NONCOMMUTATIVE FOURIER ANALYSIS AND PROBABILITY

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In this report, we study Fourier analysis on groups with applications to probability. The most general framework would be out of reach for an undergraduate project and we aim at understanding the examples which have started the theory over the past two hundred years. We will begin with a review of the cases of the Euclidean Fourier transform and of the Fourier series in Section 1 and 2. This can be generalised into the Fourier analysis on abelian locally compact groups (LCA groups) which is quite well understood, see Section 3. The non-commutative (locally compact) groups present different challenges and we present the cases of compact and finite groups in more details, see Sections 5 and 4, with some applications to probability. Throughout the duration of the project something which has become of particular interest is the structure of LCA groups, so we have looked at the theory of LCA groups with some examples.

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Convention. In this report, we will follow the following convention. dx will denote the normalised Haar measure on the groups under consideration. This means that dx is the usual Lebesgue measure on $\mathbb{R}^n$ and $\mathbb{T}^n$, and that it is a probability measure on a compact group. In particular, this corresponds to taking the average $\frac{1}{|G|}\sum_{x\in G}$ on a finite group G.

The Fourier transform of (say) an integrable function f on G will be given by

$$\widehat{f}(\rho) = \int_G f(x)\rho(x)^* dx, \tag{0.1}$$

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where ρ is a unitary representation of G. A representation is always assume to be finite dimensional and continuous unless stated otherwise.On a group equipped with a Haar measure we define $\|f\|_{L^p} = (\int_G |f(x)|^p dx)^{\frac{1}{p}}$ for $p \in [1,\infty)$ for any suitable function $f : G \to \mathbb{C}$

1. Fourier analysis on the torus

Consider the space $C(\mathbb{R}/\mathbb{Z})$, of all continuous functions with period 1. It is easily checked this is a vector space, we now equip this with an inner product

$$\langle f, g\rangle_{L^2} = \int_0^1 f(x)g(x)^* dx, \tag{1.1}$$

to make $C(\mathbb{R}/\mathbb{Z})$ into an inner product space.

Remark 1.1. It is worth noting at this point that $C(\mathbb{R}/\mathbb{Z})$ is canonically isomorphic to the space of functions $f \in C[0,1]$ where $f(0) = f(1)$.

For $k \in \mathbb{Z}$ let

$$e_k(x) = e^{2\pi kix} \quad x \in \mathbb{R}, \tag{1.2}$$

then e_k lies in $C(\mathbb{R}/\mathbb{Z})$ and e_k, $k \in \mathbb{Z}$, form an orthonormal system in $C(\mathbb{R}/\mathbb{Z})$.

For each $f \in C(\mathbb{R}/\mathbb{Z})$, we define the numbers

$$c_k(f) = \langle f, e_k\rangle_{L^2} = \int_0^1 f(x)e^{-2\pi kix} dx \tag{1.3}$$

to be the Fourier coefficients of f. The series

$$\sum_{k=-\infty}^{\infty} c_k(f)e_k(x) \tag{1.4}$$

is called the Fourier series of f. We now want to discuss the convergence of the Fourier series, as it is not immediate that the above expression converges at all.

Recall the L_2 norm, this is the norm induced by the inner product defined in 1.1 (1.1) i.e.,

$$\|f\|_{L^2} = \sqrt{\langle f, f\rangle}, \tag{1.5}$$

now $C(\mathbb{R}/\mathbb{Z})$ equiped with the L_2 norm is a normed vector space.

We now state a lemma which will help us to prove the convergence of Fourier series under certain assumptions.

Lemma 1.2. *Let $f \in C(\mathbb{R}/\mathbb{Z})$ and for $k \in \mathbb{Z}$ let c_k be its kth Fourier coefficient, then for all $n \in \mathbb{N}$*

$$\|f - \sum_{k=-n}^{k=n} c_k(f)e_k\|_{L^2}^2 = \|f\|_{L^2}^2 - \sum_{k=-n}^{k=n} |c_k(f)|^2$$

Proof. Let $g = \sum_{k=-n}^{k=n} c_k(f)e_k$. Then

$$\langle f, g\rangle = \sum_{k=-n}^{k=n} \overline{c_k(f)}\langle f, e_k\rangle = \sum_{k=-n}^{k=n} |c_k(f)|^2$$

similarly

$$\langle g, g\rangle = \sum_{k=-n}^{k=n} \overline{c_k(f)}\langle g, e_k\rangle = \sum_{k=-n}^{k=n} |c_k(f)|^2$$

Combining these two results we see

$$\|f - \sum_{k=-n}^{k=n} c_k(f)e_k\|_{L^2}^2 = \|f\|_{L^2}^2 - \sum_{k=-n}^{k=n} |c_k(f)|^2 \tag{1.6}$$

□

The following proposition is also usefull.

Proposition 1.3. *If a sequence of functions $f_n, n \in \mathbb{N}$ converges to f uniformly on $[0, 1]$, then f_n converges to f in the L_2 norm.*

Proof. Let $\epsilon > 0$. Then there is an n_0 such that for all $n_0 \leqslant n$

$$|f(x) - f_n(x)| < \epsilon \tag{1.7}$$

whenever $x \in [0, 1]$. Hence for $n_0 \leqslant n$,

$$\|f - f_n\|_{L^2}^2 = \int_0^1 |f(x) - f_n(x)|^2 dx < \epsilon^2 \tag{1.8}$$

So$\|f - f_n\| < \epsilon$. □

We now prove a huge result, that the Fourier series of every continuous function f converges to f in the L_2 norm. To do this we need the following result which we will state without proof. The interested reader is refered to [1, Sec.1.4].

Lemma 1.4. *Let $f : \mathbb{R} \to \mathbb{R}$ be periodic and such that $f|_{[0,1]}$ is a step function. Then the Fourier series of f converges to f in the L_2 norm.*

We now prove the main result, known as Plancherel's Theorem.

Theorem 1.5. *Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and periodic on $[0,1]$ then*

$$\sum_{k=-\infty}^{\infty} |c_k(f)|^2 = \int_0^1 |f(x)|^2 dx. \tag{1.9}$$

Remark 1.6. This immediately shows that the Fourier series of f converges to f in the L_2 norm, as a result of (1.6).

Definition 1.7. The convolution of two functions f_1 and f_2 which are assumed to be continuous and periodic on $[0,1]$ is denoted $f_1 * f_2$ and given by

$$f_1 * f_2(x) = \int_0^1 f_1(y) f_2(x-y) dy. \tag{1.10}$$

Theorem 1.8. *If $f_1, f_2 \in C(\mathbb{R}/\mathbb{Z})$ and $f = f_1 * f_2$, then $f \in C(\mathbb{R}/\mathbb{Z})$ and $c_k(f) = c_k(f_1)c_k(f_2)$*

Proof.

$$\begin{aligned}
c_k(f) &= \int_0^1 f(x) e^{-2\pi kix} dx \\
&= \int_0^1 \int_0^1 f_1(z) f_2(x-z) dz e^{-2\pi kix} dx \\
&= \int_0^1 \int_0^1 f_2(x-z) e^{-2\pi kix} dx f_1(z) dz \\
&= \int_0^1 \int_0^1 f_2(x') e^{-2\pi kix'} dx' f_1(z) e^{-2\pi kiz} dz \\
&= c_k(f_1) c_k(f_2).
\end{aligned}$$

note the penultimate step is intergration by substitution(where $x' = x - z$). And we justify the exchange of intergration by Fubini's theorem, the interested reader is reffered to [1, sec.7.2].

It is immediate form the definition that f is periodic, with period 1. It remains to show f is continuous.

Let $\epsilon > 0$ be arbitary and $f_1, f_2 \in C(\mathbb{R}/\mathbb{Z})$ then (as being continuous on a closed and bounded interval in $\mathbb{R}$ is equivalent to being uniformly continuous), there exists a $\delta > 0$ such that, for all $x, x' \in [0,1]$ and $f \neq 0$

$$|x - x'| < \delta \implies |f_1(x) - f_1(x')| < \frac{\epsilon}{\|f_2\|_{L^1}}.$$

Then for $|x - x_0| < \delta$ we have

$$\begin{aligned}
|f_1 * f_2(x) - f_1 * f_2(x_0)| &= \left| \int_0^1 f_1(y) f_2(x-y)dy - \int_0^1 f_1(y) f_2(x_0 - y)dy \right| \\
&\leqslant \int_0^1 |f_1(x-y) - f_1(x_0 - y)||f_2(y)|dy \\
&\leqslant \frac{\epsilon}{\|f_2\|_{L^1}} \int_0^1 |f_2(y)|dy \leqslant \epsilon
\end{aligned}$$

□

Remark 1.9. The above theorem shows that if we equipp $C(\mathbb{R}/\mathbb{Z})$ with convolution as a binary operation then, $C(\mathbb{R}/\mathbb{Z})$ may be viewed as an algebra.

If we consider the mapping $\mathcal{F}(f) = (c_k(f))_{k\in\mathbb{Z}}$ for $f \in C(\mathbb{R}/\mathbb{Z})$, then we may view this as a mapping from the convolution algebra $C(\mathbb{R}/\mathbb{Z})$ to the algebra of sequences, the latter being equipped with pointwise addition and multiplication .

Theorem 1.10. *Let $f \in C(\mathbb{R}/\mathbb{Z})$ be piecewise continuously differentiable, then $\sum |c_k(f)| < \infty$, and*

$$f(x) = \sum_{k=-\infty}^{\infty} c_k(f) e_k(x) \quad x \in [0,1). \tag{1.11}$$

Proof. Let $f \in C(\mathbb{R}/\mathbb{Z})$ be piecewise continuously differentiable, let $\phi_j : [t_{j-1}, t_j] \to \mathbb{C}$ be the continuous derivative of f and let $\phi : C(\mathbb{R}/\mathbb{Z}) \to \mathbb{C}$ be function that coincides with ϕ_j on $[t_{j-1}, t_j)$ for every j. Let $(\gamma_k)_{k\in\mathbb{Z}}$ be the Fourier coefficients of ϕ. Then by (1.6)

$$\sum_{-\infty}^{\infty} |\gamma_k|^2 \leqslant \|\phi\|_{L^2}^2 < \infty.$$

Using integration by parts we obtain

$$\int_{t_{j-1}}^{t_j} f(x) e^{-2\pi i k x} = \frac{1}{-2\pi i k x} f(x) e^{-2\pi i k x}|_{t_{j-1}}^{t_j} - \frac{1}{-2\pi i k x} \int_{t_{j-1}}^{t_j} \phi(x) e^{-2\pi i k x} dx$$

so when $k \neq 0$ we obtain

$$c_k(f) = \int_0^1 f(x) e^{-2\pi i k x} dx = \frac{1}{-2\pi i k x} \int_0^1 \phi(x) e^{-2\pi i k x} dx = \frac{1}{-2\pi i k} \gamma_k.$$

For $\alpha, \beta \in \mathbb{C}$ we have $0 \leqslant (|\alpha| - |\beta|)^2 = |\alpha| = |\beta|^2 - 2|\alpha\beta|$ so $|\alpha\beta| \leqslant \frac{1}{2}(|\alpha|^2 + |\beta|^2)$, applying this to the expression above we see that

$$|c_k(f)| \leqslant \frac{1}{2}\left(\frac{1}{4\pi^2 k^2} + |\gamma_k|^2\right),$$

which implies

$$\sum |c_k(f)| < \infty.$$

We have now proved the first part of our statement, it remains to show that $f(x) = \sum_{k=-\infty}^{\infty} c_k(f) e_k(x)$.

The first part of our statement implies that the Fourier series, $\sum c_k(f) e^{2\pi i k x}$ converges uniformly. Denote the limit function g, then g is continuous (as it is the uniform limit of continuous functions). Since the Fourier series also converges to f in the L^2 norm, and since uniform convergence implies L^2 convergence by (1.3) it follows that $f = g$, by uniqueness of limits. □

Remark 1.11. This may be viewed as a Fourier series inversion formula, in the sense that if we know the Fourier coefficients of a given function, we can then retrieve the original function.

We now consider a very interesting application of Fourier series, in a problem first posed by Pietro Mengoli in 1644 and solved by Leonhard Euler in 1734. It is known as the Basel problem. The Basel problem asks for the precise summation of the reciprocals of the squares of the natural numbers, i.e. $\sum_{k=1}^{\infty} \frac{1}{k^2}$

To solve this problem we first consider the periodic extension of the function $f : [0, 1) \to \mathbb{R}, f(x) = x$. We begin by computing the Fourier coefficients of this function to retrieve

$$c_k(f) = \begin{cases} \frac{-1}{2\pi k i} & \text{if } k \neq 0; \\ \frac{1}{2} & \text{if } k = 0. \end{cases}$$

We can apply Plancherel's theorem (1.9) to deduce

$$\frac{1}{2\pi^2 k^2} + \frac{1}{4} = \|f\|_{L^2}^2$$

Computing

$$\|f\|_{L^2}^2 = \int_0^1 x^2 dx = \left[\frac{x^3}{3}\right]_0^1 = \frac{1}{3}.$$

We finally obtain

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}. \tag{1.12}$$

2. The Fourier Transform on $\mathbb{R}$

We first define $L^1_{bc}(\mathbb{R})$ to be the set of all bounded continuous functions $\mathbb{R} \to \mathbb{R}$ satisfying

$$\|f\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}} |f(x)| dx < \infty \tag{2.1}$$

It is easily checked that this is a norm.

For $f \in L^1_{bc}(\mathbb{R})$ we define the Fourier transform of f by

$$\mathcal{F}(f)(y) = \hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i x y} dx \tag{2.2}$$

We now introduce a new set of functions called the Schwartz space denoted $\mathcal{S}(\mathbb{R})$, this is the space of all infinitely differentiable functions $f : \mathbb{R} \to \mathbb{R}$ such that for every $0 \leqslant m, n$ we have

$$\sigma_{m,n}(f) = \sup_{x \in \mathbb{R}} |x^m f^{(n)}(x)| < \infty \tag{2.3}$$

Theorem 2.1. *Let $\mathcal{F}$ denote the Fourier transform (i.e $F : f \mapsto \hat{f}$), then $\mathcal{S} \subset L^1_{bc}(\mathbb{R})$ and*

$$\mathcal{F}(\mathcal{S}(\mathbb{R})) \subset \mathcal{S}(\mathbb{R})$$

Proof. We first prove $\mathcal{S} \subset L^1_{bc}(\mathbb{R})$.

Let $f \in \mathcal{S}$. Then f is bounded and continuous and $(1 + x^2) f(x)$ is bounded, say by $C > 0$, so

$$\int_{-\infty}^{\infty} |f(x)| dx \leqslant C \int_{-\infty}^{\infty} \frac{1}{1 + x^2} dx < \infty$$

Which implies $f \in L^1_{bc}$

In order to prove the second part of this theorem, we need the following lemmas, which we will state without proof.

(1) Let $f \in L^1_{bc}(\mathbb{R})$, let $g(x) = -2\pi i x f(x)$ and $g \in L^1_{bc}(\mathbb{R})$ then $\hat{f}$ is continuously differentiable with $\hat{f}'(y) = \hat{g}(y)$.

We also need the following

(2) Let f be continuously differentiable and assume that the functions f and f' lie in $L^1_{bc}(\mathbb{R})$. Then $\widehat{f'}(y) = 2\pi i y \hat{f}(y)$, so in particular the function $y\hat{f}(y)$ is bounded.

We can now prove the main result. Iterating statement (1) above we obtain

$$(-2\pi i x)^n \widehat{f} = \widehat{f}^{(n)}$$

for every $n \in \mathbb{N}$. Now, iterating statement (2) we see that

$$\widehat{f^{(n)}}(y) = (2\pi i y)^n \widehat{f}(y)$$

for every $n \in \mathbb{N}$. Combining these two results we see that for every $f \in \mathcal{S}$ and every $n, m \in \mathbb{N}_0$ the function

$$y^m \widehat{f}^{(n)}(y)$$

is a Fourier transform of a function in $\mathcal{S}$, and hence bounded. □

Theorem 2.2 (The Fourier Inversion Formula). *Let $f, \hat{f} \in L^1_{bc}(\mathbb{R})$ then*

$$f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i x y} dy \tag{2.4}$$

Remark 2.3. The above expression can be reformulated as follows, for all $x \in \mathbb{R}$

$$\mathcal{F}^- 1 g(x) = \mathcal{F} g(-x).$$

Remark 2.4. If we know the Fourier transform of a given function, we can use the above equation to retrieve our original function.

Theorem 2.5. *$\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism of vector spaces.*

Proof. Linearity of F follows trivially. Let us show $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is injective, this follows from the Fourier inversion formulae, as given any function $g(x) \in \mathcal{S}(\mathbb{R}), g(-x)$ is the unique inverse. To show $\mathcal{F}$ is surjective we need to show $\mathcal{F}(\mathcal{S}(\mathbb{R})) = \mathcal{S}(\mathbb{R})$, but this has been shown in theorem 1.12 as given any function $g(x) \in \mathcal{S}(\mathbb{R}), g(-x)$ will map to $g(x)$. Combining these results we have shown that $\mathcal{F} : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ is an isomorphism. □

The convolution of two functions defined on $\mathbb{R}$ is defined in a similiar way to $\mathbb{R}/\mathbb{Z}$, see 1.7. We make an adjustment to definition 1.7 to be suitable for $f_1, f_2 \in L^1_{bc}(\mathbb{R})$ ie, we change the domain or our integration. So for $f_1, f_2 \in L^1_{bc}(\mathbb{R})$ the convolution of f_1 and f_2 is given by

$$f_1 * f_2(x) = \int_{\mathbb{R}} f_1(y) f_2(x-y)dy. \tag{2.5}$$

Theorem 2.6. *Let $f_1, f_2 \in L^1_{bc}(\mathbb{R})$, then $f_1 * f_2 \in L^1_{bc}(\mathbb{R})$.*

Proof. Assume $f_2(x) < C$ for all $x \in \mathbb{R}$. Then

$$\int_{\mathbb{R}} |f_1(y) f_2(x-y)|dy \leqslant C \int_{\mathbb{R}} |f_1(y)|dy = C\|f_1\|_{L^1}.$$

This implies existence and boundedness of $f * g$. Next we prove it is continuous. Let $x_0 \in \mathbb{R}$. Assume $|f_1(x)|, |f_2(x)| < C$ for all $x \in \mathbb{R}$, and assume $g \neq 0$. For a given $\epsilon > 0$ there is a $T > |x_0|$ such that

$$\int_{|y|>T} |f_2(y)|dy < \frac{\epsilon}{4C}.$$

Since a continuous function on a closed bounded interval is uniformly continuous, there exists a $\delta > 0$ such that

$$|x| < 2T, \quad |x - x'| < \delta \implies |f_1(x) - f_1(x')| < \frac{\epsilon}{2\|f_2\|_{L^1}}.$$

Then for $|x - x_0| < \delta$ we have

$$\begin{aligned}
&|\int_{-T}^{T} f_1(y) f_2(x-y)dy - \int_{-T}^{T} f_1(y) f_2(x_0 - y)dy| \\
&\qquad \leqslant \int_{-T}^{T} |f_1(x-y) - f_1(x_0 - y)||f_2(y)|dy \\
&\qquad \leqslant \frac{\epsilon}{2\|f_2\|_{L^1}} \int_{-T}^{T} |f_2(y)|dy \leqslant \frac{\epsilon}{2}.
\end{aligned}$$

And

$$\int_{|y|>T} |f_1(x-y) - f_1(x_0-y)||f_2(y)| \leqslant 2C \int_{|y|>T} |f_2(y)| < \frac{\epsilon}{2}$$

Combining these results imply that for $|x - x_0| < \delta$ we have

$$|f_1 * f_2(x) - f_1 * f_2(x_0)| < \epsilon.$$

$\square$

We now prove an analogous result of theorem 1.8 , but for the Fourier transform. This is known as the convolution theorem.

Theorem 2.7. *Let $f_1, f_2 \in L^1_{bc}(\mathbb{R})$, define $f = f_1 * f_2$ then $f \in L^1_{bc}$ and*

$$\hat{f}(y) = \hat{f}_1(y)\hat{f}_2(y). \tag{2.6}$$

Proof.

$$\begin{aligned}
\hat{f}(y) &= \int_{\mathbb{R}} f(x)e^{-2\pi ixy}dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f_1(z)f_2(x-z)dze^{-2\pi ixy}dx \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f_2(x-z)e^{-2\pi ixy}dxf_1(z)dz \\
&= \int_{\mathbb{R}}\int_{\mathbb{R}} f_2(x')e^{-2\pi ix'y}dx'f_1(z)e^{-2\pi izy}dz \\
&= \hat{f}_1(y)\hat{f}_2(y).
\end{aligned}$$

□

Note the penultimate step is intergration by substitution(where $x' = x - z$). And we justify the exchange of integration by Fubini's theorem, the interested reader is reffered to [1, Sec.7.2]

We now state a result also known as Plancherel's theorem, because it is an analogous result for the Plancherel's theorem we have seen before, but once again in a different setting.

We first define $L^2_{bc}(\mathbb{R})$ to be the set of all bounded continuous functions $\mathbb{R} \to \mathbb{R}$ satisfying,

$$\|f\|^2_{L^2(\mathbb{R})} = \int_{\mathbb{R}} |f(x)|^2 dx < \infty \tag{2.7}$$

It is easily checked that this is a norm, and $L^1_{bc}(\mathbb{R}) \subset L^2_{bc}(\mathbb{R})$ as if $f \in L^1_{bc}\mathbb{R}$, then $|f(x)| < C$ for all $x \in \mathbb{R}$ and for some C. Then

$$|f(x)|^2 \leqslant C|f(x)|, \implies \int_{\mathbb{R}} |f(x)|^2 dx \leqslant C\|f\|_{L^1}$$

Remark 2.8. L^1_{bc} is also an algebra for pointwise multiplacation, as if $f, g \in L^1_{bc}$, then $\|fg\|_{L^1} \leqslant (\sup_{x\in\mathbb{R}} f)(\|g\|_{L^1}) \leqslant \infty$

Theorem 2.9 (Plancherel's Theorem). *If $f \in L^1_{bc}(\mathbb{R})$ then $\hat{f} \in L^2_{bc}(\mathbb{R})$ and*

$$\|f\|_{L^2(\mathbb{R})} = \|\hat{f}\|_{L^2(\mathbb{R})}. \tag{2.8}$$

Consider the corresponding innner product associated with the $L^2(\mathbb{R})$ norm i.e if $f, g \in L^2_{bc}(\mathbb{R})$ then

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \int_{\mathbb{R}} f(x)\overline{g(x)}dx. \tag{2.9}$$

With this in mind we now state a corollary of Plancherel's theorem.

Corollary 2.10 (The Parseval formula). *Let $f, g \in L^2_{bc}(\mathbb{R})$ then*

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R})}. \tag{2.10}$$

We now aim to prove the Parseval formula. But first we need a lemma.

Lemma 2.11. *let V be a inner product space, and let $f, g \in V$, then*

$$\langle f, g\rangle = \frac{1}{4}(\|f+g\|^2 - \|f-g\|^2) + \frac{i}{4}(\|f+ig\|^2 - \|f-ig\|^2)$$

Proof. For the first summand, we expand to see

$$\begin{aligned}\|f+g\|^2 - \|f-g\|^2 &= \|f\|^2 + \|g\|^2 + 2\Re\langle f, g\rangle - (\|f\|^2 + \|g\|^2 - 2\Re\langle f, g\rangle) \\ &= 4\Re\langle f, g\rangle\end{aligned}$$

For the second summand, by a similar calculation we achieve

$$\|f+ig\|^2 - \|f-ig\|^2 = 4\Im\langle f, g\rangle$$

and hence we achieve the result. □

Proof of The Parseval formula. From the above lemma we see

$$\langle f, g\rangle_{L^2(\mathbb{R})} = \frac{1}{4}(\|f+g\|^2_{L^2(\mathbb{R})} - \|f-g\|^2_{L^2(\mathbb{R})}) + \frac{i}{4}(\|f+ig\|^2_{L^2(\mathbb{R})} - \|f-ig\|^2_{L^2(\mathbb{R})})$$

But also,

$$\langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R})} = \frac{1}{4}(\|\hat{f}+\hat{g}\|^2_{L^2(\mathbb{R})} - \|\hat{f}-\hat{g}\|^2_{L^2(\mathbb{R})}) + \frac{i}{4}(\|\hat{f}+i\hat{g}\|^2_{L^2(\mathbb{R})} - \|\hat{f}-i\hat{g}\|^2_{L^2(\mathbb{R})})$$

From linearity of the Fourier transform and Plancherel's theorem we can equate the above expressions to obtain $\langle f, g\rangle_{L^2(\mathbb{R})} = \langle \hat{f}, \hat{g}\rangle_{L^2(\mathbb{R})}$.

□

We pause here to look at the applications of the Fourier transform in probability.We consider a random variable X, the characteristic function is defined as the expected value of $\mathbb{E}[e^{it\underline{\mathrm{X}}}]$ i.e,

$$\begin{cases} \psi_X : \mathbb{R} \to \mathbb{C} \\ \psi_X(t) = \mathbb{E}[e^{it\underline{\mathrm{X}}}] = \int_{\mathbb{R}} e^{it\underline{\mathrm{X}}} f_X(x). \end{cases},$$

where f_X is the probability density function on X.

Now we notice that the above expression is the Fourier transform of f_X with a sign change in the exponential function.

It is a basic result of probability that the characteristic function completely determines the random variable X, so taking the fourier transform of a probability density function will often simplify a problem, as we will see in 4.

3. Fourier analysis on LCA groups

In this section we will use the convention that an LCA group is a locally compact, σ-compact abelian group.

Let A be a LCA group we will denote the set of characters i.e the dual group by $\hat{A}$ and we will use a result of representation theory that a representation of a finite abelian group is of dimension 1 and therefore given by a map $\chi : A \to \mathbb{T}$. We define $L^1_{bc}(A)$ and $L^2_{bc}(A)$ as in section 1, although now we generalise to a Haar integral on A.

Let $f \in L^1_{bc}(A)$ then $\hat{f} : \hat{A} \to \mathbb{C}$ is the Fourier transform of f defined by

$$\hat{f}(\chi) = \int_A f(x)\overline{\chi(x)}. \tag{3.1}$$

It is worth pausing at this point and checking our definition of Fourier transform here ties in with section 1. After all $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ are LCA groups.

In the case of $\mathbb{R}$ let $\phi_x(y) = e^{2\pi ixy}$ be a character, then for $f \in L^1_{bc}(\mathbb{R})$ we have

$$\hat{f}(\phi_x) = \int_{\mathbb{R}} f(y)\phi_x(y)^* dy = \int_{-\infty}^{\infty} f(y)e^{2\pi ixy} dy = \hat{f}(x)$$

Remark 3.1. It is worth noting here that $\mathbb{R}$ and $\hat{\mathbb{R}}$ are isomorphic as LCA groups via the mapping

$$\begin{cases} \mathbb{R} \to \widehat{\mathbb{R}} \\ x \mapsto \phi_x \end{cases},$$

so by convention we write $\hat{f}(x)$ instead of $\hat{f}(\phi_x)$ but this doesnt affect the fundamental idea.

For $\mathbb{R}/\mathbb{Z}$ the dual group is isomorphic to the group $\mathbb{Z}$ again via

$$\begin{cases} \mathbb{Z} \to \widehat{\mathbb{R}/\mathbb{Z}} \\ k \mapsto \phi_k \end{cases},$$

where $\phi_k(x) = e^{2\pi i k x}$, so in light of the above remark we may write

$$\hat{f}(k) = \int_{\mathbb{R}/\mathbb{Z}} f(y) e^{-2\pi i k y} dy = c_k(f)$$

So here we have recovered the $k'th$ Fourier coefficient. Our approach to Fourier analysis on groups shows us how the Fourier series and the Fourier transform are in fact very related, the only difference being the group in question.

Theorem 3.2 (Convolution Theorem)**.** *Let $f, g \in L^1_{bc}(A)$, then*

$$f * g(x) = \int_A f(xy^{-1}) g(y) dy$$

exists for every $x \in A$ and $f * g \in L^1_{bc}(A)$. Moreover

$$\widehat{f * g(\chi)} = \hat{f}(\chi)\hat{g}(\chi).$$

Proof. We only prove the second statement i.e $\widehat{f * g(\chi)} = \hat{f}(\chi)\hat{g}(\chi)$.

$$\begin{aligned}\widehat{f * g(\chi)} &= \int_A f * g(x)\chi(x)^* dx \\ &= \int_A \int_A f(xy^{-1})g(y)\chi(x)^* dydx \\ &= \int_A \int_A f(y^{-1}x)g(y)\chi(x)^* dxdy \\ &= \int_A \int_A f(x)g(y)\chi(yx)^* dxdy \\ &= \int_A f(x)\chi(x)^* dx \int_A g(y)\chi(y)^* dy \\ &= \hat{f}(\chi)\hat{g}(\chi)\end{aligned}$$

Again, the justification of exchanging integrals is Fubini's Theorem, and the interested reader is referred to [1, Sec.7.2]

□

Theorem 3.3 (Plancherel's Theorem)**.** *If* $f \in L^1_{bc}(A)$ *then* $\hat{f} \in L^2_{bc}(\hat{A})$ *and*

$$\|f\|_{L^2(A)} = \|\hat{f}\|_{L^2(\hat{A})}$$

Remark 3.4. Theorems 2.2 and 2.3 are a direct generalisation of the theorems in section 1.

Theorem 3.5 (Fourier Inversion Theorem)**.** *Let* A *be an LCA group and* f *be a function on* A*, then*

$$f(x) = \int \rho(x)\widehat{f}(\rho)d\rho$$

Theorem 3.6 (Pontryagin Duality)**.** *The map*

$$\begin{cases} A \to \hat{\hat{A}} \\ a \mapsto \delta_a \end{cases},$$

Where $\delta_a(\chi) = \chi(a)$ *is an isomorphism of LCA groups.*

We can pause here and check that the groups we have studied in chapter 1 i.e $\mathbb{R}$ and $\mathbb{R}/\mathbb{Z}$ do indeed have this property. As previously stated in remark 3.1 (3.1) $\mathbb{R}$ is isomorphic to $\hat{\mathbb{R}}$. So this directly implies $\mathbb{R}$ to isomorphic to $\hat{\hat{\mathbb{R}}}$.

We can also show that the dual group of $\mathbb{Z}$ is isomoprhic to $\mathbb{R}/\mathbb{Z}$. This is shown via the map

$$\begin{cases} \mathbb{R}/\mathbb{Z} \to \widehat{\mathbb{Z}} \\ x \mapsto \phi_x \end{cases},$$

where $\phi_x(k) = e^{2\pi i k x}$.

And similarily the dual group of $\mathbb{R}/\mathbb{Z}$ is isomorphic to $\mathbb{Z}$. This is shown in (3.1)

Combining these two results confirms the Pontryagin duality for the groups we have seen in section 1.

We can generalise these examples further, in the case of $\mathbb{R}^n$ we can see that $\mathbb{R}^n$ and $\widehat{\mathbb{R}^n}$ are isomorphic as LCA groups via the mapping

$$\begin{cases} \mathbb{R}^n \to \widehat{\mathbb{R}}^n \\ \underline{x} \mapsto \phi_{\underline{x}} \end{cases},$$

where $\phi_{\underline{x}}(y) = e^{2\pi i (x.y)}$ and $(x.y)$ denotes the usual inner product.

We now pause to think about the structure of LCA groups, with some examples. To start off any compact abelian group is clearly an LCA group. Any countable abelian group with the discrete topology is a LCA group. An exmaple of a LCA group which isnt compact is $\mathbb{R}$.

We now go on to prove two structure theorems of LCA groups, which help us break down LCA in to simpler expressions.

Theorem 3.7. *If A is a LCA group then A is isomorphic to a group of the form $\mathbb{R}^n \times H$ for some $n \in \mathbb{N}_0$, such that H is a LCA group which contains an open compact subgroup.*

Definition 3.8. A group A is said to be compactly generated if $A =< U >= \bigcup_{n\in\mathbb{N}_0} U^n$ for some compact neighbourhood U of e

For example the group $(\mathbb{R}, +)$ is compactly generated by $[-1, 1]$, $(\mathbb{Z}, +)$ is compactly generated by $-1, 0, 1$ and $(\mathbb{T}, \times)$ is compactly generated by $e^{2\pi ix}$ for $x \in [0, \frac{1}{2}]$

Theorem 3.9. *A compactly generated LCA group, A, is isomorphic to a group of the form $\mathbb{R}^n \times \mathbb{Z}^m \times K$ for some compact group K and $m, n \in \mathbb{N}_0$.*

4. Fourier analysis on finite groups

Everything in Section 1 and 2 , has been theory related to abelian groups. In this section we no longer have this restriction. So now when we have a representation of a group G, we can no longer say this is a map from G to $\mathbb{T}$. We now have a (matrix) representation is a map $G \to GL(n, \mathbb{F})$ where $GL(n, \mathbb{F})$ is the set of all $n \times n$ invertable matrices over a arbitary field $\mathbb{F}$. For this reason our definition of our Fourier transform varies slightly

Let G be a finite group with the set of corresponding matrix representations denoted $Rep(G)$. Let $f : G \to \mathbb{C}$, and $\rho \in Rep(G)$ then we define

$$\hat{f}(\rho) = \frac{1}{|G|} \sum_{g\in G} f(g)\rho(g)^*.$$

Remark 4.1. A closer inspection at the above formula shows that the Fourier transform at any given representation is indeed a matrix. We also note that as we have the restricition of only dealing with finite groups, our Haar measure becomes a summation.

Theorem 4.2 (Fourier Inversion Theorem)**.** *Let $\hat{G}$ denote the set of all equivalence classes of irreducible representations of G where two representations are said to be in the same equivalence class if their representations are equivalent. For each class in $\hat{G}$ choose a fixed representative ρ_i. Let f be a function on G, then*

$$f(s) = \frac{1}{|G|} \sum_{i} d_i Tr(\rho_i(s^{-1})\hat{f}(\rho_i))^*,$$

where d_i is the dimension of the irreducible representation ρ_i.

This theorem says if we know the values that $\hat{f}$ takes on $\hat{G}$ (or in particular just all irreducible representations) then we can retrieve the original function f.

Theorem 4.3 (Plancherel's Formula for Finite groups)**.** *Let f be a function on G, then*

$$\|f\|^2_{L^2(G)} = \frac{1}{|G|}\sum_{\rho_i} d_i \|\hat{f}(\rho_i)\|^2_{HS},$$

where d_i is the dimension of the irreducible representation ρ_i and $\|.\|_{HS}$ is the Hilbert Schmidt norm, that is $\|\pi\|^2_{HS} = tr(\pi^*\pi)$, for a square matrix π

Remark 4.4. Noting that all abelian group representations are of dimension one hence irreducible, if we restrict the above formula to abelian groups we do retrieve Plancherel's formula in the abelian case.

Theorem 4.5 (Convolution Theorem)**.** *Let f, g be functions $G \to \mathbb{C}$, then*

$$f * g(x) = \frac{1}{|G|}\sum_{y \in G} f(xy^{-1})g(y), \quad x \in G$$

and

$$\widehat{f * g(\rho)} = \hat{g}(\rho)\hat{f}(\rho), \quad \rho \in Rep(G).$$

The convolution theorem has many applications in probability, in particular on random walks.

Let $Q(g)_{g \in G}$ be a probability distrubution on G. Then define Q^{*k} to be Q convolved with itself k times. i.e $Q^{*3}(g) = Q * Q * Q(g)$. The interpretation of this in the context of probability is $Q^{*k}(g)$ is the probability that a random walk on G generated by picking elements repeatedly with weight givent by Q is at g after k steps. All walks start at the indentity.

In light of the convolution theorem we can simplify the problem of finding the probability of the state of a random walk on a group after k steps. We know the probability in question is given by $Q^{*k}(g)$, if we take the Fourier transform of this it becomes $\widehat{Q^{*k}(\rho)} = \widehat{Q}^k(\rho) \quad \rho \in Rep(G)$. The problem has now been reduced to simple matrix multiplication.

Theorem 4.6. *The uniform distribution is defined by $U(s) = \frac{1}{|G|} \quad s \in G$, where $|G|$ is the order of the group G. Then at the trivial representation $\widehat{U}(\rho) = \frac{1}{|G|}$ and at any non-trivial irreducible representation $\widehat{U}(\rho) = 0$.*

Proof. First we compute the Fourier transform at the trivial representation. Let ρ be the trivial representation, from our definition we see that

$$\widehat{U}(\rho) = \frac{1}{|G|}\sum_{s\in G}\frac{1}{|G|}.1 = \frac{1}{|G|}.$$

Now let ρ be any non-trivial representation. It suffices to show

$$\sum_{s\in G}\rho(s) = 0.$$

As a result of Schur's lemma [2, Chap.2B] we conclude that (as for any $g \in G \quad \rho(g)\sum_{s\in G}\rho(s) = (\sum_{s\in G}\rho(s))\rho(g))$

$$\sum_{s\in G}\rho(s) = \lambda I,$$

where λ is a constant and I is the identity . But

$$\rho(g)\sum_{s\in G}\rho(s) = \sum_{s\in G}\rho(s) = \lambda I,$$

so $\lambda = 0$.

□

5. Fourier analysis on compact groups

We use the convention that a compact group K is a compact topological group which is Hausdorff.

It is a result of representation theory that every representation of a compact group is a direct sum of irreducibles, and that every irreducible representation is finite dimensional. Furthermore, every finite dimensional representation may be assumed to be unitary. See [3, Sec.3.1]

We now want to discuss the Peter Weyl Theorem, but first we need some preparation.

Once again we let $\hat{K}$ denote the set of all equivalence classes of irreducible representations of K where two representations are in the same equivalence class if they are equivalent. For each class in $\hat{K}$ choose a fixed representative (τ, V_τ). Choose an orthonormal basis $e_1, .., e_n$ of V_τ and let

$$\tau_{i,j}(k) = \langle \tau(k)e_i, e_j\rangle.$$

The map $\tau_{i,j} : K \to \mathbb{C}$ is called the (i,j)th matrix coefficient of τ.

Theorem 5.1 (Peter Weyl Theorem).

The family $(\sqrt{d_\tau})\tau_{i,j})_{\tau,i,j}$ *form an orthonormal basis of the Hilbert space* $L^2(K)$.

We will now reformulate the Peter Weyl Theorem to express it in a more familiar form.

Let $f \in L^2(K)$, we express f in terms of the orthonormal basis stated above. i.e

$$f = \sum_\tau \sum_{i,j} \langle f, \sqrt{d_\tau}\tau_{i,j}\rangle_{L^2} \sqrt{d_\tau}\tau_{i,j} \tag{5.1}$$

Now we use orthonormality to compute

$$\langle f, f\rangle = \|f\|^2_{L^2} = \sum_\tau \sum_{i,j} |\langle f, \sqrt{d_\tau}\tau_{i,j}\rangle_{L^2}|^2$$

$$= \sum_\tau d_\tau \sum_{i,j} (\int f\overline{\tau_{ij}})\overline{(\int f\overline{\tau_{ij}})}$$

but

$$[\hat{f}(\tau)]_{ij} = \int f(x)\overline{\tau_{ji}}(x)dx$$

and so

$$\overline{(\int f\overline{\tau_{ji}})} = \overline{[\hat{f}(\tau)]_{ij}}$$

putting these results together we obtain

$$\|f\|^2_{L^2} = \sum_\tau d_\tau tr[\hat{f}(\tau)^* \hat{f}(\tau)].$$

This may be viewed as a generalisation of Plancherel's theorem, and if wee restrict the case to K being and LCA group or a finite group we will retrieve the Plancherel's theorem we have seen before.

Also note how (5.1) can be rearranged to give

$$= \sum_\tau d_\tau \sum_{i,j} \langle f, \tau_{i,j}\rangle_{L^2} \tau_{i,j}$$

which may be viewed as a generalisation of the Fourier inversion formulae.

We now examine the Peter Weyl theorem further, in particular the equivalences $\widehat{K}$.

For any $\tau \in Rep(K)$ let $\varepsilon_\tau = span\{\tau_{ij}\} \subset L^2(K)$ where $1 \leqslant i, j \leqslant d_\tau$. It can be shown that ε_τ depends only on $[\tau]$ i.e, if we take two equivilant representations, τ_1, τ_2 then $\varepsilon_{\tau_1} = \varepsilon_{\tau_2}$

Theorem 5.2. *Let $\varepsilon_\tau \subset L^2(K)$ be defined as above and let λ be the representation given by*

$$\begin{cases} K \to GL(L^2(K)) \\ (\lambda(k_0))f(k) = f(k_0^{-1}k) \end{cases},$$

then ε_τ is a stable subspace of $\varepsilon(K)$ under λ

Proof. It suffices to show that for all $k_0 \in K$

$$(\lambda(k_0))\tau_{ij} \in \varepsilon_\tau.$$

We begin with

$$(\lambda(k_0))\tau_{ij}(k) = \tau_{ij}(k_0^{-1}k) = [\tau(k_0^{-1}k)]_{ij}$$

Now noting that τ is a homomorphism we see that $\tau(k_0^{-1}k) = \tau(k_0)^{-1}\tau(k)$ and hence

$$= \sum_l [\tau(k_0)^{-1}]_{il}\tau_{lj}(k)$$

So overall we obtain the result

$$(\lambda(k_0))\tau_{ij} = l.i\tau_{ij}(k), \quad 1 \leqslant j \leqslant d_\tau$$

Which is a linear combination of elements of ε_τ

□

References

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2. P. Diaconis,
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