SUB-RIEMANNIAN SANTALÓ FORMULA AND APPLICATIONS

joint work with D. Prandi and L. Rizzi [arXiv:1509.05415]

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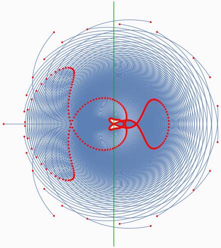
OUTLINE

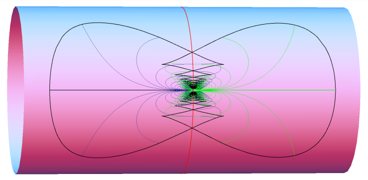


- Motivations
- Riemannian Santaló formula
- Sub-Riemannian Santaló formula
- Reduction procedure
- Applications

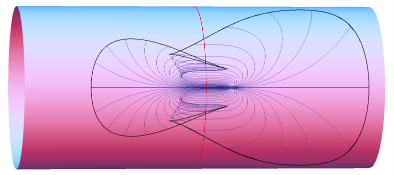
MOTIVATIONS

Understand spectral properties of Laplace-Beltrami opeartor on almost Riemannian geometries that present **classical tunnelling** with **quantum confinement** (in progress).

But most of the usual “tools” are missing.



(a) geodesics starting from the singular set (red circle)



(b) geodesics starting from the the point (.3,0)

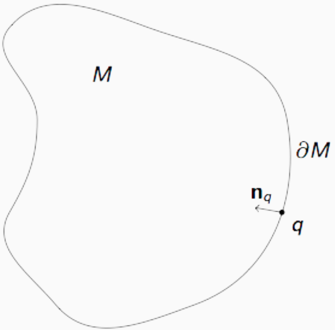
Figure 1: Geodesics for Grushin cylinder; red singular set; black wave front

RIEMANNIAN SANTALÓ FORMULA

RIEMANNIAN SANTALÓ FORMULA: INTRODUCTION

- Classical formula in integral geometry [Santaló, 1976]
- Allows to decompose integrals on a manifold with boundary in integrals along the geodesics starting from the boundary
- Deep consequences, e.g. [Croke 1980–1987]:
  - Isoperimetric inequalities
  - Geometric inequalities (Hardy, Poincarè)
  - Lower bounds for Laplace-Beltrami operator's spectrum

RIEMANNIAN SANTALÓ FORMULA: SETTING



- (M, g) Riemannian manifold
  - compact
  - with boundary $\partial M \neq \emptyset$
- $d\omega = \mathrm{vol}_{\mathrm{g}}$ Riemannian volume form
- $\mathbf{n}$ unit inward pointing vector

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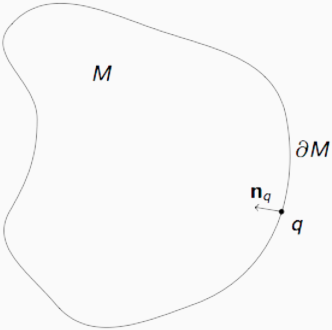

- Unit tangent bundle

$$UM = \{v \in TM \mid |v| = 1\}$$

- Geodesic flow $\phi_t : UM \to UM$,

$$\phi_t(v) = \dot{\gamma}_v(t)$$

- Exit time $\ell(v) \in [0, +\infty]$,

$$\ell(v) = \sup\{t \geq 0 \mid \gamma_v(t) \in M\}$$

- Visible unit tangent bundle

$$U^{\star}M = \{v \in UM \mid \ell(-v) < +\infty\}$$

Let $\Theta := d\dot{q} \wedge dq$ be the *Liouville measure* (natural measure on TM with coordinate $(q, \dot{q})$). From it we can derive

- the *Liouville surface measure* $d\mu$ on UM
- the measure $d\eta_q$ on fibers above q, i.e.

$$\int_{UM} F \, d\mu = \int_M \left(\int_{U_q M} F(q, \dot{q}) \, d\eta_q(\dot{q}) \right) d\omega(q)$$

In coordinates, $d\eta$ is the standard measure on $U_q M \cong \mathbb{S}^{n-1}$.

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- the measure $d\eta_q$ on fibers above q

Theorem (Santaló formula)

Let $F : UM \to \mathbb{R}$ measurable. Then

$$\int_{U^{\sunrise}M} F \, d\mu = \int_{\partial M} d\sigma(q) \int_{U_q^+ \partial M} d\eta(v) \int_0^{\ell(v)} dt \, F(\phi_t(v)) \, \mathsf{g}(v, \mathsf{n}_q)$$

where $U_q^+ \partial M := \{v \in U_q M \mid q \in \partial M, \mathsf{g}(v, \mathsf{n}_q) > 0\}$ is the set of unit inward pointing vectors on the boundary.

Choosing $F \equiv 1$ in the Santaló formula one finds

Theorem (Croke)

Let $\theta^{*} \in [0,1]$ be the visibility angle of M. Then

$$\frac{\sigma(\partial M)}{\omega(M)} \geq \frac{2\pi|\mathbb{S}^{n-1}|}{|\mathbb{S}^n|}\frac{\theta^{*}}{\mathrm{diam}(M)}.$$

Equality holds iff M is isometric to an hemisphere.

Observing that any $f : M \to \mathbb{R}$ can be lifted to $F : UM \to \mathbb{R}$ by $F(q, \dot{q}) = f(q)$, one also finds

Theorem (Croke, Derdzinski)

Let $f \in C_0^\infty(M)$. Then

$$\int_M |\nabla f|^2 \, d\omega \geq \frac{n\pi^2}{|\mathbb{S}^{n-1}|L^2} \int_M f^2 d\omega \quad \Longrightarrow \quad \lambda_1(M) \geq \frac{n\pi^2}{|\mathbb{S}^{n-1}|L^2},$$

where $L \leq \infty$ is the length of the longest Riemannian geodesic contained in M. The equality for the eigenvalue lower bound holds iff M is isometric to an hemisphere.

SUB-RIEMANNIAN SANTALÓ FORMULA

Definition

A *sub-Riemannian manifold* is a triple $(M, D, \mathbf{g})$ where

- M is an connected smooth manifold of dimension $n \geq 3$;
- $D \subset TM$ is a smooth distribution of constant rank $k \leq n$, i.e. a smooth map that associates to $q \in M$ a k-dimensional subspace $D_q \subseteq T_qM$ satisfying the *Hörmander condition*:

$$\forall q \in M, \qquad \operatorname{span}\{[X_1, [\dots [X_{j-1}, X_j]]](q) \mid X_i \in \overline{D}, j \in \mathbb{N}\} = T_qM,$$

  where $\overline{D} = \{X \in Vec(M) \mid X_q \in D_q,\ \forall q \in M\}$ denotes the set of *horizontal vector fields on M*;
- $\mathbf{g}_q$ is a smooth Riemannian metric on D_q, smooth as function of q.

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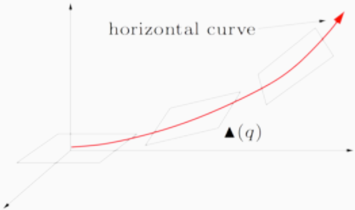
In the rest of the talk:

- M compact
- $\partial M \neq \emptyset$
- ω fixed smooth volume on M



Sub-Riemannian distance:

$$d(p,q) := \inf \left\{ \int_0^T \mathbf{g}(\dot\gamma(t), \dot\gamma(t))\, dt \;\middle|\; \dot\gamma(t) \in D_{\gamma(t)},\ \text{and } (\gamma(0), \gamma(T)) = (p,q) \right\}$$

Remark

Thanks to the Hörmander condition, (M, d) is a metric space with the same topology as the original one of M (Chow-Rashevsky theorem).

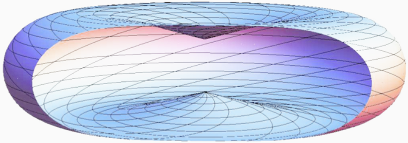


Figure 2: The Heisenberg Sub-Riemannian sphere

- spheres are highly non-isotropic
- there are geodesics loosing optimality close to the starting point $\Longrightarrow$ spheres are never smooth even for small time
- the Hausdorff dimension is bigger than the topological dimension (unless it is actually a Riemannian manifold)

SUB-RIEMANNIAN SANTALÓ FORMULA: FIRST OBSTACLES

- **Boundary**: the sub-Riemannian normal is the inward-pointing unit vector $\mathbf{n}_q \in D_q$ such that

$$\mathbf{n}_q \perp v, \qquad \forall v \in T_q\partial M \cap D_q.$$

When D_q is tangent to ∂M, $\mathbf{n}_q$ is not well defined.

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$\Rightarrow$ **Proposition** [Prandi, Rizzi, S]: the set of points such that D_q is tangent to ∂M is negligible.

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  $\Rightarrow$ **Proposition** [Prandi, Rizzi, S]: the set of points such that D_q is tangent to ∂M is negligible.
- **Geodesic flow**: since the initial velocities of the geodesics are constrained to $D \subset TM$, more than one geodesic can start with the same velocity.
  $\Rightarrow$ there is no geodesic flow $\phi_t : TM \to TM$ **(!!!)**

Consider the Hamiltonian $H : T^*M \to \mathbb{R}$,

$$H(q,p) = \frac{1}{2}\sum_{i=1}^{k}\langle p, X_i(q)\rangle^2$$

Theorem (Pontryagin Maximum Principle for normal extremals)

For any $\xi_0 \in T^*M$, the solution $\xi : [0,T] \to T^*M$ with $\xi(0) = \xi_0$ of the Hamiltonian system

$$\dot{\xi}(t) = \vec{H}(\xi(t)), \qquad \vec{H} = \frac{\partial H}{\partial q}\frac{\partial}{\partial p} + \frac{\partial H}{\partial p}\frac{\partial}{\partial q}$$

projects to a minimiser $\gamma = \pi \circ \xi : [0,T] \to M$ of the sub-Riemannian distance.

GEODESICS IN SUB-RIEMANNIAN GEOMETRY: HAMILTONIAN POV

Consider the Hamiltonian $H : T^*M \to \mathbb{R}$,

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- Normal minimizers parametrized by the initial covector
- Sub-Riemannian geodesic flow $\phi_t : T^*M \to T^*M$

Consider the sub-Riemannian geodesic flow $\phi_t : U^*M \to U^*M$ with

$$U^*M := \{\xi \in T^*M \mid H(\lambda) = 1/2\}$$

The Liouville measure on the cotangent bundle T^*M still allows to define a Liouville surface measure $d\mu$ on U^*M and a *vertical measure* $d\eta_q$ on the fibers U_q^*M.

Theorem (Prandi, Rizzi, S)

Let $F : U^*M \to \mathbb{R}$ measurable. Then

$$\int_{U^*M} F \, d\mu = \int_{\partial M} d\sigma(q) \int_{U_q^+ \partial M} d\eta(\xi) \int_0^{\ell(\xi)} dt \, F(\phi_t(\xi)) \, \langle \xi, \mathsf{n}_q \rangle$$

where $U_q^+\partial M = \{\xi \in U_q^*\partial M \mid \langle \xi, \mathsf{n}_q \rangle \geq 0\} \subset U_q^*M$ is the set of inward pointing unit covectors on the boundary.

HAMILTONIAN SANTALÓ FORMULA

Theorem (Prandi, Rizzi, S)

Let $F : U^*M \to \mathbb{R}$ measurable. Then

$$\int_{U^*M} F \, d\mu = \int_{\partial M} d\sigma(q) \int_{U_q^+\partial M} d\eta(\xi) \int_0^{\ell(\xi)} dt \, F(\phi_t(\xi)) \, \langle \xi, \mathsf{n}_q \rangle$$

where $U_q^+\partial M = \{\xi \in U_q^*\partial M \mid \langle \xi, \mathsf{n}_q \rangle \geq 0\} \subset U_q^*M$ is the set of inward pointing unit covectors on the boundary.

Problem: the typical fiber $U_q^*M \cong \mathbb{S}^{k-1} \times \mathbb{R}^{n-k}$ is not compact!
$\Longrightarrow$ Choosing $F \equiv 1$ or lifting $f : M \to \mathbb{R}$ as before gives ∞ on both sides of the equation.

REDUCTION PROCEDURE

SUB-RIEMANNIAN SANTALÓ FORMULA: REDUCTION PROCEDURE

First proposed by [Pansu 1985] in the Heisenberg group and [Chanillo-Young 2009] for 3D Sasakian manifolds.

- Assume that you can fix $\mathcal{V} \subset TM$ such that $TM = D \oplus \mathcal{V}$.
  Equiv. you can fix a Riemannian metric $\hat{\mathrm{g}}$ on M such that $\hat{\mathrm{g}} \restriction_D = \mathrm{g}$ and $\mathrm{vol}_{\mathrm{g}} = \omega$.
  $\Rightarrow T^*M = D^\perp \oplus \mathcal{V}^\perp$
- The *reduced cotangent bundle* is $T^*M^{\mathrm{r}} := T^*M \cap \mathcal{V}^\perp$ and similarly $U^*M^{\mathrm{r}} = U^*M \cap \mathcal{V}^\perp$

Example: in the Heisenberg group, given by the vector fields on $\mathbb{R}^3$

$$X = \begin{pmatrix} 1 \\ 0 \\ -\frac{1}{2}y \end{pmatrix}, \qquad Y = \begin{pmatrix} 0 \\ 1 \\ \frac{1}{2}x \end{pmatrix},$$

the geodesics with covectors in $U_q^*M^{\mathrm{r}}$ at (x, y, z) span the plane orthogonal to $(y/2, -x/2, 1)$.

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Theorem (Prandi, Rizzi, S)

Let $F : U^*M^{\mathrm{r}} \to \mathbb{R}$. Under some stability assumptions on the complement $\mathcal{V}$,

$$\int_{U^*M^{\mathrm{r}}} F \, d\mu^{\mathrm{r}} = \int_{\partial M} d\sigma(q) \int_{U_q^*\partial M^{\mathrm{r}}} d\eta^{\mathrm{r}}(\xi) \int_0^{\ell(\xi)} dt \, F(\phi_t(\xi)) \, \langle \xi, n_q \rangle.$$

Now, $U_q^*M^{\mathrm{r}} \cong \mathbb{S}^{k-1}$ is **compact**!

APPLICATIONS

ISOPERIMETRIC-TYPE INEQUALITY

Theorem (Prandi, Rizzi, S)

Let $\theta^{\ast} \in [0,1]$ *be the (reduced) visibility angle of* M*, then*

$$\frac{\sigma(\partial M)}{\omega(M)} \geq \frac{2\pi|\mathbb{S}^{k-1}|}{|\mathbb{S}^{k}|} \frac{\theta^{\ast}}{\mathrm{diam}^{\mathrm{r}}(M)}$$

where $\mathrm{diam}^{\mathrm{r}}(M)$ *is the reduced sub-Riemannian diameter of* M*.*

Remark

- it is always true that $\mathrm{diam}^{\mathrm{r}}(M) \leq \mathrm{diam}(M)$
- $\mathrm{diam}^{\mathrm{r}}(M)$ is much easier to compute than $\mathrm{diam}(M)$
- In the Riemannian setting, to get rid of the diameter one needs curvature arguments that not available right now in the sub-Riemannian setting

The sub-Riemannian gradient of $f \in C^\infty(M)$ is, for a local orthonormal frame,

$$\nabla_H f = \sum_{i=1}^{k} (X_i f) X_i$$

Theorem (Prandi, Rizzi, S)

Let $f \in C_0^\infty(M)$, then

$$\int_M |\nabla_H f|^2 d\omega \geq \frac{k\pi^2}{L^2} \int_M f^2 \, d\omega$$

*where L is the length of the longest **reduced** geodesic.*

- In a similar way one can obtain the Hardy-like inequality

$$\int_M |\nabla_H f|^2 d\omega \geq \frac{k}{4|\mathbb{S}^{k-1}|} \int_M \frac{f^2}{r^2}\, d\omega$$

  where $\frac{1}{r^2} = \int_{U_q^* M^{\mathrm{r}}} \frac{1}{\ell^2(\xi)} d\eta^{\mathrm{r}}(\xi)$ is the *harmonic mean distance from the boundary* [Davies 1999]
- and p-Hardy inequalities for $p > 1$ (omitted)
- further improvements (e.g. getting additional correction terms or a refined constant) are possible but not treated in the paper

The Dirichlet sub-Laplacian is the (Friedrichs extension) of the operator $\mathcal{L}$ s.t.

$$\int_M \mathbf{g}(\nabla_H f, \nabla_H g)\, d\omega = \int_M (-\mathcal{L}f) g \, d\omega, \qquad \forall f, g \in C_c^\infty(M)$$

- By Hörmander condition, $-\mathcal{L}$ is hypoelliptic
- For a local orthonormal frame,

$$\mathcal{L} = \sum_{i=1}^k X_i^2 + \text{lower order terms}$$

- $\mathrm{spec}(M) = \{0 < \lambda_1 \leq \lambda_2 \leq \ldots \to \infty\}$

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Theorem (Prandi, Rizzi, S)

$$\lambda_1(M) \ge \frac{k\pi^2}{L^2}$$

FINAL REMARKS

- The results apply notably to two large classes of sub-Riemannian manifolds
  - Riemannian foliations with totally geodesic fibers (submersions, contact manifold with symmetries, CR manifolds, quasi-contact manifolds)
  - All Carnot groups (left-invariant nilpotent structures on $\mathbb{R}^n$)
- The isoperimetric inequality and the lower spectral bound are sharp for the hemispheres of the complex and quaternionic Hopf fibrations on the spheres (generalizes the sharpness by Croke)

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  - All Carnot groups (left-invariant nilpotent structures on $\mathbb{R}^n$)
- The isoperimetric inequality and the lower spectral bound are sharp for the hemispheres of the complex and quaternionic Hopf fibrations on the spheres (generalizes the sharpness by Croke)
- Iterating the reduction procedure allows in some cases to improve some of the estimates (not enough to make them sharp)
- Building on these ideas it is possible to prove a sharper Hardy inequality that, in particular, holds also for almost-Riemannian manifold with non-constant rank [Prandi, Rizzi, S - in preparation]

THANK YOU FOR THE ATTENTION!