.

Finally, we observe that, since we are in a situation now where the vector spaces W_0 and W_1 have inner products, the residual symmetry, which determines when two such sets of maps give equivalent monads, is given by restricting the earlier action (A) to $U(W_0) \times U(W_1)$. As before, the automorphisms do not act on V_∞, because the data is supposed to determine a framing of the instanton at ∞.

In this section, we have shown that we can describe any instanton on $\overline{\mathbb{CP}}^2$, with a framing at ∞, in terms of a monad on F in the canonical form $(C2)$. Thus we have derived a description of the instanton moduli space over $\widetilde{\mathbb{C}}^2$ in terms of linear algebra data as follows:

Theorem 3.5.1. *Given hermitian inner product spaces W_0, W_1 of dimension k, the moduli space $\mathsf{MI}(\widetilde{\mathbb{C}}^2; r, k) = \mathsf{MI}(\overline{\mathbb{CP}}^2, \infty; r, k)$ is the quotient of the set of linear maps*

$$(a, b, c, d) \in \mathrm{Hom}(W_1, W_0)^2 \oplus \mathrm{Hom}(V_\infty, W_0) \oplus \mathrm{Hom}(W_1, V_\infty) \oplus \mathrm{Hom}(W_0, W_1)$$

satisfying conditions (I), $(N1)$, $(N2)$, $(\mu 0)$ and $(\mu 1)$, by the action (A) of $U(W_0) \times U(W_1)$.

Theorem 3.5.2. *The Hitchin-Kobayashi map $\mathbf{h} : \mathsf{MI}(\widetilde{\mathbb{C}}^2; r, k) \to \mathsf{MH}(\widetilde{\mathbb{C}}^2; r, k)$ is induced by the inclusion*

$$\{(a, b, c, d) \mid (I), (N1), (N2), (\mu 0), (\mu 1)\} \subseteq \{(a, b, c, d) \mid (I), (N1), (N2)\}.$$

Therefore, to show that the restriction map is actually a bijection, we must simply show that each orbit of $GL(W_0) \times GL(W_1)$ in the second set meets the first in precisely one orbit of $U(W_0) \times U(W_1)$. This step will be undertaken in the next chapter.

3.6 Interpretation of the Linear Data

In §3.4, we have shown how to describe the moduli space of holomorphic bundles on $\widetilde{\mathbb{C}}^2$ in terms a collection of linear maps

$$(a, b, c, d) \in \mathrm{Hom}(W_1, W_0)^2 \oplus \mathrm{Hom}(V_\infty, W_0) \oplus \mathrm{Hom}(W_1, V_\infty) \oplus \mathrm{Hom}(W_0, W_1)$$

If one restricts the corresponding monad (in canonical form) to the exceptional line $E \in |\mathcal{O}(1, -1)|$, then one can use the results in §2.3 to see that the holomorphic bundle, defined by the monad, is trivial along E if and only if $d : W_0 \to W_1$ is an isomorphism. Similarly, restricting to any of the lines in $|\mathcal{O}(0, 1)|$, i.e. the fibres of the projection $\Sigma^1 \to \mathsf{P}^1$, one sees that the bundle is trivial on the fibre over $[\mu_1, \mu_2] \in \mathsf{P}^1$ if and only if $\mu_1 a_1 + \mu_2 a_2$ is an isomorphism.

Thus we can describe the rôle of the maps a_1, a_2 and d, but they are still simply some maps which occur in a canonical form for a monad. Now the vector spaces W_0, W_1 and V_∞ actually have a more natural interpretation arising from the construction, namely

$$W_0 = H^1(\mathcal{E}(-1, 0)) \cong H^1(\mathcal{E}(-2, 0)),$$
$$W_1 = H^1(\mathcal{E}(0, -1)) \cong H^1(\mathcal{E}(-1, -1)),$$
$$V_\infty = H^0(\mathcal{E}|_{\ell_\infty}) \cong H^1(\mathcal{E}|_{\ell_\infty}(-2)).$$

We shall now show that the linear maps a_1, a_2, b, c and d can be interpreted as natural cohomology maps between these spaces.

Starting with the map $b : V_\infty \to W_0$, we claim that this is the coboundary map

$$\partial_{\mathcal{E}} : H^0(\mathcal{E}|_{\ell_\infty}) \to H^1(\mathcal{E}(-1,0))$$

associated to the short exact sequence obtained by tensoring $\mathcal{E}$ with the structure sequence

$$0 \to \mathcal{O}(-1,0) \to \mathcal{O} \to \mathcal{O}|_{\ell_\infty} \to 0 \tag{3.6.1}$$

To see that b does have this interpretation, we first observe that, combining (3.6.1) and the monad sequence $0 \to \mathcal{U} \to \mathcal{K} \to \mathcal{E} \to 0$ yields

$$\begin{array}{ccccccc}
0 \to & H^0(\mathcal{K}) & \longrightarrow & H^0(\mathcal{K}|_{\ell_\infty}) & \xrightarrow{\partial_{\mathcal{K}}} & H^1(\mathcal{K}(-1,0)) & \to \cdots \\
& \downarrow \wr & & \downarrow \wr & & \downarrow \wr & \\
0 \to & H^0(\mathcal{E}) & \longrightarrow & H^0(\mathcal{E}|_{\ell_\infty}) & \xrightarrow{\partial_{\mathcal{E}}} & H^1(\mathcal{E}(-1,0)) & \to \cdots
\end{array}$$

where the vertical maps are isomorphisms because $\mathcal{U}$ has all the relevant cohomology vanishing. Thus we identify $\partial_{\mathcal{E}}$ with $\partial_{\mathcal{K}}$. Next, if we combine (3.6.1) with the monad sequence $0 \to \mathcal{K} \to \mathcal{V} \to \mathcal{W} \to 0$ we get

$$\begin{array}{cccccccc}
 & & & & & 0 & & \\
 & & & & & \downarrow & & \\
 & & & H^0(\mathcal{K}) & \to & H^0(\mathcal{K}|_{\ell_\infty}) & \xrightarrow{\partial_{\mathcal{K}}} & H^1(\mathcal{K}(-1,0)) \\
 & & & \downarrow & & \downarrow & & \downarrow \\
 & 0 & \to & \mathcal{V} & = & \mathcal{V} & \longrightarrow & 0 \\
 & \downarrow & & \downarrow \mathcal{B} & \searrow^{\theta} & \downarrow & & \\
0 \to & H^0(\mathcal{W}(-1,0)) & \to & H^0(\mathcal{W}) & \to & H^0(\mathcal{W}|_{\ell_\infty}) & & \\
 & \downarrow & & \downarrow & & & & \\
 & H^1(\mathcal{K}(-1,0)) & \to & H^1(K) & & & & \\
 & \downarrow & & & & & & \\
 & 0 & & & & & &
\end{array}$$

This diagram naturally yields two maps $\ker\theta \to H^1(K(-1,0))$ by 'chasing' from the middle to the top right or the bottom left respectively. It is a straightforward lemma in homological algebra that these maps are actually the same. But the first map is simply the map $\partial_{\mathcal{K}}$, while the second is the part of the map $\mathcal{B} : \mathcal{V} \to H^0(W)$, whose domain is V_∞ (i.e. $H^0(\mathcal{K}|_{\ell_\infty})$) and whose codomain is the component of $H^0(W)$ corresponding to x_3. But this is just the map b, demonstrating the above claim.

Dually, the map c can be identified by observing that c^* is the coboundary map $H^0(\mathcal{E}^*|_{\ell_\infty}) \to H^1(\mathcal{E}^*(-1,0))$, so that $c : H^1(\mathcal{E}^*(-1,0))^* \to H^0(\mathcal{E}^*|_{\ell_\infty})^*$. By Serre duality, this is the same as the map $H^1(\mathcal{E}(-1,-1)) \to H^1(\mathcal{E}|_{\ell_\infty}(-2))$ which is induced on cohomology by the restriction map to ℓ_∞.

We next identify the map $d: W_0 \to W_1$ as the map $H^1(\mathcal{E}(-1,0)) \to H^1(\mathcal{E}(0,-1))$ induced by multiplication by the section $-s$ of $\mathcal{O}(1,-1)$ in the short exact sequence

$$0 \to \mathcal{E}(-1,0) \to \mathcal{E}(0,-1) \to \mathcal{E}|_E(-1) \to 0.$$

This fits d into an exact sequence

$$0 \to H^0(\mathcal{E}|_E(-1)) \to H^1(\mathcal{E}(-1,0)) \xrightarrow{d} H^1(\mathcal{E}(0,-1)) \to H^1(\mathcal{E}|_E(-1)) \to 0$$

and thus gives a more natural demonstration of the fact that $\mathcal{E}$ is trivial on E precisely when d is an isomorphism.

We start by observing that, since the monad sequence $0 \to \mathcal{E} \to \mathcal{Q} \to \mathcal{W} \to 0$ is a universal extension, the map δ in the sequence

$$0 \to H^0(\mathcal{Q}(0,-1)) \to H^0(\mathcal{W}(0,-1)) \xrightarrow{\delta} H^1(\mathcal{E}(0,-1)) \to \cdots$$

is given by the map $(s, id): W_0 \oplus W_1 \to W_1$ where s is the non-zero section of $\mathcal{O}(1,-1)$, introduced in §1.4., acting by multiplication on cohomology. Thus the image of $H^0(\mathcal{Q}(0,-1))$ is $\{(w_0, w_1) \in W_0 \oplus W_1 \mid s\,w_0 = -w_1\}$. However, restriction to ℓ_∞ yields the following commuting diagram

$$\begin{array}{ccccc}
 & & 0 & & \\
0 & & \downarrow & & \\
\downarrow & & & & \\
H^1(\mathcal{U}(0,-1)) & \xleftarrow{\sim} & H^0(\mathcal{Q}(0,-1)) & \longrightarrow & H^0(\mathcal{W}(0,-1)) \\
\downarrow{\scriptstyle i} & & \downarrow & & \downarrow\wr \\
H^1(\mathcal{U}|_{\ell_\infty}(-1)) & \xleftarrow{\sim} & H^0(\mathcal{Q}|_{\ell_\infty}(-1)) & \longrightarrow & H^0(\mathcal{W}|_{\ell_\infty}(-1))
\end{array}$$

where, given the canonical identifications, $i: W_0 \to W_1 \oplus W_0$ is the inclusion into the second factor and $D: W_0 \oplus W_1 \to W_1 \oplus W_0$ is given by the matrix $\begin{pmatrix} 0 & 1 \\ 1 & d \end{pmatrix}$. Thus $\mathrm{im}(H^0(\mathcal{Q}(0,-1)))$ is also identified with the space $\{(w_0, w_1) \in W_0 \oplus W_1 \mid w_1 = dw_0\}$. In other words, $dw_0 = (-s)w_0$ for all $w_0 \in W_0$, as required.

Finally, we note that the maps $a_1, a_2 : W_1 \to W_0$ are the maps

$$H^1(\mathcal{E}(-1,-1)) \to H^1(\mathcal{E}(-1,0))$$

induced by multiplication by the sections $-y_2$ and y_1, respectively, of $\mathcal{O}(0,1)$ in the sequence

$$0 \to \mathcal{E}(-1,-1) \to \mathcal{E}(-1,0) \to E|_{F_\mu} \to 0,$$

where F_μ is the fibre of the projection $\Sigma^1 \to \mathbb{P}^1$ over $\mu \in \mathbb{P}^1$. This fits a_i into an exact sequence

$$0 \to H^0(\mathcal{E}|_F(-1)) \to H^1(\mathcal{E}(-1,-1)) \xrightarrow{a_i} H^1(\mathcal{E}(-1,0)) \to H^1(\mathcal{E}|_F(-1)) \to 0,$$

which gives a natural demonstration of the rôle of the a_i in describing the jumping behaviour of $\mathcal{E}$ along the fibres of the projection. Indeed, this exact sequence is canonically matched to the exact sequence from §2.3,

$$0 \to H^0(\mathcal{E}|_F(-1)) \to H^1(\mathcal{U}|_F(-1)) \to H^0(\mathcal{W}|_F(-1)) \to H^1(\mathcal{E}|_F(-1)) \to 0,$$

in which a_1 and a_2, as originally defined, actually occur. We can see this using the same type of homological algebra lemma as when identifying b, but applied to the diagram

$$\begin{array}{ccccccccc}
 & & & & & & 0 & & \\
 & & & & & & \downarrow & & \\
 & & & & H^0(\mathcal{E}|_F(-1)) & \to & H^1(\mathcal{E}(-1,-1)) & \to & H^1(\mathcal{E}(-1,0)) \\
 & & & & \downarrow & & \downarrow & & \downarrow \\
 & & 0 & \to & H^1(\mathcal{U}|_F(-1)) & \to & H^2(\mathcal{U}(-1,-1)) & \to & 0 \\
 & & \downarrow & & \downarrow & \searrow^{\theta} & \downarrow & & \\
0 & \to & H^1(\mathcal{K}(-1,0)) & \to & H^1(\mathcal{K}|_F(-1)) & \to & 0 & & \\
 & & \downarrow & & \downarrow & & & & \\
 & & H^1(\mathcal{E}(-1,0)) & \to & H^1(\mathcal{E}|_F(-1)) & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

3.7 Relation to Existing Constructions

We conclude with two remarks on the nature of the constructions we have used in this chapter and their relation to constructions already in existence.

Firstly, Buchdahl [**Bu2**] has given a description of the moduli space of stable bundles on any Hirzebruch surface Σ^n using monads of the form

$$U \otimes \mathcal{O}(0,-1) \longrightarrow \begin{array}{c} V_1 \otimes \mathcal{O}(1,-1) \\ \oplus \\ V_2 \otimes \mathcal{O} \end{array} \longrightarrow W \otimes \mathcal{O}(1,0).$$

While these differ from our monads, in particular by having a non-trivial middle term, they are in some ways quite similar. Buchdahl's monads come from a degenerate Beilinson spectral sequence arising from a Koszul resolution of the diagonal in $\Sigma^n \times \Sigma^n$. Thus they are constructed in a rather more natural way than our monads have been. It seems that the way of providing a more natural construction of our monads — and at the same time relating them more closely to Buchdahl's — may lie in the newly developed theory of exceptional bundles and helixes [**GR,Ru**]. To explain very briefly, a helix is a special configuration of exceptional bundles on an algebraic variety, which, starting with any bundle on the variety, can be used to generate spectral sequences which converge to that bundle. It should be possible to find a helix on Σ^1 which would generate two degenerate spectral sequences which yield Buchdahl's monads and ours.

Secondly, Brosius **[Br1,Br2]** has described all possible rank 2 bundles on all ruled surfaces (not just the rational ones) as extensions of certain mildly singular sheaves by other basic bundles. One key ingredient of his description is the jumping behaviour of the bundle along the fibres. As we saw in the previous section, such jumping behaviour is captured by the multiplication maps on cohomology. In addition, Hurtibise **[Hu]** has shown explicitly, for holomorphic bundles over $\mathbb{C}^2$, how the linear algebra data of **[D1]** is equivalent to the jumping data long a family of parallel lines. Brosius' work thus supports the general philosophy, employed by Barth **[Ba]** in describing stable bundles on $\mathbb{P}^2$, that a stable bundle $\mathcal{E}$ on a surface S should be essentially captured by the module

$$\bigoplus_{\mathcal{L}\in \mathrm{Pic}(S)} H^1(\mathcal{E}\otimes\mathcal{L}) \quad \text{over the ring} \quad \bigoplus_{\mathcal{L}\in \mathrm{Pic}(S)} H^0(\mathcal{L}).$$

In fact, we should be able to reconstruct $\mathcal{E}$ from the 'central core' of this module. The question of what exactly constitutes this 'central core' must be closely related to the problem of constructing helixes on the surface.

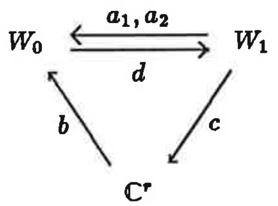
We have shown, in the previous section, that this philosophy works in classifying our bundles on Σ^1, where the 'central core' is provided by the line bundles $\mathcal{O}(-1,-1)$, $\mathcal{O}(-1,0)$ and $\mathcal{O}(0,-1)$—note that the maps b and c only really provide the extra information required to specify the framing along ℓ_∞. Indeed, had we followed Barth's approach more closely, we could have written down the maps (a_1, a_2, b, c, d) initially as maps on the cohomology of $\mathcal{E}$ and then built them into a monad to show that we could reconstruct $\mathcal{E}$ from them.

The feeling that the linear algebra data provides the most natural description of the moduli space of holomorphic bundles will be further strengthened in the next chapter, when the equivalence of the holomorphic moduli space with the instanton moduli space is proved with reference only to this data.

4 The Main Theorem

In this chapter we prove the main theorem on the equivalence of the moduli spaces of instantons and holomorphic bundles on $\tilde{\mathbb{C}}^2$.

Recall that, in the previous chapter, we derived a description of both moduli spaces in terms of the following configuration of linear maps:



where W_0 and W_1 are hermitian inner product spaces of dimension k. We now introduce the following notation:

$$\tilde{\mathcal{R}} = \mathrm{Hom}(W_1, W_0)^2 \times \mathrm{Hom}(\mathbb{C}^r, W_0) \times \mathrm{Hom}(W_1, \mathbb{C}^r) \times \mathrm{Hom}(W_0, W_1),$$

$$G_{\mathbb{C}} = GL(W_0) \times GL(W_1), \qquad G = U(W_0) \times U(W_1).$$

Note that $G_{\mathbb{C}}$ is indeed the complexification of G and has Lie algebra $\mathbf{g}_{\mathbb{C}} = \mathbf{g} \oplus i\mathbf{g}$. In particular, note that $i\mathbf{g}$ is the space of pairs of hermitian endomorphisms. Let $X \subseteq \tilde{\mathcal{R}}$ be the space of configurations $(a, b, c, d) \in \tilde{\mathcal{R}}$ which satisfy the integrability equation

$$a_1 d a_2 - a_2 d a_1 + bc = 0 \qquad (I)$$

and also the non-degeneracy conditions

$\forall\, (\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2$ such that $\lambda_1\mu_1 + \lambda_2\mu_2 = 0$ and $(\mu_1, \mu_2) \neq (0, 0)$,

$$\nexists\, v \in W_1 \text{ such that } \begin{cases} da_1\, v = \lambda_1\, v & (\mu_1 a_1 + \mu_2 a_2)v = 0 \\ da_2\, v = \lambda_2\, v & c\, v = 0 \end{cases} \qquad (N1)$$

$$\text{and } \nexists\, w \in W_0^* \text{ such that } \begin{cases} d^* a_1^*\, w = \lambda_1\, w & (\mu_1 a_1^* + \mu_2 a_2^*)w = 0 \\ d^* a_2^*\, w = \lambda_2\, w & b^*\, w = 0 \end{cases} \qquad (N2)$$

As we showed in the previous chapter, the moduli space of holomorphic bundles on $\tilde{\mathbb{C}}^2$ is the quotient of X by the canonical action of $G_{\mathbb{C}}$. Furthermore the moduli space of instantons on $\tilde{\mathbb{C}}^2$ is the quotient by G of the subset $X_\mu \subseteq X$ satisfying the real equations

$$\sum_i a_i a_i^{\star} + bb^{\star} = 1 \qquad (\mu 0)$$

$$\sum_i [da_i\, , (da_i)^{\star}] - \sum_i a_i^{\star} a_i + db(db)^{\star} - c^{\star}c = -1. \qquad (\mu 1)$$

To prove the equivalence of these two moduli spaces we use the results of §1.5. To apply these results we must find a linearisation of the $G_{\mathbb{C}}$-action on X and a suitable Kähler potential such that (N1) and (N2) imply analytic stability and $(\mu 0)$ and $(\mu 1)$ correspond to the vanishing of the associated moment map.

4.1 Non-degeneracy and Stability

We start with a lemma in linear algebra.

LEMMA 4.1.1. *Let V_0 and V_1 be finite dimensional complex vector spaces with V_1 non-trivial and* $\dim V_0 \leqslant \dim V_1$*. Suppose that we have linear maps $a_1, a_2 : V_1 \to V_0$ and $d : V_0 \to V_1$ satisfying*

$$a_1 d a_2 - a_2 d a_1 = 0. \tag{4.1.1}$$

Then there is a non-zero vector $v \in V_1$ and points $(\lambda_1, \lambda_2), (\mu_1, \mu_2) \in \mathbb{C}^2$, with $\lambda_1\mu_1 + \lambda_2\mu_2 = 0$ and $(\mu_1, \mu_2) \neq (0,0)$, such that

$$da_i(v) = \lambda_i\, v \ (i = 1, 2) \qquad \textit{and} \qquad (\mu_1 a_1 + \mu_2 a_2)\, v = 0.$$

PROOF. From (4.1.1) we see that da_1 and da_2 commute and hence have at least one pair of simultaneous eigenvalues (λ_1, λ_2). The pairs which occur will include all the eigenvalues of da_1 and da_2 (though not all pairs of eigenvalues will necessarily occur) so we can only encounter one of the two cases

i) For some eigenvector $v \neq 0$ the eigenvalue pair $(\lambda_1, \lambda_2) \neq (0,0)$. Then, applying (4.1.1) to v, we can satisfy the lemma with $(\mu_1, \mu_2) = (\lambda_2, -\lambda_1)$.

ii) Both da_1 and da_2 just have the single eigenvalue zero, so they are both nilpotent and we then need to show that there is some $(\mu_1, \mu_2) \neq (0,0)$ such that

$$\ker da_1 \cap \ker da_2 \cap \ker(\mu_1 a_1 + \mu_2 a_2) \neq 0$$

This case further subdivides into two cases:

a) There exists $v_0 \in V_1$ such that $da_1(v_0) = 0$ but $v_1 = da_2(v_0) \neq 0$. Since da_1 and da_2 commute, we know that $v_1 \in \ker da_1$ and indeed that

$$v_n = (da_2)^n(v_0) \in \ker da_1.$$

But da_2 is nilpotent so there is some m for which $v_m \neq 0$ and $v_{m+1} = 0$. Applying (4.1.1) to v_{m-1} we see that $a_1(v_m) = 0$, so we can satisfy the requirements of the lemma by choosing $v = v_m$ and $(\mu_1, \mu_2) = (1, 0)$.

b) We have $\ker da_1 \subseteq \ker da_2$ and so we simply need to show

$$\ker da_1 \cap \ker(\mu_1 a_1 + \mu_2 a_2) \neq 0 \qquad \text{for some } (\mu_1, \mu_2) \neq 0$$

Write $\tilde{a}_i : \ker da_1 \to \ker d$ for the restriction of a_i. We observe that $n(da_1) \geqslant n(d)$, where $n(d) = \dim(\ker d)$, because, by the rank-nullity formula,

$$n(da_1) - n(d) = \mathrm{rk}(d) - \mathrm{rk}(da_1) + \dim(V_1) - \dim(V_0)$$

Now, if $n(da_1) > n(d)$ then $\mu_1\tilde{a}_1 + \mu_2\tilde{a}_2$ has non-trivial kernel for all (μ_1, μ_2). Alternately, if $n(da_1) = n(d)$, we must solve the equation $\det(\mu_1\tilde{a}_1 + \mu_2\tilde{a}_2) = 0$ to find a suitable (μ_1, μ_2). Thus the lemma can be satified in this case as well. □

REMARK 4.1.2. If we suppose that the vector v, found in the above lemma, does not satisfy $a_1(v) = a_2(v) = 0$, then the values of (λ_1, λ_2) and (μ_1, μ_2) are uniquely determined, upto rescaling (μ_1, μ_2). Since $\lambda_1\mu_1 + \lambda_2\mu_2 = 0$, this means that they determine a point in $\widetilde{\mathbb{C}}^2$ (as described in §1.4), which we can think of as an 'eigenvalue' for the configuration of maps (a_1, a_2, d), just as a point in $\mathbb{C}^2$ is an 'eigenvalue' for a pair of endomorphisms. In fact, even if $a_1(v) = a_2(v) = 0$, we can still take the 'eigenvalue' of v with respect to (a_1, a_2, d) to be the 'point', in the scheme-theoretic sense, corresponding to the exceptional line in $\widetilde{\mathbb{C}}^2$.

The lemma above is the key to proving the next result.

PROPOSITION 4.1.3. *On the set* $\{\alpha \in \tilde{\mathcal{R}} \mid (I)\}$, *the open condition* $(N1)$ *is equivalent to*

$$(S1) \qquad \begin{array}{l} \nexists \text{ subspaces } V_0 \subseteq W_0,\ V_1 \subseteq W_1, \text{ with } V_1 \neq 0 \text{ and } \dim V_0 \leqslant \dim V_1, \\ \text{such that } a_i(V_1) \subseteq V_0\ (i = 1, 2),\ d(V_0) \subseteq V_1 \text{ and } V_1 \subseteq \ker c. \end{array}$$

Similarly, the condition $(N2)$ *is equivalent to*

$$(S2) \qquad \begin{array}{l} \nexists \text{ subspaces } V_0 \subseteq W_0,\ V_1 \subseteq W_1, \text{ with } V_0 \neq W_0 \text{ and } \dim V_0 \leqslant \dim V_1, \\ \text{such that } a_i(V_1) \subseteq V_0\ (i = 1, 2),\ d(V_0) \subseteq V_1 \text{ and } \operatorname{im} b \subseteq V_0. \end{array}$$

PROOF. To prove the first equivalence, we observe that "$(N1) \Rightarrow (S1)$" follows immediately from the lemma above. To show "$(S1) \Rightarrow (N1)$", suppose that we have a non-zero v contradicting $(N1)$. Then taking $V_1 = \langle v \rangle$ and $V_0 = \langle a_1 v, a_2 v \rangle$ contradicts $(S1)$, because V_0 is at most one-dimensional.

Now, we observe that, by considering dual spaces and annihilating subspaces (i.e. setting $V_0^a = \operatorname{ann} V_0 = \{\theta \in W_0^* \mid \theta(V_0) = 0\}$ and $V_1^a = \operatorname{ann} V_1$) condition $(S2)$ is equivalent to

$$(S2)^* \qquad \begin{array}{l} \nexists \text{ subspaces } V_1^a \subseteq W_1^*,\ V_0^a \subseteq W_0^*, \text{ with } V_0^a \neq 0 \text{ and } \dim V_1^a \leqslant \dim V_0^a, \\ \text{such that } a_i^*(V_0^a) \subseteq V_1^a\ (i = 1, 2),\ d^*(V_1^a) \subseteq V_0^a \text{ and } V_0^a \subseteq \ker b^*. \end{array}$$

But conditions $(N2)$ and $(S2)^*$ are just $(N1)$ and $(S1)$ with different variables, so the second equivalence follows from the first. □

REMARK 4.1.4. The requirement "$\dim V_0 \leqslant \dim V_1$" in (S1) and (S2) can be replaced by "$\dim V_0 = \dim V_1$" without, in fact, changing the condition, because we can always replace V_0 by V_0', with $V_0 \subseteq V_0' \subseteq d^{-1}(V_1)$, or V_1 by V_1', with $d(V_0) \subseteq V_1' \subseteq V_1$.

In the light of the proposition above, we can ignore the condition (I) for the moment and prove the stronger result that, on the whole of $\tilde{\mathcal{R}}$, conditions (S1) and (S2) are equivalent to analytic stability with respect to a suitable linearised action and $(\mu 0)$ and $(\mu 1)$ correspond to the vanishing of the associated moment map.

Before proceeding, we introduce a second space of linear maps

$$\mathcal{R}' = \operatorname{Hom}(W_1, W_0)^2 \oplus \operatorname{Hom}(\mathbb{C}^r, W_0) \oplus \operatorname{End}(W_1)^2 \oplus \operatorname{Hom}(\mathbb{C}^r, W_1) \oplus \operatorname{Hom}(W_1, \mathbb{C}^r)$$

and a map

$$\rho : \tilde{\mathcal{R}} \to \mathcal{R}' : (a_1, a_2, b, c, d) \mapsto (a_1, a_2, b, da_1, da_2, db, c)$$

Note that $\mathcal{R}'$ carries a canonical $G_{\mathbb{C}}$-action and that ρ intertwines this action with the $G_{\mathbb{C}}$-action on $\tilde{\mathcal{R}}$.

Now let L be the trivial line bundle over $\tilde{\mathcal{R}}$ with fibre $\mathbb{C}$. Lift the $G_{\mathbb{C}}$-action on $\tilde{\mathcal{R}}$ to L by letting (g_0, g_1) act on the fibre as multiplication by $(\det g_1)/(\det g_0)$, i.e. take the linearisation of $G_{\mathbb{C}}$ given by

$$g : \tilde{\mathcal{R}} \times \mathbb{C} \to \tilde{\mathcal{R}} \times \mathbb{C} : (\alpha, z) \mapsto \left(g \cdot \alpha, \frac{\det g_1}{\det g_0} z\right). \tag{4.1.2}$$

Define a hermitian structure on L by

$$\left\|(\alpha, z)\right\|^2 = |z|^2 e^{\|\rho(\alpha)\|^2} \tag{4.1.3}$$

where the norm on $\mathcal{R}'$ comes from the hermitian structure which is the direct sum of the standard structure, $\langle A, B\rangle = tr(A^* B)$, on each summand.

REMARK 4.1.5. The hermitian structure (4.1.3) is a global Kähler potential for the 'metric' on $\tilde{\mathcal{R}}$ pulled back under ρ from the Euclidean metric on $\mathcal{R}'$. This 'metric' is partly degenerate because ρ is not an embedding. However, ρ is injective when restricted to the open subset of $\tilde{\mathcal{R}}$ given by configurations for which a_1, a_2 and b are jointly surjective. Note that, if a_1, a_2 and b are not jointly surjective, then $V_0 = \operatorname{im} a_1 \oplus \operatorname{im} a_2 \oplus \operatorname{im} b$ and $V_1 = W_1$ gives a pair contradicting (S2). Thus, when we, in fact, restrict our attention to $\{\alpha \in \tilde{\mathcal{R}} \mid (S1), (S2)\}$ the hermitian structure on L will define a genuine Kähler metric.

PROPOSITION 4.1.6. *A configuration $\alpha \in \tilde{\mathcal{R}}$ satisfies $(S1)$ and $(S2)$, if and only if, for all real one-parameter subgroups $\lambda : \mathbb{R} \to G_{\mathbb{C}} : t \mapsto e^{tA}$ with $A \in i\mathfrak{g}$,*

$$\left\|\lambda_t(\alpha, z)\right\| \to \infty \quad \text{as} \quad t \to \infty,$$

where z is any non-zero element of $\mathbb{C}$.

PROOF. Suppose we consider the one-parameter subgroup $\lambda : \mathbb{R} \to G_{\mathbb{C}}$ given by the pair of hermitian endomorphisms $(A_0, A_1) \in i\mathfrak{g}$. We can decompose W_0 and W_1 into eigenspaces for A_0 and A_1 respectively, i.e. we can write

$$W_i = \bigoplus_{n \in \mathbb{R}} W_i^{(n)} \qquad i = 0, 1,$$

where A_i acts on $W_i^{(n)}$ as multiplication by n. Clearly, $W_i^{(n)} = 0$ for all but finitely many n and $\sum_{n \in \mathbb{R}} \dim W_i^{(n)} = k$. Let

$$\Delta = \sum_{n \in \mathbb{R}} n\left(\dim W_1^{(n)} - \dim W_0^{(n)}\right)$$

so that λ_t acts on the fibres of L as multiplication by $e^{t\Delta}$. We also introduce the notation

$$W_i^{(\leqslant n)} = \bigoplus_{m \leqslant n} W_i^{(m)}.$$

Decompose all the components of α as follows:

$$a_i^{(mn)} : W_1^{(n)} \to W_0^{(m)} \qquad b^{(m)} : \mathbb{C}^r \to W_1^{(m)}$$
$$d^{(mn)} : W_0^{(n)} \to W_1^{(m)} \qquad c^{(n)} : W_1^{(n)} \to \mathbb{C}^r$$

and also write

$$\sum_k d^{(mk)} a_i^{(kn)} = (da_i)^{(mn)} : W_1^{(n)} \to W_0^{(m)}$$
$$\sum_k d^{(mk)} b^{(k)} = (db)^{(m)} : \mathbb{C}^r \to W_0^{(m)}$$

Then, with the above notation, we have $\|\lambda_t(\alpha, z)\|^2 = |z|^2 e^{t\Delta + N(t,\alpha)}$ where

$$N(t,\alpha) = \sum_{i,m,n} e^{t(m-n)} \|a_i^{(mn)}\|^2 + \sum_{i,m,n} e^{t(m-n)} \|(da_i)^{(mn)}\|^2$$
$$+ \sum_m e^{tm} \|b^{(m)}\|^2 + \sum_m e^{tm} \|(db)^{(m)}\|^2 + \sum_n e^{-tn} \|c^{(n)}\|^2.$$

Thus we see that $\|\lambda_t(\alpha, z)\| \to \infty$ as $t \to \infty$ when $\Delta > 0$ and also when $\Delta \leqslant 0$ as long as $N(t,\alpha) \to \infty$ as $t \to \infty$. This last condition will fail if and only if

$$\forall m > n \quad a_i^{(mn)} = (da_i)^{(mn)} = 0 \quad (i = 1, 2)$$
$$\forall m > 0 \quad b^{(m)} = (db)^{(m)} = 0$$
$$\forall n < 0 \quad c^{(n)} = 0$$

These in turn are equivalent to the conditions that, for $i = 1, 2$ and $\forall n \in \mathbb{R}$,

$$\begin{aligned} a_i &: W_1^{(\leqslant n)} \to W_0^{(\leqslant n)} \\ da_i &: W_1^{(\leqslant n)} \to W_1^{(\leqslant n)} \end{aligned} \qquad \begin{aligned} b &: \mathbb{C}^r \to W_0^{(\leqslant 0)} \\ db &: \mathbb{C}^r \to W_1^{(\leqslant 0)} \\ c &: W_1^{(\leqslant -1)} \to 0 \end{aligned} \tag{4.1.4}$$

So, if we choose $V_0 = W_0^{(\leqslant n)} \cap d^{-1} W_1^{(\leqslant n)}$ and $V_1 = W_1^{(\leqslant n)}$, for any value of n, we will have

$$a_i : V_1 \to V_0,\ d : V_0 \to V_1 \quad \text{and} \begin{cases} \operatorname{im} b \subseteq V_0 & \text{if } n \geqslant 0 \\ V_1 \subseteq \ker c & \text{if } n < 0 \end{cases}.$$

Therefore, we have a contradiction to either $(S1)$ or $(S2)$ as long as we can find a value of n which makes $\dim V_0 \leqslant \dim V_1$ and also ensures that V_0 is proper $(n < 0)$ or V_1 is non-trivial $(n \geqslant 0)$. To see that this is possible, recall that we are considering the situation in which $\Delta \leqslant 0$, i.e.

$$\sum_{n \in \mathbb{R}} n \dim W_0^{(n)} \geqslant \sum_{n \in \mathbb{R}} n \dim W_1^{(n)}.$$

But we also know that

$$\sum_{n \in \mathbb{R}} \dim W_0^{(n)} = \sum_{n \in \mathbb{R}} \dim W_1^{(n)},$$

and from these last two equations it is clear — thinking of $\dim W_i^{(n)}$ as a finite measure on $\mathbb{R}$ — that there is indeed a real number n for which $\dim W_0^{(\leqslant n)} \leqslant \dim W_1^{(\leqslant n)}$, and hence $\dim V_0 \leqslant \dim V_1$, as required.

Thus we have demonstrated one half of the implication in the proposition, namely that when α is stable all orbits of real one-parameter subgroups diverge to infinity. To prove the converse, suppose that α is unstable. Choose complements V_i' to the destabilising subspaces V_i to get $W_i = V_i \oplus V_i'$ $(i = 0, 1)$. Then define a one-parameter subgroup $\lambda : \mathbb{R} \to G_{\mathbb{C}} : t \mapsto e^{tA}$ as follows:

i) If α fails $(S1)$, $A_i = \begin{pmatrix} -1 & 0 \\ 0 & 0 \end{pmatrix}$ for $i = 0, 1$.

ii) If α fails $(S2)$, $A_i = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ for $i = 0, 1$.

In both cases $\Delta = \dim V_0 - \dim V_1 \leqslant 0$ and the conditions (4.1.4) are satisfied, therefore

$$\|\lambda_t(\alpha, z)\| \not\to \infty \quad \text{as} \quad t \to \infty,$$

i.e. there is a real one-parameter subgroup whose orbit through α does not diverge to infinity. □

Corollary 4.1.7. *A configuration $\alpha \in \tilde{\mathcal{R}}$ is analytically stable with respect to the linearised $G_{\mathbb{C}}$-action* (4.1.2) *if and only if it satisfies* (S1) *and* (S2).

Proof. Follows immediately from the preceeding proposition by Lemma 1.5.4. □

4.2 The Real Equations as a Momentum Map

We now show that the the real equations $(\mu 0)$ and $(\mu 1)$ can be interpreted as the vanishing of the moment map associated to the linearised action on $\tilde{\mathcal{R}}$ described in the previous section. We thereby prove the equivalence of the two moduli spaces as a particular example of the equivalence of symplectic and algebraic quotients.

Consider a configuration $\alpha \in \tilde{\mathcal{R}}$ and define the function, as in §1.5, which measures the variation of the norm on L under the action of $G_{\mathbb{C}}$

$$M_\alpha : G_{\mathbb{C}} \to \mathbb{R} : g \mapsto \log \frac{\|g(\alpha, z)\|}{\|(\alpha, z)\|},$$

for some $z \neq 0$ (the function being otherwise independent of z). Explicitly,

$$M_\alpha(g_0, g_1) = \log \frac{|\det g_1|}{|\det g_0|} + \frac{1}{2}\|\rho(g\alpha)\|^2 - \frac{1}{2}\|\rho(\alpha)\|^2.$$

As in general, M_α descends to a function m_α on the quotient $P = G_{\mathbb{C}}/G$. The tangent space of P is naturally identified with $i\mathfrak{g}$ and the moment map is given by $\mu_\alpha = dm_\alpha(0)$. We can of course evaluate this derivative by just taking the derivative of M_α in the $i\mathfrak{g}$ directions in $G_{\mathbb{C}}$. Thus, if h_0 and h_1 are hermitian endomorphisms of W_0 and W_1 respectively,

$$\begin{aligned} dM_\alpha(0)(h_0, h_1) &= \operatorname{tr} h_1 - \operatorname{tr} h_0 + \sum_i \langle h_0\, a_i - a_i\, h_1, a_i\rangle + \langle h_0\, b, b\rangle \\ &\quad + \sum_i \langle h_1\, da_i - da_i\, h_1, da_i\rangle + \langle h_1\, db, db\rangle - \langle c\, h_1, c\rangle \\ &= \big\langle h_0, \sum_i a_i a_i^\star + bb^\star - 1\big\rangle \\ &\quad + \big\langle h_1, \sum_i [da_i, (da_i)^\star] - \sum_i a_i^\star a_i + db(db)^\star - c^\star c + 1\big\rangle. \end{aligned}$$

Thus we have proved the following

Lemma 4.2.1. *Equations $(\mu 0)$ and $(\mu 1)$ correspond to the vanishing of the moment map associated to the linearised action* (4.1.2) *of $G_{\mathbb{C}}$ on $\tilde{\mathcal{R}}$.*

4.3 Statement of the Theorem and Summary of the Proof

We now combine the various results from the previous sections and earlier chapters to prove our main theorem giving the Hitchin-Kobayashi correspondence for $\widetilde{\mathbb{C}}^2$

THEOREM 4.3.1. *The map* $\mathsf{h} : \mathsf{MI}(\widetilde{\mathbb{C}}^2; r, k) \to \mathsf{MH}(\widetilde{\mathbb{C}}^2; r, k)$*, given by taking the holomorphic part of the instanton connection, is a bijection.*

PROOF. The moduli space $\mathsf{MH}(\widetilde{\mathbb{C}}^2; r, k)$ is the quotient of the variety $X \subseteq \tilde{\mathcal{R}}$ by the complex group $G_{\mathbb{C}}$ (Theorem 3.4.1), while $\mathsf{MI}(\widetilde{\mathbb{C}}^2; r, k)$ is the quotient of a real subvariety $X_\mu \subseteq X$ by G (Theorem 3.5.1). Furthermore, the map h is induced by the inclusion of X_μ in X (Theorem 3.5.2). Now, all the orbits of $G_{\mathbb{C}}$ in X are analytically stable with respect to a suitable linearisation of the $G_{\mathbb{C}}$-action (Proposition 4.1.3 & Corollary 4.1.6) and $X_\mu = \mu^{-1}(0)$ for the corresponding moment map μ (Lemma 4.2.1). Hence, the fact that h is a bijection is an application of Theorem 1.5.3. □

4.4 The Kähler Structure of the Moduli Space

Now that we have completed the proof of Theorem 4.3.1, we can talk simply of the moduli space $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, k)$ and mean either the moduli space of holomorphic bundles or of instantons. As stated in Theorem 1.5.3, in general, this dual description means that we have more structure on the moduli space than we had a priori from either description. Specifically, the complex structure, which we would expect on $X/G_{\mathbb{C}}$, and the Reimannian structure, which we would expect on X_μ/G, are compatible and yield a Kähler structure on $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, k)$. In fact, since we started with a Hodge structure on X, we can define a Hodge structure on the quotient, i.e. we can find a global Kähler potential for the quotient metric. The linearised action enables one to take a quotient of the total space of the line bundle L to get the total space of a line bundle on the quotient. The $G_{\mathbb{C}}$ quotient gives this bundle a holomorphic structure, while the G quotient makes it hermitian. For a more details, see **[HKLR]**,**[GS]**.

REMARK 4.4.1. In our case we observe that the holomorphic line bundle $\hat{L}$ on $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, k)$, which carries the Kähler potential, has a canonical section induced by $\det d$, a $G_{\mathbb{C}}$-invariant section of L. Furthermore, the dual bundle $\hat{L}^*$ has a two-dimensional space of sections induced by $\det(\mu_1 a_1 + \mu_2 a_2)$. In the light of the discussion in §3.6, this shows that $\hat{L}$ and $\hat{L}^*$ are natural bundles to consider, in that their sections determine the 'jumping divisors' in $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, k)$, i.e. the subvarieties corresponding to holomorphic bundles over $\widetilde{\mathbb{C}}^2$ which jump over fixed lines. Such jumping divisors are important in the analysis of Donaldson's new polynomial invariants for complex surfaces **[D7]**.

REMARK 4.4.2. Itoh **[It]** has shown, in general, that the moduli space of instantons on a compact Kähler surface has a Kähler metric, induced by the L^2 metric on the space of connections. This metric is most easily described by its Kähler form

$$(a, b) \mapsto \frac{1}{2\pi^2} \int \operatorname{tr}(a \wedge b) \wedge \omega$$

where $a, b \in \Omega^1(\operatorname{End}\mathcal{E})$, the tangent space to the space of connections on $\mathcal{E}$. This metric is also defined over a non-compact Kähler surface, as long as one restricts ones attention to finite action instantons, and so we get an L^2 metric on $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, k)$. The natural question to ask is whether this is the same as — or at least a multiple of — the Kähler metric we have induced on the moduli space by our finite quotient description. While we cannot answer this question here, we can give some reasons why we might expect the answer to be "yes". Firstly, there is the naturality of the transformation to the linear algebra data, as indicated by §3.6 and the remark above. There is, further, a formal similiarity between this transformation and the Nahm transform for instantons on a four-dimensional torus, for which there is an analogous result on the equivalence of metrics [**BvB**]. Secondly, the metric has one particular property, which we shall discover in the next chapter, which is known to hold for the L^2 metric. This is the fact that the metric completion of the index one moduli space is obtained by adding an isometric copy of the base at infinity [**GP**].

5 Completing the Moduli Spaces

As shown in the previous chapter, the moduli space $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, k)$ carries a Kähler metric as a result of its description as both a complex and symplectic quotient. The metric is induced by a Euclidean metric on the configuration space $\mathcal{R}'$, restricted to a real quasi-affine subvariety $Y' \subseteq \mathcal{R}'$, under the quotient by a compact group G of isometries of $\mathcal{R}'$. Since this real subvariety is not closed, but the isometry group is compact, the metric on the quotient is not complete. However, we can easily construct its completion by taking the quotient by G of the closure $\overline{Y}'$, i.e. the (possibly singular) subvariety obtained by dropping the non-degeneracy conditions in the definition of Y'. We call the quotient $\overline{Y}'/G$ the completed moduli space $\overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, k)$.

The aim of this chapter is to show that both this completion and the analogous one $\overline{\mathsf{M}}(\mathbb{C}^2; r, k)$, over the affine plane, have the form that one expects in general (see [D5;III(iii)]). Working over a general base manifold Z, there should be a stratification

$$\overline{\mathsf{M}}(Z; r, k) = \bigcup_{l=0}^{k} S_{k,l}$$

where $S_{k,l}$ is naturally contained in $\mathsf{M}(Z; r, k) \times S^{k-l}(Z)$ and $S^{k-l}(Z)$ is the set of unordered $(k-l)$tuples of points in Z. In other words, a point in the stratum $S_{k,l}$ is an "ideal instanton", $(A_l; p_1, \ldots, p_{k-l})$, given by an ordinary instanton A_l of index l together with the (not necessarily distinct) points $p_1, \ldots, p_{k-l} \in Z$, which should be thought of as singular instantons with delta-function curvatures concentrated at those points.

Now, when $Z = \mathbb{C}^2$ or $\widetilde{\mathbb{C}}^2$, we shall see that $S_{k,l} \cong \mathsf{M}(Z; r, k) \times S^{k-l}(Z)$. In particular, when $k = 1$, we then get

$$\overline{\mathsf{M}}(Z; r, 1) \cong \mathsf{M}(Z; r, 1) \cup Z$$

and we can show further that the boundary of this moduli space, i.e. $\overline{\mathsf{M}}(Z; r, 1) \setminus \mathsf{M}(Z; r, 1)$, is actually isometric to Z, which, of course, started with a canonical Kähler metric.

We shall also show that the completions $\overline{\mathsf{M}}(\mathbb{C}^2; r, k)$ and $\overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, k)$ are algebraic, by showing that they arise as quotients in the sense of geometric invariant theory. Thus, there is an action by a complex algebraic group $G_{\mathbb{C}}$ and the quotient is formed by additionally identifying orbits whose closures intersect, i.e. it consists of equivalence classes under the relation

$$x \sim y \Leftrightarrow \overline{G_{\mathbb{C}} \cdot x} \cap \overline{G_{\mathbb{C}} \cdot y} \neq \emptyset.$$

5.1 The Index 1 Moduli Space over $\widetilde{\mathbb{C}}^2$

We begin by looking just at the incomplete moduli space $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, 1)$, regarded as the moduli space of holomorphic bundles, i.e. as the complex quotient $X/G_{\mathbb{C}}$, in the notation introduced at the beginning of §4. Identifying W_0 and W_1 with $\mathbb{C}$, we have $a \in \mathbb{C}^2$, $d \in \mathbb{C}$, $b \in (\mathbb{C}^r)^*$ and $c \in \mathbb{C}^r$. In this case, the integrability condition (I) reduces to

$$bc = 0 \tag{5.1.1}$$

and the stability conditions (S1) and (S2) reduce to

$$c \neq 0 \quad \text{and} \quad b \neq 0. \tag{5.1.2}$$

We can write the action of $G_{\mathbb{C}} = \mathbb{C}^* \times \mathbb{C}^*$ in a slightly modified form by setting $(g_0, g_1) = (\lambda_0, 1/\lambda_1)$, for $\lambda_i \in \mathbb{C}^*$, so that

$$(\lambda_0, \lambda_1) : \begin{cases} a \mapsto \lambda_0\lambda_1 a \\ b \mapsto \lambda_0 b \\ c \mapsto \lambda_1 c \\ d \mapsto (\lambda_0\lambda_1)^{-1} d \end{cases} \tag{5.1.3}$$

We see, from (5.1.1) and (5.1.2), that, up to the symmetry (5.1.3), the pair (b, c) simply determines the point $(\operatorname{im} c, \ker b)$ in the Grassmannian of partial flags

$$\mathbb{G} = \mathbb{G}_{1,r-1}(\mathbb{C}^r) = \{(V_1, V_{r-1}) \mid V_1 \subseteq V_{r-1} \subseteq \mathbb{C}^r,\ \dim V_i = i\}.$$

Furthermore, the quotient map $\{(b, c) \in (\mathbb{C}^r)^* \times \mathbb{C}^r \mid (5.1.1), (5.1.2)\} \to \mathbb{G}$ can be thought of as the principal $\mathbb{C}^* \times \mathbb{C}^*$ bundle with the associated family of line bundles

$$\mathcal{O}_{\mathbb{G}}(p, q) = \pi_1^*(\mathcal{O}(p)) \otimes \pi_2^*(\mathcal{O}(q)),$$

where the π_i are the natural projections

$$\mathbb{P}(\mathbb{C}^r) \xleftarrow{\pi_1} \mathbb{G} \xrightarrow{\pi_2} \mathbb{P}((\mathbb{C}^r)^*).$$

[c.f. the quotient $\mathbb{C}^{n+1} \setminus \{0\} \to \mathbb{P}^n$ determines a principal $\mathbb{C}^*$ bundle, with associated line bundles $\mathcal{O}_{\mathbb{P}^n}(d)$, for $d \in \mathbb{Z}$. The total space of $\mathcal{O}_{\mathbb{P}^n}(d)$ is the quotient of $(\mathbb{C}^{n+1} \setminus \{0\}) \times \mathbb{C}$ by the action $(v, \zeta) \mapsto (\lambda v, \lambda^d \zeta)$.] Thus, we see that the quotient description of the moduli space $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, 1)$ is simply the natural description of the total space of the bundle $\mathcal{O}_{\mathbb{G}}(1,1)^2 \oplus \mathcal{O}_{\mathbb{G}}(-1,-1)$. When $r = 2$, there is some simplification, because $\mathbb{G} = \mathbb{P}^1$ and the moduli space $\mathsf{M}(\widetilde{\mathbb{C}}^2; 2, 1)$ is then the total space of the bundle $\mathcal{O}_{\mathbb{P}^1}(2)^2 \oplus \mathcal{O}_{\mathbb{P}^1}(-2)$.

Now, to find the completed moduli space $\overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, 1)$, we must introduce the moment map equations $(\mu 0)$ and $(\mu 1)$. In the case of index 1, these reduce to the conditions

$$\begin{aligned} \|a\|^2 + \|b\|^2 &= 1 \\ \|b\|^2 + \|db\|^2 &= \|c\|^2 \end{aligned} \tag{5.1.4}$$

The moduli space $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, 1)$, regarded as the moduli space of instantons, is the quotient of $Y = \{(a,b,c,d) \in \tilde{\mathcal{R}} \mid (5.1.1), (5.1.2), (5.1.4)\}$ by $G = U(1) \times U(1)$, acting as in (5.1.3). The metric on $\overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, 1)$ is induced by the embedding $\rho : Y \to \mathcal{R}'$ from the Euclidean metric on $\mathcal{R}'$, so the completed moduli space, in this case, is given by

$$\overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, 1) = \overline{\rho(Y)}/G.$$

To describe the boundary of the moduli space, i.e. the set we have to add to complete it, we simply set $b = 0$ (or $c = 0$), while retaining the moment map equations (5.1.4) as well as the equations that define the image of ρ. Hence, the boundary is the quotient of the set

$$\{(a, 0, x, 0, 0) \in \mathcal{R}' \mid a_1 x_2 - a_2 x_1 = 0,\ ||a||^2 = 1\}$$

under the action $a \mapsto \lambda_0 \lambda_1\, a$, $x \mapsto x$. In other words, it is the set

$$\{([a], x) \in \mathbb{CP}^1 \times \mathbb{C}^2 \mid a_1 x_2 - a_2 x_1 = 0\},$$

where $\mathbb{CP}^1$ and $\mathbb{C}^2$ are equipped with the Fubini-Study and flat metrics respectively. This is precisely the description we had in §1.4 of the anti-self-dual Kähler metric on $\widetilde{\mathbb{C}}^2$, so we have now proved the following

PROPOSITION 5.1.1. *The boundary of the index 1 moduli space over $\widetilde{\mathbb{C}}^2$ is isometric to $\widetilde{\mathbb{C}}^2$ with its canonical anti-self-dual Kähler metric.*

We can begin to build up more of a picture of the moduli space $\overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, 1)$ with the aid of the two functions induced by the G-invariant functions $||b||$ and $|d|$, which are defined on Y, but clearly pass to $\rho(Y)$ and extend to $\overline{\rho(Y)}$. Thus, with a slight abuse of notation, we can speak of the functions

$$||b|| : \overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, 1) \to [0, 1]$$

$$|d| : \overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, 1) \to [0, \infty).$$

We can further see that $|d|$ is proper since, by (5.1.4), $||a||$ and $||b||$ are bounded and a bound on $|d|$ determines a bound on $||c||$.

As observed above, the set $||b|| = 0$ is the boundary of the moduli space and is a copy of $\widetilde{\mathbb{C}}^2$ with its canonical anti-self-dual Kähler metric. At the 'other side' of the moduli space, we can look at the set $||b|| = 1$, on which $a = 0$. From the earlier holomorphic description of $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, 1)$, we can then see that what we have is total space of the bundle $\mathcal{O}_{\mathbb{G}}(-1,-1)$ or, in the case $r = 2$, the bundle $\mathcal{O}_{\mathbb{P}^1}(-2)$. Metrically, this latter case is the quotient of the set

$$\left\{(0, b, 0, y, c) \mid \begin{matrix} y_1 b_2 - y_2 b_1 = 0,\ bc = yc = 0 \\ ||b||^2 = 1,\ ||c||^2 = 1 + ||y||^2 \end{matrix}\right\} \tag{5.1.5}$$

by the $U(1) \times U(1)$-action $b \mapsto \lambda_0 b$, $c \mapsto \lambda_1 c$ and $y \mapsto \lambda_1^{-1} y$. This determines a Kähler metric which is related to — but not equal to — the Eguchi-Hanson metric (the canonical hyper-Kähler metric on $\mathcal{O}_{\mathbb{P}^1}(-2)$), in that the latter has a quotient description similar to (5.1.5), but with the b parameter missing (see [Hi2]).

The importance of the function $|d|$ is essentially given by Remark 4.4.1, which shows that the set $|d| = 0$ (i.e. $d = 0$) is the jumping divisor for the exceptional fibre of the blowup $\widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$. In fact, as we shall see in the final chapter, the divisor $\{d = 0\}$ is itself the exceptional fibre of a blow-up.

REMARK 5.1.2. The index one moduli space of anti-self-dual instantons on $\overline{\mathbb{CP}}^2$ is known to be a cone on $\overline{\mathbb{CP}}^2$ ([**D2**],[**Bu1**]). Since $\mathsf{M}(\widetilde{\mathbb{C}}^2; r, 1) = \mathsf{MI}(\overline{\mathbb{CP}}^2, \infty; r, 1)$ is a framed moduli space, we must take the quotient by $U(r)$, acting on the fibre at ∞, to obtain the unframed moduli space. This quotient is realised, at the level of the linear algebra data (satisfying (5.1.4)), by the map

$$(a, b, c, d) \mapsto \left(\frac{a_1}{||b||}, \frac{a_2}{||b||}, \overline{d}\right),$$

which gives us the unframed moduli space as the quotient of $\mathbb{C}^3$ by the scalar action of $U(1)$, as required.

5.2 The Higher Moduli Spaces over $\mathbb{C}^2$

Before looking at the higher index moduli spaces over $\widetilde{\mathbb{C}}^2$, we first consider the analogous spaces over $\mathbb{C}^2$. The discussion over $\widetilde{\mathbb{C}}^2$ will generalise fairly directly the one given here and, once we have both, we will be able to describe explicitly the relationship between the two moduli spaces. We start by recalling Donaldson's description [**D1**] of the moduli space $\mathsf{M}(\mathbb{C}^2; r, k)$.

Given a hermitian inner product space W of dimension k consider the set of all configurations

$$(a, b, c) \in \mathcal{R} = \mathrm{End}(W)^2 \oplus \mathrm{Hom}(\mathbb{C}^r, W) \oplus \mathrm{Hom}(W, \mathbb{C}^r)$$

satisfying the integrability equation

$$a_1 a_2 - a_2 a_1 + bc = 0 \tag{5.2.1}$$

and subject to the non-degeneracy conditions:

$$\begin{aligned} &\nexists 0 \neq v \in W \text{ with } a_i(v) = \lambda_i\, v \text{ and } c(v) = 0, \\ &\nexists 0 \neq v' \in W^* \text{ with } a_i^*(v') = \lambda_i\, v' \text{ and } b^*(v') = 0. \end{aligned} \tag{5.2.2}$$

The moduli space $\mathsf{M}(\mathbb{C}^2; r, k)$ is then the quotient of this space by the natural action of $GL(W)$ or, equivalently, the quotient by $U(W)$ of the real subspace satisfying the additional moment map equation

$$[a_1, a_1^\star] + [a_2, a_2^\star] + bb^\star - c^\star c = 0. \tag{5.2.3}$$

A configuration (a, b, c) describes a holomorphic bundle on $\mathbb{C}^2$, with an extension to $\mathbb{CP}^2$ trivial at infinity, via a monad of the canonical form

$$W \xrightarrow{\mathcal{A}} (W \oplus W \oplus \mathbb{C}^r) \xrightarrow{\mathcal{B}} W$$

with

$$\mathcal{A} = \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \\ c \end{pmatrix} \qquad \mathcal{B} = \begin{pmatrix} -x_2 + a_2 & x_1 - a_1 & b \end{pmatrix} \tag{5.2.4}$$

REMARK 5.2.1. The non-degeneracy conditions (5.2.2) are precisely those required to ensure that $\mathcal{A}$ is pointwise injective and that $\mathcal{B}$ is pointwise surjective. It was observed by Helmke **[Hm]**, with reference to linear control theory, that there is a more general pair of non-degeneracy conditions on the space of all configurations, which is equivalent to (5.2.2) for integrable configurations. This pair of conditions can be written

$$\nexists V \subseteq W \text{ with } V \neq W,\ a_i(V) \subseteq V \text{ and } \operatorname{im} b \subseteq V,$$

$$\nexists V \subseteq W \text{ with } V \neq 0,\ a_i(V) \subseteq V \text{ and } V \subseteq \ker c,$$

and gives precisely the condition of stability for the action of $GL(W)$ on the space of all configurations.

In the case $k = 1$, the map $(a, b, c) \mapsto (a, cb) \in \mathbb{C}^2 \oplus \operatorname{End}(\mathbb{C}^r)$ realises the quotient and thus identifies the moduli space $\mathsf{M}(\mathbb{C}^2; r, 1)$ with $\mathbb{C}^2 \times \mathcal{N}$, where $\mathcal{N} = \{\beta \in \operatorname{End}(\mathbb{C}^r) \mid \beta^2 = 0,\ \operatorname{rk}\beta = 1\}$, equipped with a product metric in which the first factor is the Euclidean metric on $\mathbb{C}^2$. The set $\mathcal{N}$ has an obvious completion $\overline{\mathcal{N}}$ obtained by relaxing the condition $\operatorname{rk}\beta = 1$ to $\operatorname{rk}\beta \leqslant 1$, i.e. adding 0. Thus, we get the completed moduli space $\overline{\mathsf{M}}(\mathbb{C}^2; r, 1) = \mathbb{C}^2 \times \overline{\mathcal{N}}$, obtained by adding a copy of $\mathbb{C}^2$, with the Euclidean metric.

To handle the completions of the higher index moduli spaces, we must first introduce some notation, to ease the discussion, and also the notion of complete reducibility of a configuration, which provides the link between the symplectic quotient description and the algebraic quotient description. The motivating example is that of a single endomorphism $a : W \to W$ of some hermitian inner product space W. The moment map equation in this case is $\mu(a) = [a, a^\star]$ and, if $\mu(a) = 0$, then a can be diagonalised by a unitary automorphism of W. Alternatively, one has the notion of complete reducibility, which requires that, whenever a preserves a subspace $V \subseteq W$, it also preserves a complement $V' \subseteq W$. A linear endomorphism is completely reducible if and only if it is diagonalisable by some linear automorphism of W, which in turn is the case if and only if it satisfies the moment map equation with respect to some inner product on W. Since we can write any endomorphism $a : W \to W$ in lower triangular form, with respect to a suitable basis, we can apply the one-parameter subgroup $\lambda_\tau = diag(1, \tau, \tau^2, \ldots)$ and see that $\lim_{\tau\to 0} \lambda_\tau a \lambda_\tau^{-1}$ is diagonal. In other words, the $GL(W)$-orbit of any endormorphism contains, in its closure, a (unique) orbit of completely reducible endomorphisms. Thus, the algebraic quotient of $\operatorname{End}(W)$ by $GL(W)$ is equal to the set-theoretic quotient of the set of completely reducible endomorphisms, which is in turn equal to the quotient of the set $\mu^{-1}(0)$ by $U(W)$.

DEFINITION 5.2.2. A subspace $V \subseteq W$ will be called *special*, with respect to a configuration $(a, b, c) \in \mathcal{R}$, if one of the following holds:

$$a_i(V) \subseteq V \quad (i = 1, 2) \quad \text{and} \quad \operatorname{im} b \subseteq V \tag{5.2.5}$$

$$a_i(V) \subseteq V \quad (i = 1, 2) \quad \text{and} \quad V \subseteq \ker c \tag{5.2.6}$$

Thus, by Remark 5.2.1, the non-degeneracy conditions (5.2.2) are equivalent to requiring that there should be no special subspaces other than the necessary ones: W, in the case of (5.2.5), and $\{0\}$, in the case of (5.2.6). In any other situation, the subspace V will be called properly special.

Definition 5.2.3. We shall call a configuration *completely reducible* if, for every $V \subseteq W$, which is special in the sense of one of the equations (5.2.5) or (5.2.6), there is a complement $V' \subseteq W$, which is special in the sense of the other equation.

With this terminology, we have the following results:

Lemma 5.2.4. *If a configuration (a, b, c) satisfies the moment map equation (5.2.3), with respect to some hermitian inner product on W, then it is completely reducible.*

Proof. Given any orthogonal decomposition $W = V \oplus V'$, we can write a_i, b and c in block form

$$\begin{pmatrix} a_i^{(11)} & a_i^{(12)} \\ a_i^{(21)} & a_i^{(22)} \end{pmatrix} \qquad \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix} \qquad \begin{pmatrix} c^{(1)} & c^{(2)} \end{pmatrix}.$$

Taking the trace of the (11) component of the moment map equation gives

$$||a^{(12)}||^2 + ||b^{(1)}||^2 = ||a^{(21)}||^2 + ||c^{(1)}||^2,$$

where the norm comes from the usual inner product $\langle x, y\rangle = \mathrm{tr}(x^\star y)$ and we use the shorthand $||a^{(12)}||^2 = \sum_i ||a_i^{(12)}||^2$. Thus we immediately see that V is special in the sense of (5.2.5) if and only if V' is special in the sense of (5.2.6). Hence, the orthogonal complement of a special subspace is also special (in the other sense) and so the configuration is completely reducible. □

Lemma 5.2.5. *If (a, b, c) is completely reducible and integrable, then we can find a canonical decomposition $W = V \oplus V'$ with respect to which*

$$a_i = \begin{pmatrix} a_i^{red} & 0 \\ 0 & a_i^{\Delta} \end{pmatrix} \quad b = \begin{pmatrix} b^{red} \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} c^{red} & 0 \end{pmatrix}, \tag{5.2.7}$$

where a_1^{Δ} and a_2^{Δ} are simultaneously diagonalisable and the reduced configuration $\left(a^{red}, b^{red}, c^{red}\right)$ is non-degenerate and integrable.

Proof. Speciality, in the sense of (5.2.5), is preserved by intersection, so we can find a minimum such special V. Since (a, b, c) is completely reducible, there will be a complement V', which is special in the sense of (5.2.6). Writing a_i, b and c in block form with respect to the decomposition $W = V \oplus V'$ gives the required form (5.2.7). The fact that the reduced configuration is non-degenerate follows from the minimality of V and the complete reducibility of the original configuration. The integrability equation (5.2.1) reduces to exactly the same equation on V, while on V' it gives $\left[a_1^{\Delta}, a_2^{\Delta}\right] = 0$. Hence, a_1^{Δ} and a_2^{Δ} will have a simultaneous eigenvector. Complete reducibility provides a complementary subspace and so shows, by induction, that a_1^{Δ} and a_2^{Δ} can actually be simultaneously diagonalised. □

Corollary 5.2.6. *If a configuration is completely reducible and integrable, then we can find a metric on W with respect to which it satisfies the moment map equation.*

Proof. If we take the decomposition (5.2.7), the reduced configuration is integrable and non-degenerate, so we can find an appropriate metric on V. This is an alternate way of formulating our main result Theorem 4.3.1. If we extend it to W by requiring that the simultaneous eigenvectors of the a_i^{Δ} be all orthogonal (to each other and to V), then we will satisfy the moment map equation on the whole of W. □

Lemma 5.2.5 provides an explicit way of seeing the anticipated stratification of the completed moduli space:

$$\overline{\mathsf{M}}(\mathbb{C}^2;r,k) = \bigcup_{l=0}^{k} S_{k,l}.$$

In our algebraic description, the stratum $S_{k,l}$ is the quotient of the set of configurations for which the minimum special subspace V has fixed dimension l. Such a configuration determines (and is determined by) the non-degenerate reduced configuration $(a^{red}, b^{red}, c^{red})$, of index l, together with the $k-l$ points in $\mathbb{C}^2$ given by the eigenvalue pairs for a_1^{Δ}, a_2^{Δ}. In other words, as we expected

$$S_{k,l} = \mathsf{M}(\mathbb{C}^2;r,l) \times S^{k-l}(\mathbb{C}^2).$$

Lemma 5.2.4 and Corollary 5.2.6, above, show that the completed moduli space $\overline{\mathsf{M}}(\mathbb{C}^2;r,k)$ is also the set-theoretic quotient by $GL(W)$ of the space of integrable, completely reducible configurations. This then enables us to prove the promised result

Theorem 5.2.7. *The completed moduli space $\overline{\mathsf{M}}(\mathbb{C}^2;r,k)$ is the algebraic quotient by $GL(W)$ of the set of integrable configurations $(a,b,c) \in \mathcal{R}$ and, hence, is an algebraic space.*

Proof. With the above lemmas, the theorem follows from the fact that every orbit contains, in its closure, a canonical completely reducible orbit (i.e. orbit of completely reducible configurations) and that completely reducible orbits have disjoint closures.

We can prove the first of these facts by induction, taking a general integrable configuration (a,b,c) and finding a one-parameter subgroup $\lambda_\tau : \mathbb{C}^* \to GL(W)$, whose orbit converges to a completely reducible configuration. If (a,b,c) is non-degenerate then $\lambda_\tau = 1$ will do. If not, then there is a decomposition $W = W' \oplus \langle v \rangle$ such that

either $$(i)\quad a_i = \begin{pmatrix} a_i' & * \\ 0 & * \end{pmatrix} \quad b = \begin{pmatrix} b' \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} c' & * \end{pmatrix}$$

or $$(ii)\quad a_i = \begin{pmatrix} a_i' & 0 \\ * & * \end{pmatrix} \quad b = \begin{pmatrix} b' \\ * \end{pmatrix} \quad c = \begin{pmatrix} c' & 0 \end{pmatrix}.$$

Our inductive hypothesis is that we can find a one-parameter subgroup $\lambda_\tau' : \mathbb{C}^* \to GL(W')$ such that $\lim_{\tau\to 0} \lambda_\tau'(a_i', b', c')$ is completely reducible. Setting

$$\lambda_\tau = \begin{pmatrix} \lambda_\tau' & 0 \\ 0 & \tau^N \end{pmatrix},$$

we see that $\lim_{\tau\to 0} \lambda_\tau(a_i, b, c)$ is completely reducible, provided $N \in \mathbb{Z}$ is chosen so that

either (i) $N < 0$ and $\tau^{-N}\lambda_\tau' \to 0$ (in $\mathrm{End}(W')$) as $\tau \to 0$

or (ii) $N > 0$ and $\tau^N (\lambda_\tau')^{-1} \to 0$ as $\tau \to 0$.

These conditions are easily satisfied, because λ_τ' can be written as a diagonal matrix $diag(\tau^{n_i})$ and the set of n_i is finite, hence bounded above and below. □

5.3 The Higher Moduli Spaces over $\tilde{\mathbb{C}}^2$

We now present an directly analogous description of the completed moduli spaces $\overline{\mathbf{M}}(\tilde{\mathbb{C}}^2; r, k)$. Hence, we return to considering configurations

$$(a, b, c, d) \in \tilde{\mathcal{R}} = \mathrm{Hom}(W_1, W_0)^2 \times \mathrm{Hom}(\mathbb{C}^r, W_0) \times \mathrm{Hom}(W_1, \mathbb{C}^r) \times \mathrm{Hom}(W_0, W_1),$$

which satisfy the integrability condition

$$a_1 d a_2 - a_2 d a_1 + bc = 0. \tag{5.3.1}$$

In this case, we have two moment map equations

$$\sum_i a_i a_i^\star + bb^\star = 1 \tag{5.3.2}$$

$$\sum_i [da_i\,,\,(da_i)^\star] - \sum_i a_i^\star a_i + db(db)^\star - c^\star c = -1, \tag{5.3.3}$$

corresponding to the fact that the automorphism group is the product of two general linear groups. As in the previous section, we can describe the completed moduli space in a manner which is independent of the metric, using a notion of complete reducibility together with an extra condition, which we have encountered before.

Definition 5.3.1. Given a configuration $(a, b, c, d) \in \tilde{\mathcal{R}}$, we say that a pair of subspaces $V_0 \subseteq W_0$, $V_1 \subseteq W_1$ is *special* if $\dim V_0 = \dim V_1$ and one of the following conditions holds

$$a_i(V_1) \subseteq V_0,\ d(V_0) \subseteq V_1,\ \mathrm{im}\, b \subseteq V_0 \tag{5.3.4}$$

$$a_i(V_1) \subseteq V_0,\ d(V_0) \subseteq V_1,\ V_1 \subseteq \ker c \tag{5.3.5}$$

By Remark 4.1.4, the non-degeneracy conditions (S1) and (S2) of §4.1 state that there are no properly special pairs (V_0, V_1) (i.e. other than (W_0, W_1) for (5.3.4) and $(0, 0)$ for (5.3.5)).

Definition 5.3.2. A configuration will be called *completely reducible* if for every pair (V_0, V_1), which is special in the sense of one of the equations (5.3.4) or (5.3.5), there is a pair of complements (V_0', V_1'), which is special in the sense of the other equation.

Definition 5.3.3. A configuration (a,b,c,d) will be called *effective* if a_1, a_2 and b are jointly surjective, i.e. $a_1(W_1) \oplus a_2(W_1) \oplus b(\mathbb{C}^r) = W_0$.

Recall that, in §4.2, we introduced a second space of configurations $\mathcal{R}'$, and a map $\rho : \tilde{\mathcal{R}} \to \mathcal{R}'$. The completed moduli space is more naturally a quotient of a subspace of $\mathcal{R}'$. However, the space of effective configurations in $\tilde{\mathcal{R}}$ is precisely the set on which ρ is injective and so, for effective configurations, we can equally well work in $\tilde{\mathcal{R}}$ (cf. Remark 4.1.5).

We now prove the analogues of Lemma 5.2.4, Lemma 5.2.5 and Corollary 5.2.6 over the blownup plane $\tilde{\mathbb{C}}^2$.

Lemma 5.3.4. *If a configuration (a,b,c,d) satisfies the moment map equations (5.3.2) and (5.3.3), then it is completely reducible and effective.*

Proof. We immediately see that (5.3.2) implies that the configuration is effective. Indeed, given any metric on W_1, a configuration is effective if and only if there is a metric on W_0 with respect to which it satisfies (5.3.2). To see that the configuration (a,b,c,d) is completely reducible, we consider a general orthgonal decomposition $W_i = V_i \oplus V_i'$ and write the configuration in block form:

$$\begin{pmatrix} a_i^{(11)} & a_i^{(12)} \\ a_i^{(21)} & a_i^{(22)} \end{pmatrix} \qquad \begin{pmatrix} b^{(1)} \\ b^{(2)} \end{pmatrix} \qquad \begin{pmatrix} c^{(1)} & c^{(2)} \end{pmatrix} \qquad \begin{pmatrix} d^{(11)} & d^{(12)} \\ d^{(21)} & d^{(22)} \end{pmatrix}.$$

Taking the trace of suitable components of the equations (5.3.2) and "(5.3.2) + (5.3.3)" yields

$$||a^{(11)}||^2 + ||a^{(12)}||^2 + ||b^{(1)}||^2 = \dim V_0,$$
$$||a^{(21)}||^2 + ||a^{(22)}||^2 + ||b^{(2)}||^2 = \dim V_0',$$

and

$$||d^{(11)}a^{(12)} + d^{(12)}a^{(22)}||^2 + ||a^{(12)}||^2 + ||b^{(1)}||^2 + ||d^{(11)}b^{(1)} + d^{(12)}b^{(2)}||^2$$
$$= ||d^{(21)}a^{(11)} + d^{(22)}a^{(21)}||^2 + ||a^{(21)}||^2 + ||c^{(1)}||^2$$

from which we see that $a^{(21)} = d^{(21)} = c^{(1)} = 0$ if and only if $a^{(12)} = d^{(12)} = b^{(1)} = 0$. In other words, (V_0, V_1) is special in the sense of (5.3.5) if and only if (V_0', V_1') is special in the sense of (5.3.4). □

Lemma 5.3.5. *If (a,b,c,d) is completely reducible, effective and integrable, then we can find a decomposition $W_i = V_i \oplus V_i'$ with respect to which*

$$a_i = \begin{pmatrix} a_i^{red} & 0 \\ 0 & a_i^{\Delta} \end{pmatrix} \quad b = \begin{pmatrix} b^{red} \\ 0 \end{pmatrix} \quad c = \begin{pmatrix} c^{red} & 0 \end{pmatrix} \quad d = \begin{pmatrix} d^{red} & 0 \\ 0 & d^{\Delta} \end{pmatrix} \tag{5.3.6}$$

with a_i^{Δ} and d^{Δ} simultaneously diagonalisable and the reduced configuration $\left(a^{red}, b^{red}, c^{red}, d^{red}\right)$ non-degenerate and integrable.

Proof. Choose a pair of subspaces (V_0, V_1), which is special in the sense of (5.3.4) and has minimum dimension among all such special pairs. Since the configuration is completely reducible we can find a pair of complements (V_0', V_1'), which is special in the sense of (5.3.5). Thus we have decompositions of W_0 and W_1, with respect to which we can write (a,b,c,d) in the required block form.

The reduced configuration is non-degenerate because (V_0, V_1) is minimal and the original configuration is completely reducible. The integrability equation (5.3.1) reduces to the same equation for the reduced configuration, so this is also integrable. On the other hand, if we restrict (5.3.1) to V_0' and V_1', then we get the equation

$$a_1^{\Delta}\, d^{\Delta}\, a_2^{\Delta} - a_2^{\Delta}\, d^{\Delta}\, a_1^{\Delta} = 0.$$

Now, Lemma 4.1.1 and Remark 4.1.2 provide us with a generalised notion of a simultaneous eigenvalue for such a triple of maps $\left(a_1^{\Delta}, a_2^{\Delta}, d^{\Delta}\right)$. The general 'eigenvector' should be a pair of non-zero vectors $v_0 \in V_0'$, $v_1 \in V_1'$ such that $a_i^{\Delta}(v_1)$ is proportional to v_0 and $d^{\Delta}(v_0)$ is proportional to v_1.

The lemma only explicitly gives us v_1, however we can simply take v_0 to be a non-zero vector in $\langle a_1^\Delta(v_1), a_2^\Delta(v_1)\rangle$ or, if this is trivial, in $(d^\Delta)^{-1}\langle v_1\rangle$. As long as $\langle a_1^\Delta(v_1), a_2^\Delta(v_1)\rangle \neq 0$ (which is in fact the case here) such an 'eigenvector' has a well defined 'eigenvalue', which is the point in $\tilde{\mathbb{C}}^2$ given by (λ_1, λ_2) and $[\mu_1, \mu_2]$ from the lemma. Since the original configuration is completely reducible, there exist complements to $\langle v_0\rangle$ and $\langle v_1\rangle$ in V_0' and V_1', which are preserved by a_i^Δ and d^Δ and thus, by induction, these can be diagonalised. The fact that the configuration is also effective prevents the space $\langle a_1(v_1), a_2(v_1)\rangle$ from ever being trivial. □

Corollary 5.3.6. *If a configuration is completely reducible, integrable and effective, then we can find metrics on W_0 and W_1 with respect to which it satisfies the moment map equations.*

Proof. In the decomposition (5.3.6), the reduced configuration is integrable and non-degenerate, so we can find appropriate metrics on V_0 and V_1. If we adjust all the pairs (v_0, v_1) so that $a_i(v_1) = \mu_i v_0$, with $|\mu_1|^2 + |\mu_2|^2 = 1$ and then extend the metrics to W_0 and W_1 so that the v_0s and v_1s are orthonormal and orthogonal to V_0 and V_1 respectively, then the full configuration will satisfy both the moment map equations. □

Thus, we see that the completed moduli space can be described as the quotient of the space of all completely reducible, effective and integrable configurations by the full group of automorphisms. Furthermore, just as in the case of $\mathbb{C}^2$, we see an explicit description of the stratification of $\overline{\mathsf{M}}(\tilde{\mathbb{C}}^2; r, k)$. Here the stratum $S_{k,l}$ is determined by those configurations for which the spaces of the minimal equidimensional special pair (V_0, V_1) both have dimension l. For such configurations, the reduced configuration determines a true instanton with $c_2 = l$ and the 'eigenvalues' of $(a_1^\Delta, a_2^\Delta, d^\Delta)$,as described in Lemma 5.3.5, determine $k - l$ points in $\tilde{\mathbb{C}}^2$, showing that

$$S_{k,l} = \mathsf{M}(\tilde{\mathbb{C}}^2; r, l) \times S^{k-l}(\tilde{\mathbb{C}}^2).$$

We should note here that this product decomposition of the strata does not seem quite as canonical as in the case of $\mathbb{C}^2$, because the pair (V_0, V_1) is strictly just minimal, rather than minimum. However, the pair (V_0', V_1') is in fact canonical, being spanned by 'eigenvectors' of the configuration and so, in the presence of the metrics on W_0 and W_1, we have a canonical choice for (V_0, V_1) given by the orthogonal complements.

Finally, we have the analogue of Theorem 5.2.7:

Theorem 5.3.7. *The completed moduli space $\overline{\mathsf{M}}(\tilde{\mathbb{C}}^2; r, k)$ is the algebraic quotient by $GL(W_0) \times GL(W_1)$ of the set of effective integrable configurations $(a, b, c, d) \in \tilde{\mathcal{R}}$ and, hence, is an algebraic space.*

Proof. As before, we show that, for every effective integrable configuration $a, b, c, d)$, there is a one-parameter subgroup $\lambda_\tau : \mathbb{C}^* \to GL(W_0) \times GL(W_1)$, whose orbit converges to a completely reducible configuration. In addition, the completely reducible orbits have disjoint closures, so the algebraic quotient is the set-theoretic quotient of the space of completely reducible, effective, integrable configurations, as required.

If (a,b,c,d) is non-degenerate, then we can take $\lambda_\tau = 1$. Otherwise, from the original non-degeneracy conditions $(N1)$ and $(N2)$ at the start of §4, we can find decompositions $W_i = W_i' \oplus \langle v_i \rangle$ such that

either (i) $a_i = \begin{pmatrix} a_i' & * \\ 0 & * \end{pmatrix}$ $b = \begin{pmatrix} b' \\ 0 \end{pmatrix}$ $c = \begin{pmatrix} c' & * \end{pmatrix}$ $d = \begin{pmatrix} d' & * \\ 0 & * \end{pmatrix}$

or (ii) $a_i = \begin{pmatrix} a_i' & 0 \\ * & * \end{pmatrix}$ $b = \begin{pmatrix} b' \\ * \end{pmatrix}$; $c = \begin{pmatrix} c' & 0 \end{pmatrix}$ $d = \begin{pmatrix} d' & 0 \\ * & * \end{pmatrix}$.

Our induction hypothesis is that we can find a one-parameter subgroup $\lambda_\tau' : \mathbb{C}^* \to GL(W_0) \times GL(W_1)$ such that $\lim_{\tau\to 0} \lambda_\tau'(a',b',c',d')$ is completely reducible. Setting

$$\lambda_\tau = \left(\begin{pmatrix} (\lambda_\tau')_0 & 0 \\ 0 & \tau^{N_0} \end{pmatrix}, \begin{pmatrix} (\lambda_\tau')_1 & 0 \\ 0 & \tau^{N_1} \end{pmatrix} \right),$$

we see that $\lim_{\tau\to 0} \lambda_\tau(a,b,c,d)$ is completely reducible, provided that $(N_0, N_1) \in \mathbb{Z} \times \mathbb{Z}$ is chosen so that

either (i) $N_1 < 0$ and $\tau^{-N_1}(\lambda_\tau')_0 \to 0$, $\tau^{-N_0}(\lambda_\tau')_1 \to 0$ as $\tau \to 0$

or (ii) $N_0 > 0$ and $\tau^{N_0}(\lambda_\tau')_1^{-1} \to 0$, $\tau^{N_1}(\lambda_\tau')_0^{-1} \to 0$ as $\tau \to 0$.

As before, both these conditions are easily achieved. □

REMARK 5.3.8. The fact that we can complete the moduli spaces $\mathsf{M}(\mathbb{C}^2; r,k)$ and $\mathsf{M}(\widetilde{\mathbb{C}}^2; r,k)$ by adding points that look like ideal instantons does not hold simply at the level of the linear algebra data, but can also be seen when we look at the corresponding monads. A completely reducible, but degenerate, configuration defines a three-term complex which splits into the direct sum of two complexes. The first is the monad determined by the reduced configuration, which gives the "true instanton" part of the ideal instanton. The second is a complex which is exact except over those points in the base which make up the singular part of the ideal instanton. Indeed, this complex further splits into complexes (one for each point) which are resolutions of the structure sheaves of the singular points. To be more explicit, over $\mathbb{C}^2$, each summand is of the form

$$\mathcal{O} \xrightarrow{\alpha} \mathcal{O}^2 \xrightarrow{\beta} \mathcal{O}$$

with

$$\alpha = \begin{pmatrix} x_1 - a_1 \\ x_2 - a_2 \end{pmatrix}, \qquad \beta = \begin{pmatrix} -x_2 + a_2 & x_1 - a_1 \end{pmatrix},$$

which resolves the structure sheaf of the point $(a_1, a_2) \in \mathbb{C}^2$. Similarly, over $\widetilde{\mathbb{C}}^2$, each summand is of the form

$$\mathcal{O} \oplus \mathcal{O}(-1) \xrightarrow{\alpha} \mathcal{O}^4 \xrightarrow{\beta} \mathcal{O} \oplus \mathcal{O}(1)$$

where $\mathcal{O}(1)$ is the restriction of $\mathcal{O}_{\Sigma^1}(0,1)$ and, now,

$$\alpha = \begin{pmatrix} a_1 & -y_2 \\ x_1 - da_1 & 0 \\ a_2 & y_1 \\ x_2 - da_2 & 0 \end{pmatrix}, \qquad \beta = \begin{pmatrix} x_2 & a_2 & -x_1 & -a_1 \\ dy_1 & y_1 & dy_2 & y_2 \end{pmatrix}$$

This complex is a resolution of the structure sheaf of the point $([a_1, a_2], (da_1, da_2)) \in \widetilde{\mathbb{C}}^2 \subseteq \mathbb{CP}^1 \times \mathbb{C}^2$.

Remark 5.3.9. We can see from the previous remark that it would be appropriate to define an "ideal holomorphic bundle" with $c_2 = k$ to be a pair $(\mathcal{E}, \Delta)$, where $\mathcal{E}$ is a holomorphic bundle with $c_2 = l$ and Δ is the direct sum of the structure sheaves of $k - l$ (not necessarily distinct) points. Note that, as the two-term complex of coherent sheaves, with $\mathcal{E}$ in position 0, Δ in position 1 and the trivial map between them, this does indeed have $c_2 = k$ and, by the previous remark, it is the cohomology of the three term complex defined by a degenerate, completely reducible configuration. It is a trivial observation that the Hitchin-Kobayashi correspondence extends to an equivalence between the spaces of ideal instantons and of ideal holomorphic bundles. The results of this chapter have demonstrated that these spaces are, in fact, natural completions of the instanton and holomorphic bundle moduli spaces.

6 The Effect of the Blow-Up on the Moduli Spaces

In this chapter, we relate the moduli spaces over $\mathbb{C}^2$ and $\widetilde{\mathbb{C}}^2$ by exhibiting a surjective map $\pi_\bullet : \overline{\mathbf{M}}(\widetilde{\mathbb{C}}^2; r, k) \to \overline{\mathbf{M}}(\mathbb{C}^2; r, k)$, which is a birational equivalence. We show how this map should be interpreted as taking the direct image under the projection $\pi : \widetilde{\mathbb{C}}^2 \to \mathbb{C}^2$ (or its extension $\pi : \Sigma^1 \to \mathbf{P}^2$). We observe that, in the case $k = 1$, $\pi_\bullet$ is a blow-up of $\overline{\mathbf{M}}(\mathbb{C}^2; r, 1)$ at the boundary point corresponding to $0 \in \mathbb{C}^2$ and we make a suitably generalised conjecture concerning the case for general k.

6.1 The Direct Image Map

At the level of the linear algebra data, a map $\pi_\# : \tilde{\mathcal{R}} \to \mathcal{R}$ presents itself very naturally, namely

$$\pi_\# : (a, b, c, d) \mapsto (da, db, c),$$

where we take the space W in $\mathcal{R}$ to be W_1 in $\tilde{\mathcal{R}}$. Indeed, this map has already appeared as part of the map $\rho : \tilde{\mathcal{R}} \to \mathcal{R}'$ of §4.1. We can easily see that $\pi_\#$ preserves the notion of integrability and that it is equivariant with respect to the respective $GL(W_0) \times GL(W_1)$ actions (the action on $\mathcal{R}$ being trivial for the $GL(W_0)$ factor). Due to the results of Theorems 5.2.7 and 5.3.7, these two conditions are sufficient to ensure that $\pi_\#$ induces a map

$$\pi_\bullet : \overline{\mathbf{M}}(\widetilde{\mathbb{C}}^2; r, k) \to \overline{\mathbf{M}}(\mathbb{C}^2; r, k).$$

Indeed, these conditions would have to suffice, since $\pi_\#$ does not preserve the moment map equations nor the notion of complete reducibility.

To see that $\pi_\bullet$ should be interpreted as a direct image, we have to look at the relationship between the respective monads. Let us write

$$\widetilde{M}(a, b, c, d) : \widetilde{\mathcal{U}} \xrightarrow{\widetilde{\mathcal{A}}} \widetilde{\mathcal{V}} \xrightarrow{\widetilde{\mathcal{B}}} \widetilde{\mathcal{W}}$$
$$M(a, b, c) : \mathcal{U} \xrightarrow{\mathcal{A}} \mathcal{V} \xrightarrow{\mathcal{B}} \mathcal{W}$$

for the monads over $\widetilde{\mathbb{C}}^2$ and $\mathbb{C}^2$ (or Σ^1 and $\mathbb{C}\mathbf{P}^2$) in the canonical forms ($C1$) (of §3.4) and (5.2.4). For the purposes of the subsequent discussion, we shall include the cases where the linear data is degenerate and shall call the resulting three term complexes *degenerate monads.* Thus

$$\widetilde{\mathcal{U}} = W_1 \otimes \mathcal{O}_{\Sigma^1}(-1, 0) \oplus W_0 \otimes \mathcal{O}_{\Sigma^1}(0, -1)$$
$$\widetilde{\mathcal{V}} = (W_0 \oplus W_1 \oplus W_0 \oplus W_1 \oplus \mathbb{C}^r) \otimes \mathcal{O}_{\Sigma^1}$$
$$\widetilde{\mathcal{W}} = W_0 \otimes \mathcal{O}_{\Sigma^1}(1, 0) \oplus W_1 \otimes \mathcal{O}_{\Sigma^1}(0, 1)$$

$$\widetilde{\mathcal{A}} = \begin{pmatrix} a_1 x_3 & -y_2 \\ x_1 - da_1 x_3 & 0 \\ a_2 x_3 & y_1 \\ x_2 - da_2 x_3 & 0 \\ c\, x_3 & 0 \end{pmatrix} \quad \text{and} \quad \widetilde{\mathcal{B}} = \begin{pmatrix} x_2 & a_2 x_3 & -x_1 & -a_1 x_3 & b\, x_3 \\ d\, y_1 & y_1 & d\, y_2 & y_2 & 0 \end{pmatrix},$$

while

$$\mathcal{U} = W_1 \otimes \mathcal{O}_{\mathbf{P}^2}(-1)$$
$$\mathcal{V} = \left(W_1 \oplus W_1 \oplus \mathbf{C}^r\right) \otimes \mathcal{O}_{\mathbf{P}^2}$$
$$\mathcal{W} = W_1 \otimes \mathcal{O}_{\mathbf{P}^2}(1)$$

$$\mathcal{A} = \begin{pmatrix} x_1 - a_1\, x_3 \\ x_2 - a_2\, x_3 \\ c\, x_3 \end{pmatrix} \quad \text{and} \quad \mathcal{B} = \begin{pmatrix} -(x_2 - a_2\, x_3) & (x_1 - a_1\, x_3) & b\, x_3 \end{pmatrix}.$$

Given this notation, we can write down a monad map

$$\Theta : \widetilde{M}(a,b,c,d) \longrightarrow \pi^* M(da, db, c)$$

given by

$$\begin{array}{ccccc} \widetilde{\mathcal{U}} & \xrightarrow{\widetilde{\mathcal{A}}} & \widetilde{\mathcal{V}} & \xrightarrow{\widetilde{\mathcal{B}}} & \widetilde{\mathcal{W}} \\ \big\downarrow \phi & & \big\downarrow \psi & & \big\downarrow \chi \\ \pi^*\mathcal{U} & \xrightarrow{\pi^*\mathcal{A}} & \pi^*\mathcal{V} & \xrightarrow{\pi^*\mathcal{B}} & \pi^*\mathcal{W} \end{array}$$

where, noting that $\pi^*\mathcal{O}_{\mathbf{P}^2}(1) = \mathcal{O}_{\Sigma^1}(1,0)$, we take

$$\phi = \begin{pmatrix} 1 & 0 \end{pmatrix} \quad \psi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \chi = \begin{pmatrix} d \otimes 1 & 1 \otimes (-s) \end{pmatrix}.$$

Now the map Θ has the following important property.

PROPOSITION 6.1.1. *Fix $p \in \Sigma^1$ and denote $\pi(p)$ by q. Suppose that $\widetilde{M}_p(a,b,c,d)$ is non-degenerate and that either $s_p \neq 0$ or d is an isomorphism. Then $M_q(da,db,c)$ is non-degenerate and Θ_p induces an isomorphism between the respective cohomology spaces.*

PROOF. Firstly, considering the forms of $\mathcal{A}_q$ and $\widetilde{\mathcal{A}}_p$, we see that the condition that $\widetilde{\mathcal{A}}_p(a,b,c,d)$ be injective includes the condition that $\mathcal{A}_q(da,db,c)$ be so. Secondly, if $\widetilde{\mathcal{B}}_p$ and χ_p are both surjective, then $\mathcal{B}_q$ must also be. But χ_p is surjective if either $s_p \neq 0$ or d is an isomorphism. Finally, observe that $\ker \widetilde{\mathcal{B}}_p \cap \ker \psi_p$ is identified with

$$\left\{(u,v) \in W_0 \times W_0 | s(y_1 u + y_2 v) = d(y_1 u + y_2 v) = 0\right\}$$

If either $s_p \neq 0$ or d is an isomorphism, then this is just $\left\{(u,v) | y_1 u + y_2 v = 0\right\}$ and, since $(y_1, y_2) \neq 0$, this space has dimension k. However, this contains $\operatorname{im} \widetilde{\mathcal{A}}_p \cap \ker \psi_p$, which also has dimension k, so these two spaces are the same. Thus, the map $\left(\psi_p\right)_* : (\ker \widetilde{\mathcal{B}} / \operatorname{im} \widetilde{\mathcal{A}}) \longrightarrow (\ker \mathcal{B} / \operatorname{im} \mathcal{A})$ is injective and hence an isomorphism, since the dimensions agree. □

This proposition has two immediate consequences. Firstly, suppose that the bundle $\widetilde{\mathcal{E}}$ over Σ^1 is the cohomology of the non-degenerate monad $\widetilde{M}(a,b,c,d)$ and that d is an isomorphism. As shown in §3.6, this means that $\widetilde{\mathcal{E}}$ is trivial on the exceptional line E in Σ^1, but we can now see more, namely that $\widetilde{\mathcal{E}} \cong \pi^*\mathcal{E}$, where $\mathcal{E}$ is the vector bundle over $\mathbf{C}P^2$ defined by the monad $M(da,db,c)$. Hence, $\pi_*(\widetilde{\mathcal{E}}) = \mathcal{E}$ and, thus, on the open subset of $\mathsf{M}(\widetilde{\mathbf{C}}^2; r, k)$ given by $\det d \neq 0$, π_* does correspond to taking the direct image. Secondly, even if d is not an isomorphism, on the set on which $s \neq 0$ (i.e. on $\Sigma^1 \setminus E$) we still have $\tilde{\mathcal{E}} \cong \pi^*\mathcal{E}$. Of course, here π is actually a bijection, so we are really just saying that $\tilde{\mathcal{E}} \cong \mathcal{E}$ away from where the blowing-up occurs.

Observe further that, if we have an ideal holomorphic bundle/instanton $\widehat{A}_k = (A_l; p_1, \ldots, p_{k-l})$, then, by considering $\widehat{A}_k$ as given by linear algebra data in the comletely reduced form (5.3.6), we see that $\pi_*(\widehat{A}_k) = \left(\pi_*(A_l); \pi(p_1), \ldots, \pi(p_{k-l})\right)$. From the previous observation, $\pi_*(A_l)$ is an ideal bundle, which can only be singular at $0 \in \mathbb{C}^2$, and whose true part is isomorphic to A_l away from the blowing-up. At the level of ideal bundles, this is as much as one could hope to mean by saying that π_* is the direct image map.

6.2 The Relationship between the Moduli Spaces

We now look in more detail at the map $\pi_* : \overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, k) \to \overline{\mathsf{M}}(\mathbb{C}^2; r, k)$ induced by $(a, b, c, d) \mapsto (da, db, c)$ on the linear algebra data. The following facts are now easy to see

PROPOSITION 6.2.1.

(i) π_* *is a bijection between the open sets* $\tilde{U} \subseteq \overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, k)$*, determined by the condition "d is an isomorphism", and* $U \subseteq \overline{\mathsf{M}}(\mathbb{C}^2; r, k)$*, determined by the condition "a_1, a_2 and b are jointly surjective".*

(ii) π_* *maps the complement of* $\tilde{U}$ *onto the complement of* U*.*

(iii) For $\widehat{A}_k \in \overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, k)$*, the multiplicity of the singularity of* $\pi_*(\widehat{A}_k)$ *at* $0 \in \mathbb{C}^2$ *is at least the corank of* d*.*

PROOF.

(i) Restricted to the open subset of $\tilde{\mathcal{R}}$ on which d is an isomorphism, $\pi_\# : \tilde{\mathcal{R}} \to \mathcal{R}$ is onto and precisely realises the quotient by $GL(W_0)$. In other words, d gives an identification $W_0 \cong W_1$ and thus reduces the linear algebra data to precisely that in $\mathcal{R}$. Under this quotient, there is an exact correspondence between the $GL(W_1)$ actions, the integrability conditions and, especially, the notions of complete reducibility. Hence, we must simply impose the effectivity condition on $\mathcal{R}$ to see that $\tilde{U}$ and U are really identical, with the natural identification given by π_*.

(ii) If d is not an isomorphism, then clearly da_1, da_2 and db cannot be jointly surjective. To see further that π_* is onto, we observe that an ideal bundle $(A_l; q_1, \ldots, q_{k-l})$ over $\mathbb{C}^2$ is equal to $\pi_*\left(\pi^*(A_l); p_1, \ldots, p_{k-l}\right)$ over $\widetilde{\mathbb{C}}^2$, provided that $\pi(p_i) = q_i$, which can easily be arranged.

(iii) If $\widehat{A}_k \in \overline{\mathsf{M}}(\widetilde{\mathbb{C}}^2; r, k)$ is determined by the linear data (a, b, c, d) then $\operatorname{im} d$ will be a special subspace for (da, db, c) and, hence, for a completely reducible configuration (a', b', c') representing $\pi_*(\widehat{A}_k)$, there will then be a complement of $\operatorname{im} d$ on which a', b' and c' all vanish, providing at least as many singular points at $0 \in \mathbb{C}^2$ as the codimension of $\operatorname{im} d$. □

REMARK 6.2.2. The "at least" in part (iii) cannot be strengthened to "exactly", as is shown by the following completely reducible configuration for $\overline{\mathsf{M}}(\tilde{\mathbb{C}}^2; 2, 3)$:

$$a_1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad a_2 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \quad d = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$b = \begin{pmatrix} 0 & 0 \\ -1 & 0 \\ 0 & 1 \end{pmatrix} \quad c = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The completely reducible configuration equivalent to (da, db, c) is $(0,0,0)$, so that the multiplicity of the singularity at 0 is 3, while the corank of d is 2. At the level of sheaves, this means that, to determine the singularity of the direct image of a bundle down a blow-up, it is not sufficient to know the restriction of the bundle to the exceptional line. Friedman & Morgan **[FM;Rem 5.4]** have observed that, in the case of rank 2 bundles, when the splitting type $\mathcal{E}|_E = \mathcal{O}(-n) \oplus \mathcal{O}(n)$ is determined by $n = \operatorname{corank} d$, the multiplicity of the singularity at 0 lies between n and n^2.

Proposition 6.2.1 shows that $\overline{\mathsf{M}}(\tilde{\mathbb{C}}^2; r, k)$ and $\overline{\mathsf{M}}(\mathbb{C}^2; r, k)$ are birationally equivalent and that the subset of $\overline{\mathsf{M}}(\mathbb{C}^2; r, k)$, on which this equivalence fails to be a bijection, consists precisely of those ideal instantons which have at least one singular point at $0 \in \mathbb{C}^2$. This set is just a copy of $\overline{\mathsf{M}}(\mathbb{C}^2; r, k-1)$ embedded by adding a singularity at 0. Denote this subset by $\overline{\mathsf{M}}(\mathbb{C}^2; r, k-1; 0)$. Then we actually believe that a much more precise result is true.

CONJECTURE. The map $\pi_\bullet : \overline{\mathsf{M}}(\tilde{\mathbb{C}}^2; r, k) \to \overline{\mathsf{M}}(\mathbb{C}^2; r, k)$ is a blow-up of the latter space along the subspace $\overline{\mathsf{M}}(\mathbb{C}^2; r, k-1; 0)$.

This conjecture can certainly be verified in the case $k = 1$, when $\overline{\mathsf{M}}(\mathbb{C}^2; r, 0; 0)$ is just the boundary point 0.

PROPOSITION 6.2.3. *The map* $\pi_\bullet : \overline{\mathsf{M}}(\tilde{\mathbb{C}}^2; r, 1) \to \overline{\mathsf{M}}(\mathbb{C}^2; r, 1)$ *is a blow-up of the latter space at the point 0 in the boundary* $\mathbb{C}^2$.

PROOF. We need to show that the link of 0 in $\overline{\mathsf{M}}(\mathbb{C}^2; r, 1)$ is the same as $\pi_\bullet^{-1}(0)$. Now, from §5.1, this set is simply the quotient of

$$\{(a, b, 0, 0, c) \in \mathcal{R}' \mid bc = 0,\ (a, b) \neq (0, 0)\}$$

by the $\mathbb{C}^* \times \mathbb{C}^*$-action $a \mapsto \lambda_0\lambda_1 a$, $b \mapsto \lambda_0 b$ and $c \mapsto \lambda_1 c$. If we apply the transformation $(a, b, c) \mapsto (a, cb)$, then we see that we have described precisely the $\mathbb{C}^*$ quotient of the space $(\mathbb{C}^2 \times \overline{\mathcal{N}}) \setminus (0, 0)$, which is precisley the required link, since $\overline{\mathsf{M}}(\mathbb{C}^2; r, 1) = \mathbb{C}^2 \times \overline{\mathcal{N}}$. □

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