

# ADJOINT ACTION OF AUTOMORPHISM GROUPS OF PROJECTIVE REPRESENTATIONS OF DYNKIN QUIVERS

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ABSTRACT. Let  $\Delta$  be a quiver and let  $P$  be a projective representation of  $\Delta$ . We study the adjoint action of  $AutP$  on the space of radical endomorphisms and show that there is a dense open orbit for all  $P$ , if and only if  $\Delta$  is a Dynkin quiver. Our main application is to show the existence of dense open orbits for the adjoint action of subgroups of parabolics in  $Gl_n$  on the nilpotent radical of corresponding subalgebras.

## INTRODUCTION

Let  $\Delta$  be a quiver and let  $P$  be a projective representation. We study generic orbits for the adjoint action of  $AutP$  on the radical endomorphisms  $radEndP$ . If  $\Delta$  is of type  $\mathbb{A}$  with linear orientation, then  $EndP$  is a parabolic subalgebra in  $\mathfrak{gl}_n$  and a dense open orbit exists by a theorem of Richardson [12]. If  $\Delta$  is of type  $\mathbb{A}$  with arbitrary orientation, then  $EndP$  is a seaweed Lie algebra, and a dense open orbit exists by a theorem of Jensen-Su-Yu [5]. The following is the main result of this paper.

**Theorem 1.** *Let  $\Delta$  be a quiver, then there is a dense open  $AutP$ -orbit in  $radEndP$  for all projective representations  $P$ , if and only if  $\Delta$  is a Dynkin quiver.*

The study of  $AutP$ -orbits in  $radEndP$  can be transferred to the study of good representations of the double quiver  $\tilde{\Delta}$  of  $\Delta$  modulo some relations  $J$ , see [3, 9]. We investigate relative sections (see Section 1.2) of representation varieties of Dynkin quivers and their double quivers. The main technique of proving Theorem 1 is to show that varieties of good representations of the double quiver  $(\tilde{\Delta}, J)$  are generically equivalent to representation varieties of  $\Delta'$ , where  $\Delta'$  is a Dynkin quiver with the same underlying graph as  $\Delta$ . That is, generic good representations of  $(\tilde{\Delta}, J)$  have the same parameter space as representations of  $\Delta'$ , which are well understood, although in general  $(\tilde{\Delta}, J)$  is of wild representation type. Since  $\Delta'$  is of finite representation type we can thus conclude the existence of dense open orbits in the varieties of good representations.

Our main application of Theorem 1 is to study the generic orbits for the adjoint action of subgroups of parabolics in  $\mathfrak{gl}_t$  on nilpotent radicals of the corresponding Lie subalgebra. Let  $\mathfrak{l} \subseteq \mathfrak{b} \subseteq \mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ , where  $\mathfrak{b}$  is a Borel subalgebra, and  $\mathfrak{l}$  is a Cartan subalgebra of  $\mathfrak{g}$ . Let  $\mathfrak{n} \subseteq \mathfrak{b}$  be a nilpotent subalgebra such that  $\mathfrak{s} = \mathfrak{l} + \mathfrak{n} \subseteq \mathfrak{b}$  is a subalgebra. To each vector  $d \in \mathbb{N}^n$  we associate Lie algebras  $\mathfrak{l}(d) \subseteq \mathfrak{s}(d) \subseteq \mathfrak{b}(d) \subseteq \mathfrak{gl}_t$ , for  $t = \sum_i d_i$ , where  $\mathfrak{b}(d)$  is a parabolic Lie subalgebra with Levi factors  $\mathfrak{l}(d)$  and  $\mathfrak{s}(d) = \mathfrak{l}(d) + \mathfrak{n}(d) \subseteq \mathfrak{b}(d)$  for the nilpotent subalgebra  $\mathfrak{n}(d) \subseteq rad\mathfrak{b}(d)$ .

If  $\mathfrak{s} = \mathfrak{b}$ , then  $\mathfrak{s}(d) = \mathfrak{b}(d)$  is a parabolic subalgebra, and it is known by a theorem of Richardson [12] that there is an open dense  $S(d)$ -orbit in  $\mathfrak{n}(d)$ . The theorem of Richardson holds for all reductive Lie algebras, but the proof is not constructive and some effort has been made to explicitly construct elements with dense orbits. This has been completed in the classical types by Baur [1], and in type  $\mathbb{A}$ , Brüstle, Hille, Ringel and Röhle [4] constructed open orbits using representations of quivers. These results and methods have been extended in various directions, see for example [2, 7, 8, 13].

We associate to  $\mathfrak{s}$  a certain quiver  $\Delta(\mathfrak{s})$  with relations  $\mathcal{I}(\mathfrak{s})$  and realize  $\mathfrak{s}(d)$  as the endomorphism ring of a projective  $(\Delta(\mathfrak{s}), \mathcal{I}(\mathfrak{s}))$ -representation  $P(d) = \bigoplus_i P(i)^{d_i}$ , where  $P(i)$  is the indecomposable projective representation associated to the vertex  $i$ . The following is

the main application of Theorem 1. Let  $S(d)$  be the Lie group corresponding to  $\mathfrak{s}(d)$ . We prove the following theorem.

**Theorem 2.** *If  $\Delta(\mathfrak{s})$  is a tree, then there is an open dense  $S(d)$  orbit in the nilpotent radical  $\mathfrak{n}(d)$  for all  $d$ , if and only if  $\Delta(\mathfrak{s})$  is a Dynkin quiver.*

The remainder of this paper is organized as follows. In Section 1 we recall basic facts on quivers and their representations. In Section 2 we give a definition of the class of Lie algebras  $\mathfrak{s}(d)$  using root spaces in  $\mathfrak{gl}_t$ , and in Section 3 we show how these algebras can be realized as endomorphism algebras of projective representations of a quiver with relations. In Section 4 we recall results of Brüstle-Hille [3] and Hille-Vossieck [9] on the use of double quivers to parameterize  $AutP$ -orbits of radical endomorphisms of projective representations  $P$ . In Section 5 we recall a result of Voigt and prove a fundamental inequality on the dimensions of stabilizers of radical endomorphisms. In Section 6 we recall the construction in type  $\mathbb{A}$ , and simplify the proofs using the fundamental inequality. Using the construction in type  $\mathbb{A}$  we prove our main technical results in Section 7 and prove Theorems 1 and 2 in Section 8.

## 1. REPRESENTATION VARIETIES AND ALGEBRAIC GROUP ACTIONS

**1.1. Representations of quivers.** A quiver  $\Delta$  consists of a finite set of vertices  $\Delta_0$  and a finite set of arrows  $\Delta_1$  and two functions  $s, t : \Delta_1 \rightarrow \Delta_0$  sending an arrow to its starting and terminating vertex, respectively. A vertex  $i$  in  $\Delta$  is called a sink if there are no arrows terminating at  $i$ , and a source if there are no arrows starting at  $i$ . It is called admissible if it is either a sink or a source, and interior if there are at least two arrow incident to  $i$ .

A representation  $V$  of  $\Delta$  consists of vector spaces  $\{V_i\}_{i \in \Delta_0}$  and linear maps  $\{f_\alpha : V_{s(\alpha)} \rightarrow V_{t(\alpha)}\}_{\alpha \in \Delta_1}$ . A homomorphism of representations  $h : (V, f) \rightarrow (W, g)$  is a collection of maps  $h_i : V_i \rightarrow W_i$  satisfying  $h_j f_\alpha = g_\alpha h_i$  for each arrow  $\alpha : i \rightarrow j \in \Delta_1$ . The direct sum of two representation is obtained by taking direct sum of vector spaces and linear maps. A representation is indecomposable if it is not isomorphic to the direct sum of two non-zero representations.

For a representation  $V$  we let  $dimV = (dimV_i)_{i \in \Delta_0}$  denote the dimension vector of  $V$ . Let

$$Rep(\Delta, d) = \prod_{\alpha \in \Delta_1} Hom(k^{d_{s(\alpha)}}, k^{d_{t(\alpha)}})$$

be the space of representations. The group  $Gl(d) = \prod_i Gl_{d_i}$  acts on  $Rep(\Delta, d)$  by change of basis, and we have a bijection between  $Gl(d)$ -orbits in  $Rep(\Delta, d)$  and isomorphism classes of representations of  $\Delta$  with dimension vector  $d$ . We fix a basis and view elements in  $Rep(\Delta, d)$  and  $Gl(d)$  as tuples of matrices.

The path algebra  $k\Delta$  of  $\Delta$  is the algebra with basis the set of paths in  $\Delta$ . For two paths  $p$  and  $q$ , their product is defined to be the composition  $pq$  if  $q$  ends where  $p$  starts, and zero otherwise. Let  $e_i$  denote the trivial path of length zero at vertex  $i$ . The trivial paths form a complete set of orthogonal idempotents for  $k\Delta$ . There is an equivalence of categories between left  $k\Delta$ -modules and representations of  $\Delta$ .

Let  $\langle, \rangle : \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{Z}$  be the Ringel form defined by

$$\langle d, e \rangle = \sum_{i \in \Delta_0} d_i e_i - \sum_{\alpha \in \Delta_1} d_{s(\alpha)} e_{t(\alpha)}$$

Let  $q_\Delta$  defined by  $q_\Delta(d) = \langle d, d \rangle$  be the corresponding quadratic form, called the Tits form of  $\Delta$ .

If  $\Delta$  is a Dynkin quiver, that is the underlying graph of  $\Delta$  is one of the Dynkin graphs  $\mathbb{A}_i, \mathbb{D}_j, \mathbb{E}_l$  for  $i \geq 1, j \geq 4, l = 6, 7, 8$ , it follows from Gabriel's Theorem [6] that there is a dense orbit in  $Rep(\Delta, d)$  for any dimension vector  $d$ .

**Theorem 3.** *[Gabriel] There is a dense open orbit in  $Rep(\Delta, d)$  for all  $d$ , if and only if  $\Delta$  is a Dynkin quiver. Moreover, if  $\Delta$  is Dynkin, then the number of orbits in  $Rep(\Delta, d)$  is finite.*

We also consider representations satisfying relations. Let  $\mathcal{I} \subseteq \Delta_1 \supseteq k\Delta$  be an ideal contained in the ideal generated by the arrows in  $\Delta$ . The corresponding subset  $Rep(k\Delta/\mathcal{I}, d) = Rep(\Delta, \mathcal{I}, d) \subseteq Rep(\Delta, d)$  consisting of representations that are annihilated by  $\mathcal{I}$  is a  $Gl(d)$ -stable Zariski closed subvariety, which is called the representation variety of  $(\Delta, \mathcal{I})$  with dimension vector  $d$ .

**1.2. Relative sections.** Let  $(G, V)$  consist of an algebraic group  $G$  which acts regularly on an irreducible variety  $V$ .

**Definition 1.** A pair  $(H, W)$  is called a (relative) section of  $(G, V)$  if  $W \subseteq V$ ,  $H \subseteq G$  is contained in the set-wise stabilizer of  $W$ , and  $H \cdot w = (G \cdot w) \cap W$ . We say that  $(H, W)$  is a generic section if in addition  $G \cdot W$  contains an open subset of  $V$ .

If  $(H', W')$  is equivariantly isomorphic to  $(H, W)$  and  $(H, W)$  is a (generic) section of  $(G, V)$ , then we also say that  $(H', W')$  is a (generic) section of  $(G, V)$ . If  $(G, V)$  and  $(G', V')$  have a common generic section, then we say they are generically equivalent.

Note that  $(1, x)$  is a section in  $(G, V)$  for any point  $x \in V$ , where 1 denotes the trivial group with one element, and that  $(1, x)$  is a generic section in  $(G, V)$  if and only if the orbit  $G \cdot v \subseteq V$  is open. By the theorem of Gabriel we see that there exists  $x \in Rep(Q, d)$  such that  $(1, x)$  is a generic section in  $(Gl(d), Rep(Q, d))$  for any dimension vector  $d$ , if and only if  $Q$  is Dynkin. In general, for any quiver and dimension vector  $d$ , the canonical decomposition defined by Kac [11],  $d = \sum_i d_i$  gives us a generic section  $(\prod_i Gl_{d_i}, \prod Rep(Q, d_i))$  of  $(Gl(d), Rep(Q, d))$ .

## 2. A CLASS OF LIE SUBALGEBRAS

Let  $\mathfrak{g} = \mathfrak{gl}_n(\mathbb{C})$ , let  $\mathfrak{l} \subseteq \mathfrak{b}$  be the Cartan subalgebra of diagonal matrices, let  $\mathfrak{b} \subseteq \mathfrak{g}$  be the Borel subalgebra of upper triangular matrices. Let  $\mathfrak{m} \subset \mathfrak{b}$  be the nilpotent radical of  $\mathfrak{b}$ , consisting of strictly upper triangular matrices. Let  $\mathfrak{n} \subseteq \mathfrak{b}$  be a nilpotent subalgebra such that  $\mathfrak{s} = \mathfrak{l} \oplus \mathfrak{n} \subseteq \mathfrak{b}$  is a subalgebra. Note that  $\mathfrak{n} = \mathfrak{s} \cap \mathfrak{m}$  is the nilpotent radical of  $\mathfrak{s}$ .

Let  $\Phi = \Phi^+ \cup \Phi^-$  be the root system determined by  $\mathfrak{l}$  such that  $\mathfrak{m} = \bigoplus_{\alpha \in \Phi^+} \mathfrak{g}_\alpha$ , where  $\mathfrak{g}_\alpha$  is the root space of  $\alpha$  in  $\mathfrak{g}$ . A subset  $\Sigma \subseteq \Phi$  is closed under addition if  $\alpha, \beta \in \Sigma$  and  $\alpha + \beta \in \Phi$  imply that  $\alpha + \beta \in \Sigma$ . A subset  $\Sigma$  closed under addition is generated by a subset  $\Sigma' \subseteq \Sigma$  if  $\Sigma$  is the smallest subset of roots closed under addition and containing  $\Sigma'$ . That is, the roots in  $\Sigma$  are given as  $\mathbb{N}$ -linear positive linear combinations of roots from  $\Sigma'$ . For a subset  $\Sigma \subseteq \Phi$  which is closed under addition, we let

$$\mathfrak{g}_\Sigma = \bigoplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha \subseteq \mathfrak{gl}_n.$$

The following fact is not difficult to verify and the proof is skipped.

**Lemma 1.** *There is a bijection between subsets  $\Sigma \subseteq \Phi^+$  closed under addition, and nilpotent subalgebras  $\mathfrak{n} \subseteq \mathfrak{b}$  such that  $\mathfrak{l} \oplus \mathfrak{n} \subseteq \mathfrak{b}$  is a Lie subalgebra, given by  $\Sigma \mapsto \mathfrak{g}_\Sigma$ .*

Let  $\Sigma \subseteq \Phi^+$  be a subset closed under addition and let  $\mathfrak{s} = \mathfrak{l} \oplus \mathfrak{g}_\Sigma$ . Let  $d \in \mathbb{N}^n$  be a dimension vector and let  $t = d_1 + \dots + d_n$ . We associate to  $d$  the Lie subalgebras  $\mathfrak{l}(d) \subseteq \mathfrak{s}(d) \subseteq \mathfrak{b}(d) \subseteq \mathfrak{gl}_t$  as follows. Let  $\Phi_t$  denote the root system of  $gl_t$ . Let  $\alpha_1, \dots, \alpha_{n-1}$  denote the simple roots in  $\Phi^+$ , ordered such that any root  $\alpha \in \Phi^+$  is of the form  $\alpha = \alpha_h + \alpha_{h+1} \dots + \alpha_j$  for  $j \geq h$ , and similarly, let  $\beta_1, \dots, \beta_{t-1}$  denote the simple roots in  $\Phi_t^+$ . Let  $\Sigma(d) = \Sigma(d)^- \cup \Sigma(d)^+ \subseteq \Phi_t$  be the set of roots closed under addition, where

- $\Sigma(d)^-$  is generated by all simple negative roots except  $\{-\beta_{i_1}, \dots, -\beta_{i_{n-1}}\}$ , where  $d_j = i_j - i_{j-1}$  and where we define  $i_0 = 0$  and  $i_n = t$ ,
- $\Sigma(d)^+$  is generated by  $-\Sigma(d)^-$  and all roots of the form  $\sum_{s=i_h}^{i_j} \beta_s$  for  $\alpha_h + \alpha_{h+1} + \dots + \alpha_j \in \Sigma$ .

We define  $\mathfrak{s}(d) = (\mathfrak{gl}_t)_{\Sigma(d)} \oplus \mathfrak{l}_t$ , where  $\mathfrak{l}_t$  are the diagonal matrices in  $\mathfrak{gl}_t$ . Here the root space of  $\alpha_h + \dots + \alpha_j$  in  $\mathfrak{s}$  correspond to a  $d_h \times d_j$  block of root spaces in  $\mathfrak{s}(d)$ .

Note that  $\mathfrak{b}(d) = (\mathfrak{gl}_t)_{\Sigma(d)-\cup\Phi(d)^+} \oplus \mathfrak{l}_t$  is a parabolic Lie subalgebra in  $\mathfrak{gl}_t$  with Levi factors  $\mathfrak{l}(d) = (\mathfrak{gl}_t)_{\Sigma(d)-\cup-\Sigma(d)^-} \oplus \mathfrak{l}_t$ . We clearly have,  $\mathfrak{l}(d) \subseteq \mathfrak{s}(d) \subseteq \mathfrak{b}(d)$ .

Finally, we remark that the description of  $\mathfrak{s}(d)$  given in this section generalizes to subalgebras of parabolic subalgebras of any reductive Lie algebra.

### 3. LIE SUBALGEBRAS AND ENDOMORPHISM RINGS OF PROJECTIVE REPRESENTATIONS

A seaweed Lie subalgebra  $\mathfrak{q} \subseteq \mathfrak{gl}_n$  is the intersection  $\mathfrak{q} = \mathfrak{p} \cap \mathfrak{p}'$  of two parabolic Lie subalgebras  $\mathfrak{p}$  and  $\mathfrak{p}'$ , where  $\mathfrak{p} + \mathfrak{p}' = \mathfrak{gl}_n$ . It is known [10] that seaweed Lie subalgebras in  $\mathfrak{gl}_n$  can be realized as endomorphism algebras of projective representations of a quiver of type  $\mathbb{A}$ , generalizing the fact that endomorphism algebras of projective representations of quivers of type  $\mathbb{A}$  with linear orientation realize parabolic subalgebras [4]. In this section we generalize these results to the class of Lie algebras  $\mathfrak{s}(d)$  defined in the previous section.

Let  $\Delta$  be a quiver without oriented cycles with vertices  $\Delta_0 = \{1, \dots, n\}$ . Let  $s(\Delta)$  be the quotient of the path algebra of  $\Delta$  by the relations  $p = q$  for all paths  $p$  and  $q$  in  $\Delta$  with the same starting and ending vertex. Note that we allow  $p$  or  $q$  to be arrows in  $\Delta$ , and that  $s(\Delta) = k\Delta$  when  $\Delta$  is a tree.

If there are no arrows in the relations defining  $s(\Delta)$ , we say that  $\Delta$  is minimal. Given  $\Delta$ , there is always a unique minimal  $\Delta' \subseteq \Delta$  such that  $s(\Delta') = s(\Delta)$ . In this case, we say that  $\Delta'$  is the quiver of  $s(\Delta)$ .

The quiver  $\Delta$  is said to have a standard orientation if whenever there is a path from  $i$  to  $j$  in  $\Delta$  then  $j \geq i$ . Any quiver without oriented cycles is isomorphic to a quiver with a standard orientation. For the remainder of this section we assume that  $\Delta$  has a standard orientation.

Let  $\mathfrak{s}(\Delta)$  be the Lie algebra  $End(s(\Delta))$ , where  $End(s(\Delta))$  denotes the endomorphisms of  $s(\Delta)$  as a left module. There is an embedding  $\mathfrak{s}(\Delta) \subseteq \mathfrak{b} \subseteq \mathfrak{gl}_n$ , given by identifying  $Hom(s(\Delta)e_j, s(\Delta)e_i) = e_j s(\Delta)e_i$  with the root space  $\mathfrak{g}_p$ , where the root  $p = \alpha_i + \dots + \alpha_j$  corresponds to the path that starts at  $i$  and ends at  $j$ . We have  $End(s(\Delta)) \cong s(\Delta)^{op}$  as associative algebras.

**Lemma 2.** *There is a bijection between quivers  $\Delta$  with standard orientation, and nilpotent subalgebras  $\mathfrak{n} \subseteq \mathfrak{b}$  such that  $\mathfrak{l} \oplus \mathfrak{n} \subseteq \mathfrak{b}$  is a subalgebra, given by  $\Delta \mapsto rad\mathfrak{s}(\Delta)$ .*

*Proof.* Consider the set  $\Psi = \{p|p \text{ a non-trivial path in } \Delta\}$ . Clearly, paths are closed under composition. We define a map  $\Psi \rightarrow \Phi^+$  given by sending a path  $p$  from  $i$  to  $j$ , to the root  $\alpha_i + \dots + \alpha_j$ . This map induces a bijection between quivers  $\Delta$  with standard orientation and subsets of positive roots closed under addition. The lemma now follows from Lemma 1.  $\square$

Note that  $\mathfrak{s}(\vec{\mathbb{A}}) = \mathfrak{b} \subseteq \mathfrak{g}$ , where  $\vec{\mathbb{A}}$  is a quiver of type  $\mathbb{A}$  with the standard orientation  $i \rightarrow i + 1$  for  $i = 1, \dots, n - 1$ , and  $\mathfrak{s}(\Delta_0) = \mathfrak{l}$ . Clearly,  $\mathfrak{l} \subseteq \mathfrak{s}(\Delta) \subseteq \mathfrak{b}$ . Also in the above bijection, nilpotent Lie ideals  $\mathfrak{n} \subseteq \mathfrak{b}$  correspond to quivers  $\Delta$  such that  $rad\mathfrak{s}(\Delta) \subseteq \mathfrak{s}(\vec{\mathbb{A}})$  is an ideal. We therefore obtain a bijection between nilpotent ideals in  $\mathfrak{b}$  and quotient algebras of  $k\vec{\mathbb{A}}$  by nilpotent ideals.

Now let  $d \in \mathbb{N}^n$  be a vector and let  $P(d) = \bigoplus_{i=1}^n P_i^{d_i}$ , where  $P_i = s(\Delta)e_i$  is the indecomposable projective  $s(\Delta)$ -module at vertex  $i$ . The embedding  $\mathfrak{s}(\Delta) \subseteq \mathfrak{gl}_n$  induces an embedding  $End(P(d)) \subseteq \mathfrak{gl}_t$  where  $t = \sum_i d_i$ , and moreover  $End(P(d)) = \mathfrak{s}(\Delta)(d)$ ,  $\mathfrak{l}(d) = \mathfrak{s}(\Delta_0)(d)$  and  $\mathfrak{b}(d) = \mathfrak{s}(\vec{\mathbb{A}})(d)$ .

### 4. ADJOINT ACTIONS AND DOUBLE QUIVERS

Let  $\Delta$  be a quiver without oriented cycles, with vertices  $\Delta_0 = \{1, \dots, n\}$ . Let  $P = P(d) = \bigoplus_{i=1}^n P_i^{d_i}$  be a projective  $\Delta$ -representation, where  $P_i = k\Delta e_i$  is the indecomposable projective  $k\Delta$ -module associated to the vertex  $i \in \Delta_0$ , and where  $d = (d_1, \dots, d_n) \in \mathbb{N}^n$  is a dimension vector. The group  $AutP$  of  $\Delta$ -automorphisms of  $P$  acts adjointly on the radical

$radEndP$  of the Lie-algebra  $EndP$ . By a result of Brüstle-Hille [3] and Hille-Vossieck [9] we may parameterize the orbits of this action using the isomorphism classes of good modules of a finite dimensional quasi-hereditary  $k$ -algebra which we denote by  $D = D(\Delta)$ .

We recall the construction of  $D$  given by Hille-Vossieck [9]. The quiver of  $D$  is the double quiver  $\tilde{\Delta}$ . That is,  $\tilde{\Delta}_0 = \Delta_0$  and  $\tilde{\Delta}_1 = \Delta_1 \cup \Delta_1^*$ , where for every arrow  $\alpha : i \rightarrow j$  in  $\Delta_0$  there is a corresponding arrow  $\alpha^* : j \rightarrow i$  in  $\Delta_1^*$ . The relations defining  $D$  are

$$\alpha^* \alpha - \sum_{\beta \in \Delta_1, t(\beta)=s(\alpha)} \beta \beta^*$$

for any arrow  $\alpha \in \Delta_1$ , and  $\alpha^* \beta$  where  $\alpha \neq \beta$  is a pair of arrows in  $\Delta$  terminating at the same vertex. By the relations we see that  $D$  has a multiplicative basis  $\{pq^* | s(p) = s(q) \text{ for paths } p, q \in \Delta\}$ .

By the results of Brüstle-Hille and Hille-Vossieck there is a natural correspondence between the  $AutP$ -orbits in  $radEndP$  and the isomorphism classes of  $D$ -modules  $X$  with  $k_\Delta X \cong P$ . Indeed, let  $e$  be the dimension vector of  $P$ , let  $Rep(D, e)$  be the variety of  $\tilde{\Delta}$ -representations satisfying the relations of  $D$  and let  $Gl(e)$  be the group acting by change of basis on  $Rep(D, e)$ . Let  $Rep(D, P) \subseteq Rep(D, e)$  consist of representations  $X$  with  $k_\Delta X = P$ . Then the stabilizer  $AutP$  acts on  $Rep(D, P)$  and the orbits correspond to isomorphism classes of  $D$ -modules  $X$  with  $k_\Delta X \cong P$ . We let  $G(d) = AutP(d)$ .

**Lemma 3.** *The following are true.*

- i)  $Rep(D, P) \subseteq Rep(D, e)$  is an affine subspace.
- ii)  $Gl(e) \cdot Rep(D, P) \subseteq Rep(D, e)$  is irreducible and open.

*Proof.*  $Rep(D, P)$  is an affine subspace since the relations defining  $D$  are linear in  $\Delta_1^*$ . This proves i) and that  $Gl(e) \cdot Rep(D, P)$  is irreducible. There is a projection map  $Rep(D, e) \rightarrow Rep(\Delta, e)$  by forgetting the  $\Delta_1^*$ -structure. Now  $Gl(e) \cdot Rep(D, P)$  is the preimage of the open orbit of  $P$ , and so ii) follows.  $\square$

Note that there is an equivariant isomorphism between  $radEndP$  and  $Rep(D, P)$ , which gives us the correspondence between  $Gl(e)$ -orbits of  $k_\Delta$ -projective representations in  $Rep(D, e)$  and  $AutP$ -orbits in  $radEndP$ .

The algebra  $D$  is quasi-hereditary with Verma modules  $P_1, \dots, P_n$ . A  $D$ -module  $X$  is good if  $k_\Delta X$  is projective, which is equivalent to that  $X$  has a filtration with subfactors isomorphic to Verma modules. If  $d_i$  is the multiplicity of  $P_i$  in  $k_\Delta X$ , then  $d = (d_i)_i$  is called the  $\Delta$ -dimension vector of  $X$ , and is denoted by  $d = dim_\Delta X$ . The full sub-quiver of  $\Delta$  given by vertices where  $d_i > 0$  is called the  $\Delta$ -support of  $X$ , and is denoted by  $Supp_\Delta X$ . The quasi-hereditary structure on  $D$  will not play an important role in the sequel.

The algebra  $D$  has a basis of paths  $ab^*$ , with  $a, b$  paths in  $\Delta$ . Let  $p$  be a path in  $\tilde{\Delta}$ . If  $p = \sum ab^*$  modulo the defining relations, then the number of arrows in  $p$  from  $\Delta_1^*$  is equal to the length of  $b$  for an  $b^*$  in the sum. This follows since the relations defining  $D$  are homogeneous in  $\Delta_1^*$ . Similarly, the relations are homogeneous in  $\Delta_1$ , and so the number of arrows in  $p$  from  $\Delta_1$  is equal to the length of  $a$  in any term of the sum.

## 5. AN EXACT SEQUENCE AND A RELATIVE VOIGT'S LEMMA

Let  $J$  be the ideal in  $D$  generated by the arrows in  $\Delta_1^*$ . Note that  $D$  is a split extension of  $k_\Delta$  by  $J$ , and that  $J$  is projective as a left  $D$ -module. In this section, let  $A = k_\Delta$ .

**Lemma 4.** *Let  $X$  be a  $k_\Delta$ -projective  $D$ -module. Then the following sequence is a projective resolution of  $X$ ,*

$$0 \longrightarrow J \otimes_A X \xrightarrow{f} D \otimes_A X \xrightarrow{g} X \longrightarrow 0$$

where  $g(d \otimes m) = dm$  and  $f(d\beta \otimes x) = d\beta \otimes x - d \otimes \beta x$  for  $d\beta \in J, \beta \in \Delta^*$ , and  $x \in X$ .

*Proof.* Since  ${}_D J$  is projective, we only need to prove that the sequence is exact. By the definition of  $f$  and  $g$ , the map  $g$  is surjective and  $gf = 0$ . For any projective  $A$ -module  $M$ , the sequence

$$0 \longrightarrow J \otimes_A M \xrightarrow{i \otimes 1} D \otimes_A M \xrightarrow{g} M \longrightarrow 0,$$

obtained by applying  $- \otimes_A M$  to

$$0 \longrightarrow J \xrightarrow{i} D \longrightarrow A \longrightarrow 0$$

is a projective resolution of  ${}_D M$ . So what remains is to show that  $f$  is injective. Let  $p$  be a path in  $\tilde{\Delta}$  and define  $\mu(p \otimes x)$  to be the number of arrows from  $\Delta_1^*$  in  $p$ . Let  $\sum pb^* \otimes x$ , for  $b \in \Delta_1$  be in the kernel of  $f$ . We may assume that  $\mu(pb^* \otimes x)$  is constant for the terms in the sum. Now  $f(\sum pb^* \otimes x) = \sum(pb^* \otimes x - p \otimes b^*x) = 0$  which shows that  $\sum pb^* \otimes x = 0$  since  $\mu(p \otimes b^*x) < \mu(pb^* \otimes x)$ . Hence  $f$  is injective, and we are done.  $\square$

**Lemma 5.** *Let  $X$  be an  $A$ -projective  $D$ -module. Then we have the following exact sequence.*

$$0 \longrightarrow \text{End}_D X \longrightarrow \text{End}_A X \longrightarrow \text{radEnd}_A X \longrightarrow \text{Ext}_D^1(X, X) \longrightarrow 0$$

*Proof.* By applying  $\text{Hom}_D(-, X)$  to the sequence in Lemma 4, we have the following exact sequence:

$$0 \longrightarrow \text{End}_D X \longrightarrow \text{Hom}_D(D \otimes X, X) \longrightarrow \text{Hom}_D(J \otimes X, X) \longrightarrow \text{Ext}_D^1(X, X) \longrightarrow 0.$$

Note that we have the isomorphism  $\phi : \text{Hom}_D(D \otimes_A X, X) \rightarrow \text{End}_A X$  given by  $\phi(f)(x) = f(1 \otimes x)$ . Moreover we can show that  $\text{Hom}_D(J \otimes_A X, X) \cong \text{radEnd}_A X$ , which completes the proof of the lemma.  $\square$

As a corollary of Lemma 5, we give a direct proof of a version of Voigt's Lemma [14] for  $D$ -modules. This lemma includes an inequality for the dimension of stabilizers, which is an important tool for showing that a given good  $D$ -module is rigid.

**Lemma 6.** *Let  $X$  be an  $A$ -projective  $D$ -module  $X$  with  ${}_A X = P(d)$ . Then  $\dim \text{End}_D X \geq \sum_i d_i^2$ . Moreover, the following are equivalent.*

- i)  $G(d) \cdot X \subseteq \text{Rep}(D, P(d))$  is open.
- ii)  $X$  is rigid.
- iii)  $\dim \text{End}_D X = \sum_i d_i^2$ .

*Proof.* We have  $\dim \text{End}_A X - \dim \text{radEnd}_A X = \sum d_i^2$ , and so  $\dim \text{End}_D X \geq \sum_i d_i^2$  by the previous lemma, with equality if and only if  $\text{Ext}_D^1(X, X) = 0$ . This shows the equivalence of ii) and iii). Note that

$$\dim G(d) \cdot X = \dim \text{End}_A X - \dim \text{End}_D X = \dim \text{Rep}(D, P(d)) - \dim \text{Ext}_D^1(X, X),$$

and so i) and ii) are equivalent.  $\square$

## 6. TYPE $\mathbb{A}$

**6.1. Linear orientation.** In this subsection, let  $\Delta$  be of type  $\mathbb{A}$  with orientation  $\alpha_i : i \rightarrow i + 1$  for each vertex  $i = 1, \dots, n - 1$ . In this case  $\text{Aut} P$  is a parabolic and the existence of dense orbits follows from the classical result of Richardson [12]. We now recall an explicit construction of dense orbits using quivers due to Brüstle, Hille, Ringel and Röhrlé [4].

The projective  $D$ -module  $Q_n = De_n$  at vertex  $n$  is injective and has a multiplicative basis consisting of paths  $pq^*$ , where  $q$  is a path in  $\Delta$  ending at the vertex  $n$ . By a multiplicative basis of a module  $M$ , we mean that  $pv$  is zero or a basis element of  $M$  for any path  $p$ , and any basis element  $v$  of  $M$ . By [4] there is a bijection between the isomorphism classes of indecomposable rigid  $k\Delta$ -projective  $D$ -modules and submodules of  $Q_n$ . A submodule  $X$  of  $Q_n$  has a multiplicative basis given by a subset of the paths  $pq^*$ , and it is uniquely determined by its  $k\Delta$ -structure  ${}_k \Delta X \cong \bigoplus_{i=1}^n P_i^{\delta_i}$  with  $\delta_i \in \{0, 1\}$ . In fact, there is a bijection between subsets  $I \subseteq \Delta_0$  and  $k\Delta$ -projective indecomposable rigid  $D$ -modules  $X = X(I)$ ,

where  $I = \{i \mid \delta_i = 1\}$ . For a dimension vector  $d = (d_1, \dots, d_n)$ , define  $X(d) = \sum_{i=1}^t X(I_i)$ , with  ${}_{k\Delta}X(d) \cong P(d)$ , and  $I_1 \subseteq I_2 \subseteq \dots \subseteq I_t$ .

**Theorem 4.** [Brüstle-Hille-Ringel-Röhrle] *The  $k\Delta$ -projective  $D$ -module  $X(d)$  is rigid.*

The theorem gives a bijection between isomorphism classes of rigid  $k\Delta$ -projective  $D$ -modules and descending chains of subsets of  $\{1, \dots, n\}$ .

We will give a short proof of the above theorem, but first we need the following lemma on the dimension of homomorphism spaces between indecomposable rigid  $k\Delta$ -projective  $D$ -modules.

**Lemma 7.** *Let  $X(I)$  and  $X(J)$  be two indecomposable rigid  $k\Delta$ -projective  $D$ -modules, associated to the subsets  $I \subseteq J \subseteq \Delta_0$ . Then*

$$\dim_k \text{Hom}_D(X(I), X(J)) = \dim_k \text{Hom}_D(X(J), X(I)) = |I|.$$

*Proof.* The projective  $Q_n$  is generated by  $e_n$ . We have  $\text{End}_D(Q_n)$  is  $n$ -dimensional with basis  $f_i$ , where  $i = 0, \dots, n-1$ , defined by  $f_i(e_n) = (\alpha_{n-1} \alpha_{n-1}^*)^i$ . Assume  ${}_{k\Delta}X(I) = \bigoplus_i P_i^{\delta_i}$ . Suppose that  $I = J$ . Then any  $f_i : Q_n \rightarrow Q_n$  induces  $f_i|_{X(I)} : X(I) \rightarrow X(I)$ . Moreover  $f_i|_{X(I)} = 0$  if and only if  $i \geq \sum \delta_i$ . Therefore  $\dim_k \text{Hom}(X(I), X(I)) = \sum_i \delta_i = |I|$ . Note that  $X(I)$  is a submodule of  $X(J)$ , and  $\text{Hom}(X(I), X(J)) \cong \text{Hom}(X(I), X(I))$  and so  $\dim_k \text{Hom}_D(X(I), X(J)) = |I|$ . Similarly,  $\dim_k \text{Hom}_D(X(J), X(I)) = |I|$ .  $\square$

The theorem of Brüstle, Hille, Ringel and Röhrle [4] is a consequence of the following lemma.

**Lemma 8.** *Let  $X(I)$  and  $X(J)$  be two indecomposable rigid  $k\Delta$ -projective  $D$ -modules, associated to  $I \subseteq J \subseteq \Delta_0$ . Then  $X(I) \oplus X(J)$  is rigid.*

*Proof.* Let  $X = X(I) \oplus X(J)$  and assume that  ${}_{k\Delta}X \cong \bigoplus_i P(i)^{x_i}$ . Then  $x_i^2 = 4$  if  $i \in I \cap J$ ,  $x_i^2 = 1$  if  $i \in J \setminus I$  and  $x_i^2 = 0$  if  $i \notin J$ . By the previous lemma we see that  $\dim_k \text{End}(X) = \sum x_i^2$ , and so  $X$  is rigid by Lemma 6.  $\square$

**6.2. Gluing modules at sinks and sources.** Let  $\Delta$  be a quiver of type  $\mathbb{A}_n$ , with vertices  $\{1, \dots, n\}$ , and arrows  $\alpha_i : i \rightarrow i+1$  or  $\alpha_i : i \leftarrow i+1$  for  $i = 1, \dots, n-1$ . We recall how to glue a pair of  $k\Delta$ -projective  $D$ -modules at an admissible vertex to obtain a new  $k\Delta$ -projective  $D$ -module, which usually has higher dimension [10].

Let  $i_1 < i_2 < \dots < i_t$  be a complete list of interior admissible vertices in  $\Delta$ . Let  $i_0 = 1$  and  $i_{t+1} = n$  be the end vertices of  $\Delta$ . Let  $i = i_l$  be an interior admissible vertex  $\Delta$ . Let  $M'$  and  $M''$  be two indecomposable  $k\Delta$ -projective  $D$ -modules with  $\text{Supp}_\Delta(M')_0 \subseteq \{1, \dots, i\}$ ,  $\text{Supp}_\Delta(M'')_0 \subseteq \{i, \dots, n\}$  and  $(\dim_\Delta M')_i = 1 = (\dim_\Delta M'')_i$ .

Assume that  $i$  is a sink in  $\Delta$ . Then  $M'$  and  $M''$  have unique quotient modules isomorphic to the simple  $D$ -module  $P_i$ . We have short exact sequences

$$0 \longrightarrow \ker(f') \longrightarrow M' \xrightarrow{f'} P_i \longrightarrow 0$$

and

$$0 \longrightarrow \ker(f'') \longrightarrow M'' \xrightarrow{f''} P_i \longrightarrow 0$$

Let  $M$  be given by the pullback of  $f'$  and  $f''$ , that is, we have a short exact sequence

$$0 \longrightarrow M \longrightarrow M' \oplus M'' \longrightarrow P_i \longrightarrow 0$$

Similarly, if  $i$  is a source, we have a pushout sequence

$$0 \longrightarrow P_i \longrightarrow M' \oplus M'' \longrightarrow M \longrightarrow 0$$

In both of these cases, we say that  $M$  is obtained by gluing  $M'$  and  $M''$  at  $i$ . Gluing of homomorphisms is done similarly.

6.3. **The construction of rigid modules.** Let  $\Delta$  be as in the previous section. Let  $d$  be a dimension vector and let  $(d^s)_j = d_j$  for  $j = i_s, \dots, i_{s+1}$  and  $(d^s)_j = 0$  otherwise. Using Theorem 4 we can construct a rigid  $D$ -module  $Y(d^s)$  which has  $\Delta$ -dimension vector  $d^s$  when considered as a module for the double quiver supported on  $\{i_s, \dots, i_{s+1}\}$  subject to the corresponding relations, and which may not be  $k\Delta$ -projective. However, if  $i_s$  is an interior source, we extend  $Y(d^s)$  by  $P_{i_s-1}^{d_{i_s}}$ , and if  $i_{s+1}$  is an interior source, we extend  $Y(d^s)$  by  $P_{i_{s+1}+1}^{d_{i_{s+1}}}$ , to obtain a  $k\Delta$ -projective  $D$ -module, which we denote by  $X(d^s)$ . Since the extension preserve  $\Delta$ -dimension vector and dimension of endomorphism ring, by Lemma 6 the module  $X(d^s)$  is rigid. Clearly,  $X(d^s)$  can be given a multiplicative basis by extending the basis for  $Y(d^s)$ .

The construction of a rigid  $k\Delta$ -projective module  $X(d)$  with  $\Delta$ -dimension vector  $d$  is done by induction on the number of interior admissible vertices in  $\Delta$ . Clearly, if there are no interior admissible vertices, then  $\Delta$  has linear orientation, and we are done.

Let  $s > 0$  and let  $e^s$  be given by  $(e^s)_j = d_j$  if  $j \leq i_s$  and let  $(e^s)_j = 0$  otherwise. Assume by induction we have constructed an rigid  $k\Delta$ -projective  $D$ -module  $X(e^s)$  with dimension vector  $e^s$ . Also, by induction, we may assume that the summands of  $X(e^s)$  which have  $\Delta$ -support at  $i_s$  are totally ordered according to the following order  $\leq_{i_s}$ : Let  $M$  and  $N$  be two indecomposable summands of  $X(e^s)$  with  $(\dim_{\Delta} M)_{i_s} = 1 = (\dim_{\Delta} N)_{i_s}$ . Assume that  $M$  and  $N$  are obtained by gluing indecomposables with supports  $I_a, \dots, I_{s-1}$  and  $J_b, \dots, J_{s-1}$ , respectively. Let  $t$  be the smallest number such that  $I_{s-t} \neq J_{s-t}$ , then  $M \leq N$  if  $J_{s-t} \subseteq I_{s-t}$ , if  $t$  is even, and  $I_{s-t} \subseteq J_{s-t}$ , if  $t$  is odd. Note that  $M \cong N$  if  $t$  does not exist.

We want to glue  $X(e^s)$  with  $X(d^s)$ , by gluing together indecomposable summands. The gluing leaves unchanged all summands not  $\Delta$ -supported at  $i_s$ .

**Lemma 9.** *Let  $M, N$  and  $t$  be as above, and assume that  $M \leq N$ , let  $i_s \in I \subseteq J \subseteq \{i_s, \dots, i_{s+1}\}$ , let  $X$  be obtained by gluing  $M$  with  $X(J)$  and let  $Y$  be obtained by gluing  $M$  with  $X(I)$ . Then  $X \oplus Y$  is rigid.*

*Proof.* We first show that  $\dim_k \text{Hom}(X, Y) = |I| + \dim_k \text{Hom}(M, N) - 1$ . Let  $i_s$  be a source. Any map  $f : M \rightarrow N$  extends to a map  $f' : X \rightarrow Y$ , with  $f'|_{X(I)}$  equal to the unique injection  $X(I) \subset X(J)$  if  $f|_{M_{i_s}}$  is non-zero, and  $f'|_{X(I)} = 0$  otherwise. In addition there are  $|I| - 1$  non injective maps  $X(I) \rightarrow X(J)$ , which gives us  $\dim_k \text{Hom}(X, Y) = |I| + \dim_k \text{Hom}(M, N) - 1$  in this case. The case of sink is similar.

Now the case  $Y = X$  follows, and by similar arguments

$$\dim_k \text{Hom}(Y, X) = |I| + \dim_k \text{Hom}(N, M) - 1.$$

Then since  $M \oplus N$  is rigid,  $\dim_k \text{End}(X \oplus Y) = 4|I| + \sum_{i \leq i_s} d_i^2 - 4 = \sum_i d_i^2$ , and therefore  $X \oplus Y$  is rigid, by Lemma 6.  $\square$

We also need to consider summands without  $\Delta$ -support at  $i_s$ . Let  $X \oplus Y$  be as in the previous lemma. If  $L$  is a summand of  $X(e^s)$  without  $\Delta$ -support at  $i_s$ , then  $\text{Hom}(L, X \oplus Y) = \text{Hom}(L, M \oplus N)$  and  $\text{Hom}(X \oplus Y, L) = \text{Hom}(M \oplus N, L)$  and so  $L \oplus X \oplus Y$  is rigid, by Lemma 6. Similarly, if  $L'$  is a summand of  $X(d^s)$  without  $\Delta$ -support at  $i_s$ , then  $L' \oplus X \oplus Y$  are rigid. Therefore, by induction we construct a rigid  $k\Delta$ -projective  $D$ -module  $X(e^{s+1})$ , with  $\Delta$ -dimension vector  $e^{s+1}$ . Finally, using that  $d = e^{t+1}$  we have the following.

**Theorem 5.** *[Jensen-Su-Yu]  $X(d)$  is a rigid  $k\Delta$ -projective  $D$ -module with  $\Delta$ -dimension vector  $d$ .*

By the construction we see that  $X(d)$  has a multiplicative basis obtained by gluing the multiplicative bases of the linear pieces  $X(e^s)$ .

We use this basis to give  $X(d)$  a grading as follows. Let  $E$  be the quotient of  $D$  by the ideal  $\mathcal{J}$  generated by all paths of the form  $\alpha\beta^*$  for  $\alpha, \beta \in \Delta_1$ . Then  $E$  is a string algebra [5]. We note that  $D = \bigoplus_{i \geq 0} D^i$  is a graded algebra by letting the degree of a path  $p$  be the biggest  $i$  such that  $p \in \mathcal{J}^i$ . This grading on  $D$  induces a grading on any indecomposable

rigid  $k\Delta$ -projective  $D$ -module  $X = \bigoplus_{i \geq 0} X^i$ , in such a way that the graded component  $(X^i)_u$  identifies with  $(\mathcal{J}^i X / \mathcal{J}^{i+1} X)_u$ , in particular,  $X^0 \cong E \otimes_D X$ . Furthermore, the basis of a graded component is a subset of the multiplicative basis of  $X$ . Also,  $\text{Hom}_D(X, Y) = \bigoplus_i \text{Hom}_D^i(X, Y)$ , where  $\text{Hom}_D^i(X, Y)$  denotes the degree  $i$  homomorphisms from  $X$  to  $Y$ , where  $X$  and  $Y$  are indecomposable rigid  $k\Delta$ -projective  $D$ -modules  $X$  and  $Y$ .

**6.4. Descending chains.** The construction for the linear case shows that rigid modules are in bijection with descending chains of subsets of the set of vertices  $\{1, \dots, n\}$ . We generalize this fact to quivers of type  $\mathbb{A}$  with arbitrary orientation.

Let  $u \in \{1, \dots, n\}$  with  $i_{v-1} \leq u \leq i_v$  and let  $I \subseteq \{1, \dots, n\}$  such that  $I$  contains at least one element  $z \in I$  with a path in  $\Delta$  from  $z$  to  $u$ . Let  $I_{v-t} = \{i_{v-t-1}, \dots, i_{v-t}\} \cap I$  if  $t$  is even and  $I_{v-t} = (\{i_w | i_w \in I\} \cap \{i_{v-t}, i_{v-t-1}\}) \cup (\{i_{v-t-1} + 1, \dots, i_{v-t} - 1\} \cap I^c)$  if  $t$  is odd. Here  $I^c$  denotes the complement of  $I$  in  $\{1, \dots, n\}$ . We say that  $I$  is connected if  $I_{w-1} \neq \emptyset \neq I_{w+1}$  implies  $i_{w-1} \in I_{w-1}$  and  $i_w \in I_{w+1}$  for all  $w$ . Let  $X_I$  be the indecomposable rigid  $k\Delta$ -projective  $D$ -module obtained by gluing the modules  $X(I_{v-t})$ . The following lemma follows immediately from Theorem 5.

**Lemma 10.** *Let  $u \in \{1, \dots, n\}$ . The correspondence  $I \mapsto X_I$  induces a bijection between descending chains of subsets of  $\{1, \dots, n\}$  where each subset is connected and contains at least one element  $z$  with a path in  $\Delta$  from  $z$  to  $u$ , and rigid modules with all indecomposable summands supported at  $u$ .*

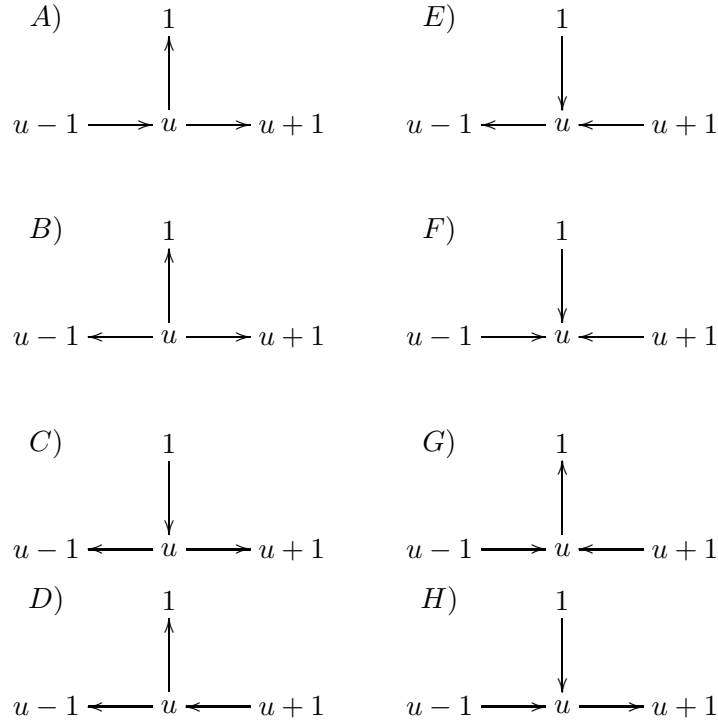
We recover the linear case by considering by letting  $u$  be the sink vertex. We now define a total order which will be important in the next section.

**Definition 2.** *If  $J \subseteq I$  then  $X_I \leq_u X_J$ .*

## 7. ONE POINT EXTENSION OF $\mathbb{A}$

Let  $\Delta$  be a quiver obtained from a quiver of type  $\mathbb{A}_{n-1}$ , with vertices  $\{2, \dots, n\}$ , by attaching a vertex 1 to an interior vertex  $u$  in  $\mathbb{A}_{n-1}$  with one arrow which we denote by  $\gamma$ .

There are eight possible orientations at the vertex  $u$ . These are



We will study generic  $\text{Aut}P$ -orbits in the three cases A), B) and C). This is sufficient, since D) follows from A), and E), F), G) and H) are dual to A), B), C) and D), respectively.

7.1. **Case A.** Denote by  $\Gamma$  the full subquiver of  $\Delta$  supported on the vertices  $\{2, \dots, n\}$ . Also let  $\Delta'$  be the quiver with the same underlying graph as  $\Delta$ , and orientation  $i \rightarrow i+1$  for  $i = 2, \dots, u-1$  and  $i \leftarrow i+1$  for  $i = u, \dots, n-1$ , and the orientation of  $\gamma$  is from  $u$  to 1.

Let  $d \in \mathbb{N}^n$  be a vector and let  $d'$  be the vector given by  $(d')_1 = 0$  and  $(d')_i = d_i$  for  $i \neq 1$ . By the previous section we can construct a rigid module  $Y(d')$  for the double quiver of  $\Gamma$  with  $\Delta$ -dimension vector  $d'$ . Let  $N^1, \dots, N^p$  be representatives of isomorphism classes of the indecomposable summands of  $Y(d')$  supported at  $u$  and ordered such that  $N^i <_u N^{i+1}$ , and let  $n_i$  denote the multiplicity of  $N^i$  as a summand in  $Y(d')$ .

We extend  $Y(d')$  to a rigid  $k\Delta$ -projective  $D$ -module  $X(d')$  with  $\Delta$ -dimension vector  $d'$  as follows. For each indecomposable summand  $Y = N^i$  of  $Y(d')$ , we construct a corresponding indecomposable  $k\Delta$ -projective  $D$ -module  $X = M^i$ . Let  $X_1 = Y_u$ ,  $X_w = Y_w$  for  $w \neq 1$ ,  $X_\alpha = Y_\alpha$  for any  $\alpha \in \Gamma_1 \cup \Gamma_1^*$ ,  $X_\gamma = Id$ . If  $\beta : u-1 \rightarrow u \in \Delta$ , then the equation  $\gamma^* \gamma = \beta \beta^*$  uniquely determines the action of  $\gamma^*$ , and so  $X_{\gamma^*}$  maps  $(X^i)_u = (Y^i)_u$  onto  $(X^{i+1})_u$  for all  $i$ , where  $Y^i$  is the  $i$ th graded component in the grading induced by the ideal  $\mathcal{J}$ . By the construction we see that  $X$  is  $k\Delta$ -projective. Finally, let

$$X(d') = Y' \oplus (\oplus_i (M^i)^{n_i}),$$

where  $Y'$  consist of all summands in  $Y(d')$  not supported at  $u$ . The extension of  $Y(d')$  to  $X(d')$  preserves the  $\Delta$ -dimension vector and the dimension of the endomorphism ring, and so  $X(d')$  is rigid by Lemma 6. Moreover,  $X(d')$  has a grading induced by the grading on  $Y(d')$ .

Let  $End^0(X(d'))_u$  denote the subspace of maps which are restrictions  $f|_{X(d')_u^0} : X(d')_u^0 \rightarrow X(d')_u^0$  for  $f : X(d') \rightarrow X(d')$  with  $f(X(d')_u^0) \subseteq X(d')_u^0$ . The action of  $Aut(X(d'))$  on  $X(d')$  induces an action of  $Aut^0(X(d'))_u$  on  $X(d')_u^0$ , where  $Aut^0(X(d'))_u \subseteq End^0(X(d'))_u$  consists of the invertible maps.

**Lemma 11.** *The pair  $(Aut^0(X(d'))_u \times Gl_{d_1}, Hom(k^{d_1}, X(d')_u^0))$  is a generic section of  $(G(d), Rep(D, P(d)))$ .*

*Proof.* A  $k\Delta$ -projective  $D$ -module  $X$  has a unique submodule  $X' \subseteq X$  with  $\Delta$ -support in  $\{2, \dots, n\}$ , generated by  $X_i$  for  $2 \leq i \leq n$ . We want to parameterize the  $k\Delta$ -projective  $D$ -modules  $X$  with  ${}_\Delta X(d) = P(d)$  and  $X' = X(d')$ .

By choosing a basis the structure of  $X = X(c', c'')$  at the subquiver supported at 1 and  $u$  is

$$X(d')_u^0 \oplus (JX(d'))_u \begin{array}{c} \xrightarrow{\begin{pmatrix} 0 & Id & 0 \\ 0 & 0 & Id \end{pmatrix}^{tr}} \\ \xleftarrow{\begin{pmatrix} c' & 0 & 0 \\ c'' & z_1 & z_2 \end{pmatrix}} \end{array} k^{d_1} \oplus X(d')_u^0 \oplus (JX(d'))_u,$$

where  $X(d')_{\gamma^*} = \begin{pmatrix} 0 & 0 \\ z_1 & z_2 \end{pmatrix}$  and  $(z_1 \ z_2)$  is surjective by the construction of  $X(d')$ . The set-wise stabilizer at  $u$  and 1 of the subset containing all modules  $X(c', c'')$  is

$$\{(a, \begin{pmatrix} g & 0 \\ b & a \end{pmatrix}) \mid a \in Aut(X(d'))_u, g \in Gl_{d_1} \text{ and } b : k^{d_1} \rightarrow X(d')_u\}$$

Then using column operations we see that  $X(c', c'') \cong X(c', 0)$ . Moreover  $X(c', 0) \cong X(c, 0)$  if and only if  $c$  and  $c'$  are conjugate under the action by  $(Aut^0(X(d'))_u \times Gl_{d_1})$ . Any element in  $(Aut^0(X(d'))_u$  induces an automorphism of  $X'$ , and so we can view  $(Aut^0(X(d'))_u$  as a subgroup of  $(Aut(X(d'))_u$ . This shows that  $(Aut^0(X(d'))_u \times Gl_{d_1}, Hom(k^{d_1}, X(d')_u^0))$  is a section of  $(G(d), Rep(D, P))$ . It is generic because  $X(d')$  is rigid.  $\square$

We compute  $Aut^0(X(d'))_u$ .

**Lemma 12.** Let  $i, j \in \{1, \dots, p\}$ . Then  $\text{Hom}^0(M^i, M^j)_u = k$  if and only if, either

- a)  $i \geq j$  and  $(\dim_{\Delta}(M^i))_w = (\dim_{\Delta}(M^j))_w$  for all  $w \leq u$ , or
- b)  $i < j$  and  $(\dim_{\Delta}(M^i))_w = (\dim_{\Delta}(M^j))_w$  for all  $w > u$ ,

otherwise  $\text{Hom}^0(M^i, M^j)_u = 0$ . Moreover, any composition of non-zero maps is non-zero.

We construct a  $\Delta'$ -representation  $Z(d')$  with  $\text{End}(Z(d'))_u \cong \text{End}^0(X(d')_u)$  and dimension vector denoted by  $e(d')$ , with  $e(d')_1 = 0$ . For each segment  $[i, j]$  for  $2 \leq i \leq j \leq n$  there is an associated indecomposable  $\Delta'$ -representation  $M[i, j]$  supported on  $\{i, \dots, j\}$ . We construct the  $\Delta'$ -representation  $Z(d') = \bigoplus Z_i^{n_i}$  as follows. Let  $Z^1 = M[u, n]$ . Given  $Z^i = M[j, j']$ , let  $Z^{i+1} = M[j-1, j']$  if there is a map  $M^i \rightarrow M^{i+1}$ ,  $Z^{i+1} = M[j, j'-1]$  if there is a map  $M^i \leftarrow M^{i+1}$ , and  $Z^{i+1} = M[j-1, j'-1]$  otherwise.

**Lemma 13.** We have  $2 \leq j \leq u \leq j' \leq n$  for any summand  $Z^i = M[j, j']$  in  $Z(d')$ .

*Proof.* The inequalities  $j \leq u$  and  $j' \leq n$  are trivial. Now by the description of  $X(d')$  from Lemma 10 there are at most  $u-2$  cases where either, there is a map  $M^i \rightarrow M^{i+1} \in \text{Hom}^0(M^{i+1}, M^i)_u$ , or there is no map in either direction. Therefore  $2 \leq j$ . Similarly,  $u \leq j'$ .  $\square$

Let  $e(d)$  denote the dimension vector  $e(d) = (d_1, e(d)_2, \dots, e(d)_n)$ . Let  $\Gamma'$  be the full subquiver of  $\Delta'$  supported on  $\{2, \dots, n\}$ .

**Lemma 14.** The pair  $(\text{Aut}(Z(d')_u) \times \text{Gl}_{d_1}, \text{Hom}(k^{d_1}, Z(d')_u))$  is a generic section of  $(\text{Gl}(e(d)), \text{Rep}(\Delta', e(d)))$ .

*Proof.* The  $\text{Aut}(Z(d')) \times \text{Gl}_{d_1}$  orbits in  $\text{Hom}(k^{d_1}, Z(d')_u)$  parameterize the representations of  $\Delta'$  with restriction to  $\Gamma'$  equal to  $Z(d')$ . The lemma follows if we can prove that  $Z(d')$  is a rigid  $\Delta'$ -representation.

The full subquiver  $\Gamma'$  has a sink  $u$ . Let  $M = M[i, j] \oplus M[i', j']$ , for  $1 \leq i, i' \leq u \leq j, j' \leq n$ . Then  $\text{Ext}^1(M[i, j], M[i', j']) \neq 0$  if and only if  $i < u < j$  and  $[i', j'] \subseteq [i+1, j-1]$ . The lemma follows.  $\square$

Finally we have that

**Lemma 15.** The pairs  $(\text{Gl}(e(d)), \text{Rep}(\Delta', e(d)))$  and  $(G(d), \text{Rep}(D, P(d)))$  are generically equivalent.

*Proof.* There are isomorphisms  $(Z^i)_u \rightarrow (M^i)_u^0$ , which extend to an isomorphism

$$\text{Hom}(k^{d_1}, Z(d')_u) \rightarrow \text{Hom}(k^{d_1}, X(d')_u^0)$$

of vector spaces. By the construction  $\text{Hom}^0(M^i, M^j)_u \cong \text{Hom}(Z^i, Z^j)_u$ , and by Lemma 12 we have an isomorphism  $\text{Aut}(Z(d')_u) \rightarrow \text{Aut}^0(X(d')_u)$ . Therefore there is a commutative diagram

$$\begin{array}{ccc} (\text{Aut}(Z(d') \times \text{Gl}_{d_1}) \times \text{Hom}(k^{d_1}, Z(d')_u)) & \longrightarrow & (\text{Aut}^0(X(d')_u \times \text{Gl}_{d_1}) \times \text{Hom}(k^{d_1}, X(d')_u^0)) \\ \downarrow & & \downarrow \\ \text{Hom}(k^{d_1}, Z(d')_u) & \longrightarrow & \text{Hom}(k^{d_1}, X(d')_u^0) \end{array}$$

where the vertical maps are actions and the horizontal maps are isomorphisms, and the two actions are equivariantly isomorphic. The lemma now follows from Lemmas 11 and 14.  $\square$

**7.2. Case B.** Let  $\Gamma, \gamma, \Delta', d, d'$  and  $Y(d')$  be similarly defined as in Case A. We extend  $Y(d')$  to a rigid  $k\Delta$ -projective  $D$ -module  $X(d')$  with  $\Delta$ -dimension vector  $d'$  as follows. Let  $Y = N^i$  be an indecomposable summand of  $Y(d')$  supported at  $u$ . We construct a corresponding indecomposable  $k\Delta$ -projective  $D$ -module  $X = M^i$  and let  $X(d') = Y' \oplus (\bigoplus (M^i)^{n_i})$ , where  $Y'$  consist of all summands of  $Y(d')$  not supported at  $u$ . Let  $X_1 = Y_u$ ,  $X_w = Y_w$  for  $w \neq 1$ ,  $X_{\alpha} = Y_{\alpha}$  for any  $\alpha \in \Gamma_1 \cup \Gamma_1^*$ ,  $X_{\gamma} = Id$ , and  $X_{\gamma^*} = 0$ . The extension of  $Y(d')$  to  $X(d')$

preserves the  $\Delta$ -dimension vector and the dimension of the endomorphism algebra, and so  $X(d')$  is rigid by Lemma 6. Moreover,  $X(d')$  has a grading induced by the grading on  $Y(d')$ .

The following lemma is proved similarly to Lemma 14 in Case A, and so we skip the details.

**Lemma 16.** *The pair  $(\text{Aut}^0(X(d'))_u \times \text{Gl}_{d_1}, \text{Hom}(k^{d_1}, X(d')_u^0))$  is a generic section of  $(G(d), \text{Rep}(D, P(d)))$ .*

We compute  $\text{Aut}^0(X(d'))_u$ . The vertex  $u$  is a source in  $\Gamma$  and so we may assume that  $u = i_v$  is equal to the  $v$ th admissible vertex of  $\Gamma$ .

**Lemma 17.** *Let  $i, j \in \{1, \dots, p\}$ . Then  $\text{Hom}^0(M^i, M^j)_u = k$  if and only if, either*

- a)  $i < j$  and  $(\dim_{\Delta} M^i)_w = (\dim_{\Delta} M^j)_w$  for all  $w < u$ , or
- b)  $i \geq j$  and  $(\dim_{\Delta} M^i)_w = (\dim_{\Delta} M^j)_w$  for all  $w > u$ ,

otherwise  $\text{Hom}^0(M^i, M^j)_u = 0$ . Moreover, any composition of non-zero maps is non-zero.

Using a similar procedure as in Case A, we construct a  $\Delta'$ -representation  $Z(d')$  such that  $\text{End}(Z(d'))_u \cong \text{End}^0(X(d'))_u$ , its dimension vector  $e(d')$  is zero at vertex 1, and each indecomposable summand is of the form  $M[i, j]$  for a segment  $[i, j]$  in  $2, \dots, n$ . Again, let  $e(d) = (d_1, e(d)_2, \dots, e(d)_n)$ .

Similar to Case A we have the following lemma.

**Lemma 18.** *The pairs  $(\text{Gl}(e(d)), \text{Rep}(\Delta', e(d)))$  and  $(G(d), \text{Rep}(D, P(d)))$  are generically equivalent.*

**7.3. Case C.** Let  $\Gamma$  be the full subquiver of  $\Delta$  supported on the vertices  $\{1, \dots, u\}$ . Moreover, in this section we assume that  $n = u + 2$ . In particular  $\Delta$  could be of type  $\mathbb{E}_6, \mathbb{E}_7$  or  $\mathbb{E}_8$ . Let  $\alpha$  denote the arrow  $u \rightarrow u + 1 \in \Delta$ , and let  $\beta \in \Delta_1$  denote the arrow between  $u + 1$  and  $u + 2$ , which could be of either orientation. That is, either  $u + 2$  is a sink, or it is a source. Let  $\Delta'$  be the quiver with the same underlying graph as  $\Delta$ , and with orientation  $i - 1 \rightarrow i$  for  $i \leq u$ ,  $u \leftarrow u + 1 \leftarrow u + 2$  and  $1 \rightarrow u$ .

Let  $d \in \mathbb{N}^n$  be a dimension vector, and let  $d'$  be given by  $d'_i = 0$  for  $i = u + 1, u + 2$  and  $d'_i = d_i$  otherwise. Let  $Y(d')$  be a rigid module for the double quiver of  $\Gamma$ . Let  $N^1, \dots, N^p$  be representatives of isomorphism classes of the indecomposable summands of  $Y(d')$  supported at  $u$  and ordered such that  $N^i <_u N^{i+1}$ , and let  $n_i$  denote the multiplicity of  $N^i$  as a summand in  $Y(d')$ .

We extend  $Y = Y(d')$  to a rigid  $k\Delta$ -projective  $D$ -module  $X = X(d')$  with  $\Delta$ -dimension vector  $d'$  as follows. Let  $X_v = Y_v$ , for  $v \leq u$  and  $X_{u+1} = Y_u$ ,  $X_{\sigma} = Y_{\sigma}$  for any  $\sigma \in \Gamma_1 \cup \Gamma_1^*$ ,  $X_{\alpha} = \text{Id}$ , and  $X_{\alpha^*} = Y_{\gamma} Y_{\gamma^*}$ . If  $u + 2$  is a source, then  $X_{u+2} = 0$ , and if  $u + 2$  is a sink then  $X_{u+2} = Y_u$ ,  $X_{\beta} = \text{Id}$  and  $X_{\beta^*} = X_{\alpha^*}$ . For each indecomposable summand  $Y = N^i$  of  $Y(d')$ , we let  $X = M^i$  denote the corresponding indecomposable summand of  $X(d')$ . The extension of  $Y(d')$  to  $X(d')$  preserves the  $\Delta$ -dimension vector and the dimension of the endomorphism ring, and so  $X(d')$  is rigid by Lemma 6. Moreover,  $X(d')$  has a grading induced by the grading on  $Y(d')$ .

Let  $d''$  be the dimension vector supported on  $\{u + 1, u + 2\}$ , given by  $d''_i = 0$  for  $i \leq u$  and  $d''_i = d_i$  otherwise. Let  $X(d'')$  be the rigid  $D$ -module with  $\Delta$ -dimension vector  $d''$ , which is supported on the vertices  $\{u + 1, u + 2\}$ .

If  $u + 2$  is a sink, let  $V = \text{soc}(X(d''))$ , let  $H_V$  consist of restrictions  $f|_V$  for  $f \in \text{Aut}(X(d''))$ , let  $W = X(d'')_{u+1}^0$  and let  $H_W = \text{Aut}^0(X(d''))_{u+1}$ . If  $u + 2$  is a source, let  $V = (X(d'')/\text{rad}X(d''))_{u+1} \cong k^{d_{u+1}}$ , let  $H_V$  consist of induced maps  $\bar{f} : (X(d'')/\text{rad}X(d''))_{u+1} \rightarrow (X(d'')/\text{rad}X(d''))_{u+1}$  for  $f \in \text{Aut}(X(d''))$ , let  $W = X(d'')_u^0$  and let  $H_W = \text{Aut}^0(X(d''))_u$ .

**Lemma 19.** *The pair  $(H_V \times H_W, \text{Hom}_k(V, W))$  is a generic section in  $(G(d), \text{Rep}(D, P(d)))$ .*

*Proof.* We only consider the case where  $u + 2$  is a sink, as the other case is similar.

Associated to any rigid  $k\Delta$ -projective module  $X$  with  $\Delta$ -dimension vector  $d$  there is a unique submodule  $X' \subseteq X$  with  $X' \cong X(d')$  and quotient  $X/X' \cong X(d'')$ , since  $X(d')$  and  $X(d'')$  are rigid.

We decompose  $X(d') = M \oplus N \oplus L$ , where  $M$  consists of all summands of  $X(d')$  with  $\Delta$ -support at both 1 and  $u$ ,  $L$  consists of all summands not supported at  $u$ , and  $N$  consists of all other summands. That is,  $N$  consists of all summands of  $X(d')$  with  $\Delta$ -support at either 1 or  $u$ , but not both. Let  $W_1 = (M^0)_{u+1}$ ,  $W_2 = (M^1)_{u+1}$  and  $W_3 = (N^0)_{u+1}$ . We have  $W = W_1 \oplus W_3$ . Let  $X(d'') = R \oplus S$ , where  $R$  consist of all summands with  $\Delta$ -support at both  $u+1$  and  $u+2$ , and  $S$  consist of all other summands. Let  $V_1 = (R^0)_{u+2}$ ,  $V_2 = (R^1)_{u+2}$  and  $V_3 = S_{u+2}$ , the linear space of  $S$  at vertex  $u+2$ . We have  $V = V_2 \oplus V_3$ .

By the relations of  $D$ , we see that  $X = X(c)$  is determined by the maps between the vertices  $u+1$  and  $u+2$  which up to isomorphism have the form

$$\begin{array}{ccc} & \begin{pmatrix} Id & 0 \\ 0 & X(d'')_{\beta} \end{pmatrix} & \\ & \xrightarrow{\hspace{10em}} & \\ X(d')_{u+1} \oplus X(d'')_{u+1} & & X(d')_{u+2} \oplus X(d'')_{u+2} \\ & \xleftarrow{\hspace{10em}} & \\ & \begin{pmatrix} X(d')_{\beta^*} & c \\ 0 & X(d'')_{\beta^*} \end{pmatrix} & \end{array}$$

where  $c = (c_{ij})_{ij} : V_1 \oplus V_2 \oplus V_3 \longrightarrow W_1 \oplus W_2 \oplus W_3$ .

A computation shows that  $X(c) \cong X(c')$  where

$$c' = \begin{pmatrix} c'_{11} & c_{12} & c_{13} \\ 0 & 0 & 0 \\ 0 & c_{32} & c_{33} \end{pmatrix}.$$

Let  $a : W_1 \rightarrow W_2$  and  $b : V_1 \rightarrow V_2$  be restrictions of automorphisms of  $X(d')$  and  $X(d'')$ , respectively. Then  $X(c') \cong X(c'')$  with  $c'_{11} = c'_{11} + (M_{\alpha}M_{\alpha^*})^{-1}ac_{12}R_{\beta}R_{\beta^*} - c_{12}b$ , and  $c''_{ij} = c'_{ij}$  otherwise. By choosing bases we may assume that the matrices of  $M_{\alpha}M_{\alpha^*}$  and  $R_{\beta}R_{\beta^*}$  are identities, i.e. we identify  $W_1, V_1$  with  $W_2, V_2$ , respectively. Moreover,  $b$  can be any quadratic matrix, and that the set of matrices  $a$  include all upper triangular matrices, since whenever  $i \leq j$  we have  $Hom^0(M^j, M^i)_{u+1} \neq 0$  and  $Hom^1(M^i, M^i)_{u+1} \neq 0$ .

The map

$$\phi : Gl_{dim V_1} \times B_{dim W_1} \rightarrow Mat_{dim V_1 \times dim W_1}, (b, a) \mapsto ac_{12}b^{-1}$$

is dominant when  $c_{12}$  is generic, where  $B_{dim W_1}$  consists of all invertible upper triangular  $dim W_1 \times dim W_1$ -matrices. This is because  $(\mathfrak{gl}_{dim V_1} \times \mathfrak{b}_{dim W_1}, Mat_{dim V_1 \times dim W_1})$  is a generic section in  $(Gl(f), Rep(\mathbb{A}_{dim W_1+1}, f))$  for a dimension vector  $f$ . Then the induced map of  $\phi$  on tangent spaces  $(b, a) \mapsto ac_{12} - c_{12}b$  is surjective. Then since  $X(d')$  is rigid, therefore  $c_{12}$  is generic, and so there exists  $a$  and  $b$  such that  $c'_{11} + ac_{12} - c_{12}b = 0$ .

So  $X(c'') \cong X(c''')$  where  $c'''_{11} = 0$  and  $c'''_{ij} = c''_{ij}$  otherwise. That is, we may consider  $c'''$  as a map  $V \rightarrow W$ . Now for  $c_1, c_2 : V \rightarrow W$  we have  $X(c_1) \cong X(c_2)$  if and only if they are conjugate under the action of  $H_V \times H_W$ . Then the lemma follows.  $\square$

Similar to Case A, we construct a rigid  $\Delta'$ -representation  $Z(d')$  with dimension vector denoted by  $e(d')$ , where  $e(d')_{u+1} = 0 = e(d')_{u+2}$  and  $e(d')_u = dim_k W$ , and with summands supported on intervals in  $2, 3, \dots, u-1, u, 1$ . The construction gives us a commutative diagram

$$\begin{array}{ccc} Aut(Z(d'))_u \times Z(d')_u & \longrightarrow & H_W \times W \\ \downarrow & & \downarrow \\ Z(d')_u & \longrightarrow & W \end{array}$$

where the horizontal maps are isomorphisms and the vertical maps are actions. Similarly, we construct a rigid  $\Delta'$ -representation  $Z(d'')$  supported at  $u + 1, u + 2$ . Let  $e(d)$  be the dimension vector of  $Z(d') \oplus Z(d'')$ . As before we have the following lemma.

**Lemma 20.** *( $Aut(Z(d'')_{u+1} \times Aut(Z(d')_u), Hom(Z(d'')_{u+1}, Z(d')_u))$  is a generic section in  $(Gl(e(d)), Rep(\Delta', e(d)))$ .*

Similar to Case A, we may conclude with the following lemma.

**Lemma 21.** *The pairs  $(Gl(e(d)), Rep(\Delta', e(d)))$  and  $(G(d), Rep(D, P(d)))$  are generically equivalent.*

## 8. MAIN THEOREM

We prove Theorem 1 stated in the introduction

**Theorem 6.** *Let  $\Delta$  be a quiver. Then there is a dense open  $AutP$ -orbit in  $radEndP$  for all projective representations  $P$  if and only if  $\Delta$  is a Dynkin quiver.*

*Proof.* First assume that  $\Delta$  is a Dynkin quiver. By Theorem 5 we may assume that  $\Delta$  is not of type  $\mathbb{A}$  and we assume that  $\Delta$  is obtained from the quiver of type  $\mathbb{A}_{n-1}$ , with vertices  $\{2, \dots, n\}$ , by attaching an vertex 1 to an interior vertex  $u$  in  $\mathbb{A}_{n-1}$  with an arrow which we denote by  $\gamma$ . We consider the possible orientations  $A, B, C, D, E, F, G$  and  $H$  given at the beginning of the previous section. In cases  $A, B$  and  $C$ , the theorem follows from Theorem 3 which is due to Gabriel and Lemmas 15, 18 and 21, respectively

Case  $D$  follows from Case  $A$  by symmetry.

Finally, if  $\Delta^{op}$  is the opposite quiver of  $\Delta$ , then there is a commutative diagram

$$\begin{array}{ccc} AutP \times radEndP & \longrightarrow & (AutP)^{op} \times rad(EndP)^{op} \\ \downarrow & & \downarrow \\ radEndP & \longrightarrow & rad(EndP)^{op} \end{array}$$

where the vertical arrows are actions and the horizontal arrows are isomorphisms  $f \mapsto f^{op}$  and  $(g, f) \mapsto ((g^{op})^{-1}, f^{op})$ . Therefore the case  $E, F, G$  and  $H$  follow from the cases  $A, B, C$  and  $D$ , respectively.

Conversely, let  $\Delta$  be a non-Dynkin quiver. If there is a dense open orbit for the action of  $AutP$  on  $radEndP$ , then there is a dense open orbit for the induced action of  $AutP$  on  $radEndP/(radEndP)^2$ . But,  $AutP$ -orbits in  $radEndP/(radEndP)^2$  are naturally isomorphism classes of representations of  $\Delta^{op}$ , with dimension vector equal to  $d$  for  $P = P(d)$ . Since  $\Delta^{op}$  is not Dynkin, there are dimension vectors such that the associated representation varieties do not have dense open orbits. Hence, there are projective  $\Delta$ -representations  $P$  without dense open orbits in  $radEndP$ .  $\square$

Now Theorem 2 follows.

**Corollary 22.** *Let  $\Delta$  be a tree. There is a open dense  $S(\Delta)(d)$ -orbit in the nilpotent radical  $\mathfrak{n}(\Delta)(d)$  of  $\mathfrak{s}(\Delta)(d)$  for all  $d$ , if and only if  $\Delta$  is a Dynkin quiver.*

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