Exercise Sheet 4— Diffusion processes

4.1: Let $\mathbf{W}$ be d-dimensional Brownian motion, starting at $\mathbf{a} \in \mathbb{R}^d$, and write $B_R := \{\mathbf{x} \in \mathbb{R}^d : |\mathbf{x}| < R\}$ for the open ball with radius R. By applying the Dynkin formula to W, with $f(x) = |x|^2$, find $\mathbb{E}[\tau_R]$, where $\tau_R := \inf\{t \geq 0 : \mathbf{W}_t \notin B_R\}$.

[Hint: Apply Dynkin's formula to $\tau_R \wedge t$, and use monotone and bounded convergence theorems.]

4.2: Let $A(a, b) = \{\mathbf{x} \in \mathbb{R}^d : a < |\mathbf{x}| < b\}$ be the annulus in $\mathbb{R}^d$, for $0 \leq a < b < \infty$ and $d \geq 2$. Write $\tau_{a,b}$ for the exit time from $A(a, b)$.

By considering the function

$$f(\mathbf{x}) = \begin{cases} -\log|\mathbf{x}| & d = 2 \\ |\mathbf{x}|^{2-d} & d > 2 \end{cases}$$

find $\mathbb{E}^{\mathbf{x}}\left[f(\mathbf{W}_{\tau_{a,b}})\right]$, for $\mathbf{x} \in A(a, b)$.

Hence find $\mathbb{P}^{\mathbf{x}}(|\mathbf{W}_{\tau_{a,b}}| = a)$. What happens to this probability as $b \to \infty$, for fixed a? What happens as $a \to 0$ for fixed b? Show that

$$\mathbb{P}^{\mathbf{x}}(\mathbf{W}_t = \mathbf{0}, \text{ some } t \geq 0) = 0$$

for $\mathbf{x} \neq \mathbf{0}$.

4.3: (a) Let X_t be a time-homogenous Itô diffusion in one dimension, so

$$\mathrm{d}X_t = \mu(X_t)\,\mathrm{d}t + \sigma(X_t)\,\mathrm{d}W_t, \qquad X_0 = x.$$

Suppose f is a $C^2(\mathbb{R})$ function such that $\mathcal{A}f = 0$. Let $a < x < b$ and set $H_{a,b} := \inf\{t \geq 0 : X_t \in \{a, b\}\}$. We assume $H_{a,b} < \infty$, almost surely. Show that

$$\mathbb{P}^x(X_{H_{a,b}} = b) = \frac{f(x) - f(a)}{f(b) - f(a)}.$$

(b) Hence find the exit distribution from the interval (a, b) for a Brownian motion started at x.

(c) Find the exit distribution from an interval when $X_t = x + \mu t + \sigma W_t$, for $\mu, \sigma \in \mathbb{R}$.

4.4: Let W be one-dimensional Brownian motion. Let $0 \leq a \leq b \leq 1$, and suppose $x_0 \in (0, 1)$. By considering the function

$$f(x) = \begin{cases} 0 & x \leq a \\ (x - a)^2 & x \in (a, b] \\ (b - a)^2 + 2(x - b)(b - a) & x \in (b, 1] \end{cases},$$

find the expected amount of time the Brownian motion started at x_0 spends in the interval $(a, b]$ up to $H_{0,1} := \inf\{t \geq 0 : W_t \in \{0, 1\}\}$.

[Hint: Recall from Q4.3 that the probability that Brownian motion started at x first exits the interval (a, b) at b is $\frac{x-a}{b-a}$. You may also assume in this question that Dynkin's formula extends to functions whose second derivative is only piecewise continuous.]

4.5: The *Feynman-Kac formula* says that, if $f \in C_0^2(\mathbb{R}^n)$, $q \in C_b(\mathbb{R}^n)$ (the set of continuous and bounded functions), and if X_t is an Itô diffusion with generator $\mathcal{A}$, then

$$v(t,x) = \mathbb{E}^x \left[\exp\left(-\int_0^t q(X_s)\,\mathrm{d}s \right) f(X_t) \right]$$

solves

$$\begin{cases} \frac{\partial v}{\partial t} = \mathcal{A}v - qv & t > 0, x \in \mathbb{R}^n \\ v(0,x) = f(x) & x \in \mathbb{R}^n \end{cases}. \tag{1}$$

Prove this, and show that if $w \in C_b^{1,2}$ is another solution to (1), then $v = w$.

4.6: Let $\mathbf{W}_t$ be a d-dimensional Brownian motion, and suppose D is a bounded open set in $\mathbb{R}^d$. Suppose $h > 0$ satisfies $\Delta h = 0$ in D. Let X_t solve the SDE

$$\mathrm{d}X_t = \nabla(\ln h)(X_t)\,\mathrm{d}t + \mathrm{d}W_t.$$

(a) Show that the generator $\mathcal{A}$ of X_t satisfies:

$$\mathcal{A}f(x) = \frac{\Delta(hf)}{2h},$$

and hence that if $f = \frac{1}{h}$, $\mathcal{A}f = 0$.

(b) Using this, show that if there exists $x_0 \in \partial D$ such that

$$\lim_{x \to y \in \partial D} h(x) = \begin{cases} 0 & y \neq x_0 \\ \infty & y = x_0 \end{cases}.$$

then $\lim_{t \to \tau_D} X_t = x_0$.
[Hint: Consider $\mathbb{E}^x[f(X_\rho)]$ for suitable stopping times ρ, with $f = h^{-1}$.]

(c) Consider two-dimensional Brownian motion on the unit disc, $D = \{|x| \in \mathbb{R}^2 : |x| < 1\}$. By considering the function

$$h(x^1, x^2) = \frac{1 - (x^1)^2 - (x^2)^2}{1 + (x^1)^2 + (x^2)^2 - 2x^2},$$

find a diffusion on the unit disk which guaranteed to exit at $(0,1)$.