HARMONIC ANALYSIS ON THE HEISENBERG GROUP AND RELATED TOPICS

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The Heisenberg group $\mathbb{H}_n$ plays a fundamental rôle in several branches of analysis, especially in non-commutative harmonic analysis, but also in sub-Riemannian geometric analysis. Indeed, $\mathbb{H}_n$ may be viewed as the simplest example of non-commutative nilpotent Lie group and sub-Riemannian manifold. Moreover, it is related to Euclidean phase-space analysis via the Schrödinger representation.

In this series of lectures, we will discuss the following topics:

- We will start with the definition of the Heisenberg group $\mathbb{H}_n$, in particular various equivalent realisations.
- We will then introduce important objects and structures considered on the Heisenberg group coming from sub-Riemannian geometry (e.g. the canonical sub-Laplacian, the horizontal distributions, the CC-distance etc.).
- We will discuss some harmonic analysis on the Heisenberg group in relation to Singular Integral theory and possibly representation theory.
- At the end of the lectures, we will discuss the recent progress on non-commutative phase-space analysis on the Heisenberg group and beyond.

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1. Definition(s) of the Heisenberg group

In this section, we discuss the usual definition(s) of the Heisenberg group as a Lie group.

Recall that a Lie group is a smooth manifold M equipped with a compatible group structure; the compatibility means that the multiplication and inverse operations $(x, y) \mapsto xy$ and $x \mapsto x^{-1}$ are smooth maps $M \times M \to M$ and $M \to M$ respectively.

1.1. Our preferred definition.

In these lectures, our preferred definition for the Heisenberg group $\mathbb{H}_n$ will be the following: it is the manifold $\mathbb{R}^{2n+1}$ equipped with the law

$$(x, y, t)(x', y', t') := (x + x', y + y', t + t' + \frac{1}{2}(xy' - x'y)), \tag{1.1}$$

where (x, y, t) and (x', y', t') are in $\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \sim \mathbb{H}_n$. In these lecture notes, we adopt the following convention: if x and y are two vectors in $\mathbb{R}^n$ for some $n \in \mathbb{N}$, then xy denotes their standard scalar product

$$xy = \sum_{j=1}^{n} x_j y_j \quad \text{if} \quad x = (x_1, \ldots, x_n),\ y = (y_1, \ldots, y_n).$$

It is straightforward to check that (1.1) defines a group law, with inverse $(x, y, t)^{-1} = (-x, -y, -t)$ and neutral element $0 = (0, 0, 0)$. These product and inverse operations are smooth, so $\mathbb{H}_n$ is indeed a Lie group.

1.2. Lie groups isomorphic to $\mathbb{H}_n$.

First we remark that the factor $\frac{1}{2}$ in the group law given by (1.1) is irrelevant in the following sense. Let $\alpha \in \mathbb{R}^* = \mathbb{R}\backslash\{0\}$. Consider the Lie group $\mathbb{H}_n^{(\alpha)}$ given by the manifold $\mathbb{R}^{2n+1}$ equipped with the law

$$(x, y, t)(x', y', t') := (x + x', y + y', t + t' + \frac{1}{\alpha}(xy' - x'y)).$$

Then the Lie groups $\mathbb{H}_n^{(\alpha)}$ and $\mathbb{H}_n = \mathbb{H}_n^{(2)}$ are isomorphic via

$$\left\{ \begin{array}{ccc} \mathbb{H}_n & \longrightarrow & \mathbb{H}_n^{(\alpha)} \\ (x, y, t) & \longmapsto & (x, y, \frac{2}{\alpha} t) \end{array} \right. . \tag{1.2}$$

In the same way, consider the *polarised Heisenberg group* $\tilde{\mathbb{H}}_n$ given by the manifold $\mathbb{R}^{2n+1}$ equipped with the law

$$(x, y, t)(x', y', t') := (x + x', y + y', t + t' + xy').$$

Then the Lie groups $\tilde{\mathbb{H}}_n$ and $\mathbb{H}_n$ are isomorphic via

$$\left\{ \begin{array}{ccc} \mathbb{H}_n & \longrightarrow & \tilde{\mathbb{H}}_n \\ (x, y, t) & \longmapsto & (x, y, t + \frac{1}{2} xy) \end{array} \right. .$$

Note that the Heisenberg group $\mathbb{H}_n$ can be also viewed as a matrix group. For simplicity, we consider $n = 1$, in which case the group $\tilde{\mathbb{H}}_1$ is isomorphic to T_3, the group of 3-by-3 upper triangular real matrices with 1 on the diagonal:

$$\left\{ \begin{array}{ccc} \tilde{\mathbb{H}}_1 & \longrightarrow & T_3 \\ (x, y, t) & \longmapsto & \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \end{array} \right. .$$

All the statements above can be readily checked by a straightforward computation. Combining two isomorphisms above, we obtain the identification $\mathbb{H}_1 \longrightarrow \tilde{\mathbb{H}}_1 \longrightarrow T_3$ given by

$$\left\{ \begin{array}{ccc} \mathbb{H}_1 & \longrightarrow & T_3 \\ (x,y,t) & \longmapsto & \begin{bmatrix} 1 & x & t+\frac{1}{2}xy \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix} \end{array} \right. .$$

1.3. **The Heisenberg group as the boundary of the complex sphere.** Here we consider the case $n = 1$ for the sake of clarity.

The Heisenberg group can be also realised as a group of transformations: for each

$$h = (x, y, t) \in \mathbb{H}_1,$$

the affine (holomorphic) map given by

$$\phi_h : \mathbb{C} \times \mathbb{C} \ni (z_1, z_2) \longmapsto (z_1 + x + iy, z_2 + t + 2iz_1(x - iy) + i(x^2 + y^2)) \in \mathbb{C} \times \mathbb{C},$$

sends the (Siegel) domain

$$\mathscr{U} := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \ : \ \Im z_2 > |z_1|^2\} \quad (= \mathrm{SU}(2, 1)/\mathrm{U}(2))$$

to itself, and the (Shilov) boundary of $\mathscr{U}$,

$$b\mathscr{U} := \{(z_1, z_2) \in \mathbb{C} \times \mathbb{C} \ : \ \Im z_2 = |z_1|^2\},$$

also to itself. One can check that $\mathbb{H}_1 \ni h \mapsto \phi_h$ defines an action of $\mathbb{H}_1$ on $\mathscr{U}$ and on $b\mathscr{U}$. Furthermore, the action of $\mathbb{H}_1$ on $b\mathscr{U}$ is simply transitive. The Cayley type transform

$$(w_1, w_2) \longmapsto (z_1, z_2) \quad \text{with} \quad z_1 = \frac{w_1}{1 + w_2}, \quad z_2 = i\frac{1 - w_2}{1 + w_2},$$

is a biholomorphic bijective mapping which sends $\mathscr{U}$ onto the unit complex ball of $\mathbb{C}^2$. It also send $b\mathscr{U}$ to the unit complex sphere $\mathbb{S}^3$, more precisely onto $\mathbb{S}^3\backslash\{S\}$ where $S = (0, -1)$ is the south pole (which may be viewed as the image of ∞). Hence the Heisenberg group acts simply transitively on $\mathbb{S}^3\backslash\{S\}$.

1.4. **The Heisenberg group from its Lie algebra.** *Recall* that a Lie algebra is a vector space $\mathfrak{L}$ equipped with a Lie bracket $[\cdot, \cdot]$, that is, a skew-symmetric bilinear operation $\mathfrak{L} \times \mathfrak{L} \to \mathfrak{L}$ satisfying the Jacobi identity:

$$\forall Z_1, Z_2, Z_3 \in \mathfrak{L} \qquad [Z_1, [Z_2, Z_3]] + [Z_3, [Z_1, Z_2]] + [Z_2, [Z_3, Z_1]] = 0.$$

Convention: In these lecture notes, unless otherwise stated, the Lie algebras are over the fields $\mathbb{R}$ of real numbers and are finite dimensional.

The (abstract) Heisenberg Lie algebra $\mathfrak{h}_n$ is the vector space $\mathbb{R}^{2n+1}$ with Lie brackets given on the canonical basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ by

$$[X_j, Y_j] = T, \quad j = 1, \ldots, n, \tag{1.3}$$

and all other relations being zero. These relations are known as *the canonical commutation relations.*

In particular $[T, X_j] = [T, Y_j] = 0$ for $j = 1, \ldots, n$ so $\mathbb{R}T$ is the centre of the Lie algebra $\mathfrak{h}_n$. Moreover, $[\cdot, [\cdot, \cdot]] = 0$. Consequently, $\mathfrak{h}_n$ is a nilpotent Lie algebra of step 2.

The recap in Section 1.5 below implies that there is a unique simply connected Lie group with Lie algebra $\mathfrak{h}_n$. It is the Lie group with underlying manifold $\mathbb{R}^{2n+1}$ with group law given by

$$X \star Y = X + Y + \frac{1}{2}[X, Y], \qquad X, Y \in \mathbb{R}^{2n+1}.$$

1.5. **Recap from elementary differential geometry and some Lie theory.** *Recall the following definitions*:

(1) What is a *vector field*?
   - Analysts' definition: A vector field $\tilde{X}$ on an open subset $U \subset \mathbb{R}^N$ is a field of vectors, i.e. a smooth map $\tilde{X} : U \to \mathbb{R}^N$, given in components by $(c_1, \ldots, c_N)$. Equivalently, it is the differential operator of degree 1 with no constant term given by $\tilde{X} = \sum_{j=1}^N c_j \partial_j$. Roughly speaking, a vector field $\tilde{X}$ on a manifold M is a differential operator on M such that in every chart, it is given by a vector field on $\mathbb{R}^N$ with $N = \dim M$.
   - Geometers' definition: A vector field is a smooth section over the tangent bundle TM, i.e. $\tilde{X}_p$ is a tangent vector at a point p depending smoothly on $p \in M$.
   Recalling the following notions may help with this definition. Given a point p_0 in M and a path $\gamma(t)$ with $\gamma(0) = p_0$ (i.e a smooth function $\gamma : I \to M$ defined on a small interval I about $t \in 0$ and satisfying $\gamma(0) = p_0$), consider the operation $f \mapsto \frac{d}{dt} f(\gamma(t))\Big|_{t=0}$ on the space of smooth functions $f : M \to \mathbb{R}$. Many paths yield the same operation, and modulo this equivalence, this defines a *tangent vector* at p_0. The set of tangent vectors at p_0, known as the *tangent space* $T_{p_0}M$ at p_0. This is a finite dimensional vector space whose dimension is the dimension of the manifold. The *tangent bundle* of M is defined as $TM := \cup_{p_0} T_{p_0} M$.

   Naturally, the two definitions coincide, although the geometers' justifies more intrinsically the fact that a vector field makes sense as an object on a manifold.

(2) The space of vector fields $\Gamma(TM)$ on M is equipped with the *commutator bracket*

$$[\tilde{X}, \tilde{Y}] := \tilde{X}\tilde{Y} - \tilde{Y}\tilde{X}, \qquad \tilde{X}, \tilde{Y} \in \Gamma(TM).$$

   This definition requires to check that the commutator bracket of two vector fields is a vector field. The analysts' viewpoint will start with the setting $\mathbb{R}^N$ where the commutator bracket of two vector fields viewed as differential operators is clearly a differential operator of degree ≤ 2, and a (simple but lengthy) computation shows that the terms with degrees $= 2$ and $= 0$ vanish, so it is a vector field in $\mathbb{R}^N$. Consequently, this is also true on manifold.

   A more geometric proof is the following. The *flow* $\phi^{\tilde{X}}$ of $\tilde{X}$ is the solutions of the ODE $\frac{d}{dt}\phi_t^{\tilde{X}}(p) = \tilde{X}\phi_t^{\tilde{X}}(p)$ where $\tilde{X}$ is viewed here as a field of vectors in the case of $\mathbb{R}^N$ and more generally as a smooth section of TM in the manifold case. We have

$$[\tilde{X}, \tilde{Y}] = \frac{d}{dt}\Big|_{t=0} \phi^{\tilde{X}}_{\sqrt{t}} \phi^{\tilde{Y}}_{\sqrt{t}} \phi^{\tilde{X}}_{-\sqrt{t}} \phi^{\tilde{Y}}_{-\sqrt{t}} = \frac{1}{2}\frac{d^2}{dt^2}\Big|_{t=0} \phi^{\tilde{X}}_t \phi^{\tilde{Y}}_t \phi^{\tilde{X}}_{-t} \phi^{\tilde{Y}}_{-t}.$$

   The fact that this indeed defines a vector field that can alternatively be defined with the commutator bracket is one of the first results of differential geometry.

   Note that the space of vector fields $\Gamma(TM)$ equipped with the commutator bracket is then a Lie algebra of infinite dimension.

(3) Assume that $M = G$ is a Lie group.

   By definition, a vector field $\tilde{X}$ is *left-invariant* when it commutes with the left multiplication $p \mapsto g_0 p$ by every element g_0 of the group G, that is, $\tilde{X}(f(g_0 \cdot)) = (\tilde{X}f)(g_0 \cdot)$ on $M = G$.

   One checks readily that the commutator bracket of two left-invariant vector fields is also left-invariant. This allows us to define the *Lie algebra* $\mathfrak{L}$ of the Lie group G as the space of left-invariant vector fields equipped with the commutator bracket. Note that $\mathfrak{L}$ is a Lie sub-algebra of $\Gamma(TM)$ with this definition.

   There is an alternative description of $\mathfrak{L}$ that shows that it is finite dimensional. Indeed, we observe that the tangent space $T_{e_G}G$ at the origin e_G of the group is in one-to-one correspondence with the left-invariant vector fields

$$X \longleftrightarrow \tilde{X}$$

via

$$\tilde{X} f(g) = \frac{d}{dt}\Big|_{t=0} f(g\gamma(t)), \text{for any path } \gamma(t) \text{ with } \gamma(0) = e_G,\ \frac{d}{dt}\Big|_{t=0}\gamma(t) = X.$$

Hence, $\mathfrak{L}$ may be viewed as the tangent space $T_{e_G}G$ at the origin equipped with the Lie bracket given by

$$[X, Y] = [\tilde{X}, \tilde{Y}](e_G).$$

Consequently, $\dim \mathfrak{L} = \dim G < \infty$.

Note that the analysis above may be also done with right-invariant vector fields.

If $X \in \mathfrak{L}$ then the time-one flow of $\tilde{X}$ defines the exponential of X:

$$\exp X := \phi^{\tilde{X}}_{t=1}.$$

This defines the smooth map

$$\exp : \mathfrak{L} \to G,$$

called *the exponential map*. Note that this is the exponential map from Lie theory - not to be confused with the exponential map on a Riemannian manifold and its generalisations.

References for an introduction to differential geometry with basic aspects of Lie theorey include for instance the textbook [27]. References on Lie groups and Lie algebras include for instance [28, 13] and furthermore for the very motivated readers [15, 16].

Recall the following facts in Lie theory:

- The exponential map is a smooth diffeomorphism from a neighbourhood of $0 \in \mathfrak{L}$ onto a neighbourhood of $e_M \in G$. However, for a simply connected nilpotent Lie group M, this a (global) diffeomorphism from $\mathfrak{L}$ onto M.
  Convention: A simply connected topological space is assumed connected in these lecture notes.
- The Baker-Campbell-Hausdorff formula

$$\exp(X)\ \exp(Y) = \exp(X + Y + \frac{1}{2}[X, Y] + \ldots) = \exp(X \star Y),$$

  with $X \star Y$ given in (1.4) below, holds on any Lie group for X and Y in a small enough neighbourhood of $0 \in \mathfrak{L}$. The sum in the exponential on the right-hand side of the Baker-Campbell-Hausdorff formula is

$$X \star Y := \sum_{\ell=1}^{\infty} \frac{(-1)^{\ell-1}}{\ell} \sum_{\substack{r,s\in\mathbb{N}_0^\ell \\ r_1+s_1>0,\ldots,r_\ell+s_\ell>0}} c_{r,s} \mathrm{ad}^{r_1} X \mathrm{ad}^{s_1} Y \ldots \mathrm{ad}^{r_\ell} X \mathrm{ad}^{s_\ell - 1} Y(Y). \tag{1.4}$$

  Above, ad denotes the operation $\mathrm{ad}(Z_1)(Z_2) = [Z_1, Z_2]$; this is known as the adjoint representation of $\mathfrak{L}$. The coefficients $c_{r,s}$ are known:

$$c_{r,s}^{-1} = \Big(\sum_{j=1}^{\ell}(r_j + s_j)\Big)\, \Pi_{i=1}^{\ell} r_i! s_i!$$

  In (1.4), the term for which $s_\ell > 1$ or $s_\ell = 0$ and $r_\ell > 1$ is zero, while the term $\mathrm{ad}X\mathrm{ad}^{-1}Y(Y)$ for $s_\ell = 0, r_\ell = 1$ is understood as X. On any Lie group, the sum in (1.4) converges when

X and Y are in a small enough neighbourhood of $0 \in \mathfrak{L}$ and then the Baker-Campbell-Hausdorff formula holds. In the case of a nilpotent Lie group, the sum over ℓ is finite and the formula holds globally when G is a connected nilpotent Lie group.

The Baker-Campbell-Hausdorff formula holds in more general settings [22, Appendix].

- To any Lie algebra corresponds at least one Lie group. In fact, there are often many Lie groups with the same Lie algebra. However, the correspondence is known to be one-to-one between nilpotent Lie algebras and simply connected nilpotent Lie groups.
- Let us describe more precisely the correspondence:

  nilpotent Lie algebra $\mathfrak{L} \longleftrightarrow$ connected simply connected nilpotent Lie group G.

  $\leftarrow$ If G is a nilpotent Lie group then $\mathfrak{L}$ is its Lie algebra (see Part (3) of the definition above).

  $\rightarrow$ If $\mathfrak{L}$ is a nilpotent Lie algebra, then consider the manifold G given by the same underlying vector space $\mathcal{V}$ as $\mathfrak{L}$. As $\mathfrak{L}$ is nilpotent, the sum in (1.4) is finite and we take this to define the operation sometimes known as the Dynkin or star product. One checks readily that this defines a group law and that G is a simply connected Lie group with Lie algebra $\mathfrak{L}$. Moreover, in this case, the exponential mapping is given via

$$V \sim \mathfrak{L} \ni V \mapsto \exp V \cong V \in G \sim V.$$

Recall that a left Haar measure μ_G on a Lie group G (or more generally on a locally compact group G) is a positive Radon measure that is invariant under left translations, i.e. $\mu_G(g_0 A) = \mu_G(A)$ for any $g_0 \in G$, A measurable subset of G, or equivalently $\int_G f(g_0 g) d\mu_G(g) = \int_G f(g) d\mu_G(g)$ for any function $f \in C_c(G)$ and $g_0 \in G$. A left Haar measure always exists, and is unique up to a constant. A similar notion exists for right Haar measure with a similar property. A nilpotent Lie group is unimodular, meaning that a left Haar measure is also a right one, and that a right Haar measure is also a left one. On a unimodular group, there is no confusion about the meaning of measurability or L^p-integrability with respect to 'the Haar measure(s)' as they all differ from one another by a (fixed) constant. Moreover, $\int_G f(g^{-1}) d\mu_G(g) = \int_G f(g) d\mu_G(g)$.

On a connected simply connected nilpotent Lie group G, a choice of basis on its Lie algebra $\mathfrak{L}$ yields a natural choice for the Lebesgue measure on $\mathfrak{L}$, and therefore a natural choice for the Haar measure μ_G on G via the exponential mapping. For instance, on the Heisenberg group, the Haar measure associated with our canonical basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ is given by

$$\int_{\mathbb{H}_n} f \, d\mu_{\mathbb{H}_n} = \int_{\mathbb{H}_n} f(x, y, t) dx dy dt, \qquad f \in C_c(\mathbb{H}_n). \tag{1.5}$$

1.6. **The realisations in Sections 1.1 and 1.4 coincide.** Our preferred definition for the Heisenberg group $\mathbb{H}_n$ was given in Section 1.1. Let us make the difference between the vector space $\mathcal{V} = \mathbb{R}^{2n+1}$ underlying the Heisenberg group $\mathbb{H}_n$, and the Heisenberg group itself which is the manifold $\mathcal{V}$ equipped with the law (1.1).

Let us show that this coincides with the construction of the simply connected nilpotent Lie group whose Lie algebra is the Heisenberg Lie algebra in Section 1.4. Again, we will make a difference between the vector space $\mathcal{V}$ and the Heisenberg Lie algebra $\mathfrak{h}_n$ which is $\mathcal{V}$ together with Lie bracket defined via the canonical commutation relations (1.3) given on the canonical basis $X_1, \ldots, X_n, Y_1, \ldots, Y_n, T$ of $\mathcal{V}$. Elements of the group $\mathbb{H}_n$ and of the Lie algebra $\mathfrak{h}_n$ are both identified with elements of the underlying subspace $\mathcal{V}$ via the exponential mapping:

$$\exp(\sum_{j=1}^n x_j X_j + y_j Y_j \ + \ tT) = (x, y, t) \in \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R} \sim \mathbb{H}_n.$$

Now let us consider two elements $(x,y,t)=\exp V$ and $(x',y',t')=\exp V'$ with writing as above and compute $V\star V' = V+V'+\frac{1}{2}[V,V']$, especially

$$[V,V'] = \left[\sum_{j=1}^n x_jX_j + y_jY_j + tT, \sum_{j'=1}^n x'_{j'}X_{j'} + y'_{j'}Y_{j'} + t'T\right]$$
$$= \sum_{j=1}^n\sum_{j'=1}^n \left[x_jX_j + y_jY_j, x'_{j'}X_{j'} + y'_{j'}Y_{j'}\right] = \left(\sum_{j=1}^n x_jy'_j - y_jx'_j\right)T.$$

Therefore,

$$(x,y,t)(x',y',t') = \exp(V\star V') = \exp(V+V'+\frac{1}{2}[V,V']).$$

This shows that our preferred definition for the Heisenberg group $\mathbb{H}_n$ in Section 1.1. coincides with the construction of the simply connected nilpotent Lie group whose Lie algebra is the Heisenberg Lie algebra in Section (1.4).

Let us give a concrete expression for the left-invariant vector fields corresponding to the canonical basis of $\mathfrak{h}_n$. We know that $\tilde{X}_jf(p) = \partial_{t=0}f(p\gamma(t))$ for any smooth function $f:\mathbb{H}_n\to\mathbb{R}$ and a path $\gamma(t)$ with $\frac{d}{dt}\Big|_{t=0}\gamma(t) = X_j$. Here, let us take $\gamma(t)=\exp(tX_j)$ and compute

$$(x,y,t)\gamma(t) = \exp\left((\sum_{j'=1}^n x_{j'}X_{j'} + y_{j'}Y_{j'} + tT)\star tX_j\right)$$
$$= \exp\left(\sum_{j'=1}^n x_{j'}X_{j'} + y_{j'}Y_{j'} + tT \ + \ tX_j + \frac{1}{2}y_jtT\right),$$

hence

$$(x,y,t)\gamma(t) = (x_1,\ldots,x_j+t,\ldots,x_n,y_1,\ldots,y_n,t-\frac{1}{2}y_jt)$$

and

$$\partial_{t=0}f((x,y,t)\gamma(t)) = (\partial_{x_j} - \frac{y_j}{2}\partial_t)f(x,y,t).$$

We have obtained:

$$\tilde{X}_j = \partial_{x_j} - \frac{y_j}{2}\partial_t.$$

We compute similarly

$$\tilde{Y}_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \qquad \tilde{T} = \partial_t.$$

We check easily that these vector fields indeed satisfy the canonical commutation relations (1.3).

For comparison, the right-invariant vector fields corresponding to the canonical basis of $\mathfrak{h}_n$ are

$$\check{X}_j = \partial_{x_j} + \frac{y_j}{2}\partial_t, \ \check{Y}_j = \partial_{y_j} - \frac{x_j}{2}\partial_t, \ j=1,\ldots,n, \ \text{and}\ \check{T}=\partial_t.$$

1.7. **Links with the realisations of the Heisenberg group in Section 1.2.** The tangent space of $\mathbb{H}_n^{(\alpha)}$ at the origin 0 is $\mathbb{R}^{2n+1}$ with canonical basis

$$X_{1,\alpha},\ldots,X_{n,\alpha},Y_{1,\alpha},\ldots,Y_{n,\alpha},T_\alpha.$$

The left-invariant vector field corresponding to $X_{1,\alpha}$ is $\tilde{X}_{1,\alpha}f(p) = \partial_{t=0}f(p\gamma(t))$ for any smooth function $f:\mathbb{H}_n^{(\alpha)}\to\mathbb{R}$ and a path $\gamma(t)$ with $\frac{d}{dt}\Big|_{t=0}\gamma(t) = X_{1,\alpha}$. Here, let us take $\gamma(t)=(t,0,\ldots,0)$ and compute

$$(x,y,t)\gamma(t) = (x_1+t,\ldots,x_n,y_1,\ldots,y_n,t-\frac{1}{\alpha}y_1t)$$

so

$$\tilde X_{1,\alpha} = \partial_{x_j} - \frac{y_j}{\alpha}\partial_t.$$

More generally, the left-invariant vector field corresponding to the canonical basis are

$$\tilde X_{j,\alpha} = \partial_{x_j} - \frac{y_j}{\alpha}\partial_t, \qquad \tilde Y_{j,\alpha} = \partial_{y_j} + \frac{x_j}{\alpha}\partial_t, \qquad \tilde T = \partial_t.$$

We then compute their commutator brackets: they are all zero except for:

$$[\tilde X_{j,\alpha}, \tilde Y_{j,\alpha}] = \frac{2}{\alpha}\tilde T_\alpha, \quad j = 1, \ldots, n.$$

The map

$$\sum_{j=1}^n x_j X_j + y_j Y_j \ + \ tT \longmapsto \sum_{j=1}^n x_j X_{j,\alpha} + y_j Y_{j,\alpha} \ + \ \frac{2}{\alpha} tT_\alpha$$

is an isomorphism between the Lie algebras $\mathfrak{h}_n$ and $\mathfrak{h}_n^{(\alpha)}$ of $\mathbb{H}_n$ and $\mathbb{H}_n^{(\alpha)}$. Taking the exponential on both sides yields the isomorphism of Lie groups in (1.2).

We have a similar analysis for the polarised Heisenberg group $\tilde{\mathbb{H}}_n$.

Remark 1.1.
- A physicist's viewpoint on this analysis would be that there is only one realisation of 'the' Heisenberg group up to bad choices of normalisation for the generators of the Lie algebra. Meanwhile, a mathematician would say that the isomorphisms between all these realisations are easily deduced from the Lie brackets on the canonical bases.
- Apart from a choice of canonical basis and canonical commutation relations, we have used the fact that the exponential mapping is a global diffeomorphism from the vector space underlying the Lie algebra to the Heisenberg group viewed as a manifold. It is possible to then compose with other maps and obtain other groups isomorphic to $\mathbb{H}_n$, for instance see [4, Remark 2.2.4 and Example 2.2.5].

1.8. **Some comments.**

1.8.1. *Generalisation of the above construction.* Any simply connected nilpotent Lie group of step 2 can be constructed as above: take a vector space $\mathcal{V}$ and assume that there exists a Lie bracket $[\cdot,\cdot]$ on $\mathcal{V}$ which turns it into a step-two nilpotent Lie algebra $\mathfrak{L}$. The star product in (1.4) reduces to $X \star Y = X + Y + \frac{1}{2}[X,Y]$, $X, Y \in \mathcal{V}$. The space $\mathcal{V}$ viewed as a manifold equipped with the star product is a simply connected nilpotent Lie group G with step 2. Particular cases of these are of course the Heisenberg group, but also the groups of Heisenberg types and more generally the Metivier groups.

Naturally, this extends to any step, but the star product takes longer to write down.

1.8.2. *Why so many realisations?*
- *Lie theory.* As explained above, our preferred realisation of the Heisenberg group is rooted in Lie theory via the Baker-Campbell-Hausdorff formula and the exponential mapping.

  Note that the Heisenberg group appears in another part of Lie theory, namely in the Iwasawa or KAN decomposition of the semi-simple Lie group

  $$\mathrm{SU}(n,1) = \{M \in \mathrm{SL}_{n+1}(\mathbb{C}), M^T DM = D\}, \quad \text{where} \quad D = \mathrm{diag}(1, \ldots, 1, -1).$$

  because $\mathrm{SU}(n,1) = KAN$ with $K = \mathrm{S}(\mathrm{diag}(\mathrm{U}(n), \mathrm{U}(1)) \sim \mathrm{U}(n)$ its maximal compact subgroup, $A \sim \mathbb{R}$ and $N \sim \mathbb{H}_n$.

  Note that the compact analogue of $\mathrm{SU}(n,1)$ is $\mathrm{SU}(n+1)$, and that its quotient $\mathrm{SU}(n+1)/K$ by K is naturally identified with the sphere $\{z \in \mathbb{C}^{n+1}, |z| = 1\}$ of $\mathbb{C}^{n+1}$ or equivalently the odd dimensional sphere $\mathbb{S}^{2n+1}$. This is related to the description in Section 1.3, see [9].

The classic references (perhaps reserved for motivated readers) regarding Iwasawa decompositions and symmetric spaces are Helgason's textbooks [15, 16, 14].

However, other viewpoints may lead to favour other realisations.

- *Complex analysis.* An important motivation for studying the Heisenberg group comes from analysis on complex manifolds (as mentioned below in relation to contact manifolds in Section 2.3). In this context, the Heisenberg group is often described as the real manifold $\mathbb{C}^n \times \mathbb{R}$ equipped with the group law
$$(z,t)(z',t') = (z+z', t+t' - \frac{1}{2}\Im(z\bar{z}')),$$
or more generally with any $\alpha \in \mathbb{C}$
$$(z,t)(z',t') = (z+z', t+t' - \alpha\Im(z\bar{z}')),$$
where
$$z\bar{z}' = z_1\bar{z}_1' + \ldots + z_n\bar{z}_n' \qquad \text{so} \qquad \Im(z\bar{z}') = xy' - x'y,$$
having written $z = x+iy$ and $z' = x'+iy'$. This choice of description for the Heisenberg group tends to take $\alpha = 1$ above and to consider the complexification of the Heisenberg Lie algebra, in particular the elements
$$Z_j := X_j - iY_j, \qquad \bar{Z}_j = X_j + iY_j.$$
- *Other approaches.* Other realisation of the Heisenberg than our preferred one may be more suitable for certain approaches. For instance, it is immediate that the realisation of the Heisenberg group as the matrix group T_3 shows that 'the' Heisenberg group contains discrete subgroups; indeed, it suffices to take the entries to be integers, for instance
$$\mathbb{H}_1(\mathbb{Z}) := \left\{ \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, x, y, t \in \mathbb{Z} \right\}.$$
The quotient $\mathbb{H}_1/\mathbb{H}_1(\mathbb{Z})$ is well understood [19, 26]. More generally, the existence of closed discrete subgroups of a nilpotent Lie groups and their quotient (aka nilmanifolds) are well understood [6].

  In a direction which would see us exit the realm of smooth manifolds and Lie groups, we can consider the Heisenberg group
$$\mathbb{H}_1(R) := \left\{ \begin{bmatrix} 1 & x & t \\ 0 & 1 & y \\ 0 & 0 & 1 \end{bmatrix}, x, y, t \in R \right\},$$
associated with any commutative ring R with an identity, for example $\mathbb{Z}/p\mathbb{Z}$.

2. Which kind of analysis can / should we do on $\mathbb{H}_n$?

The underlying manifold of $\mathbb{H}_n$ is $\mathbb{R}^{2n+1}$. The type of analysis we would like to perform is not related to this Euclidean structure, but to the structure of the group (e.g. analysis of left-invariant operators) as well as other structures coming from the stratification of the Heisenberg Lie algebra.

2.1. The stratified structure of the Heisenberg Lie algebra.

From now on, we will not distinguish between an element of the Lie algebra and a left-invariant vector field. In other words, the Lie algebra $\mathfrak{h}_n$ of $\mathbb{H}_n$ is identified with the vector space of left-invariant vector fields. We have already computed that the canonical basis of $\mathfrak{h}_n$ is given by the left-invariant vector fields

$$X_j = \partial_{x_j} - \frac{y_j}{2}\partial_t, \quad Y_j = \partial_{y_j} + \frac{x_j}{2}\partial_t, \ j = 1, \ldots, n, \quad \text{and } T = \partial_t.$$

and the canonical commutation relations

$$[X_j, Y_j] = T, \quad j = 1, \ldots, n,$$

while T is in the centre of $\mathfrak{h}_n$.

The Heisenberg Lie algebra admits the following stratification

$$\mathfrak{h}_n = \mathfrak{g}_1 \oplus \mathfrak{g}_2, \quad \text{where} \quad \mathfrak{g}_1 := \mathbb{R}X_1 \oplus \ldots \oplus \mathbb{R}X_n \oplus \mathbb{R}Y_1 \oplus \ldots \oplus \mathbb{R}Y_n, \ \mathfrak{g}_2 := \mathbb{R}T.$$

Therefore, the Heisenberg group can be equipped with three related structures:

- dilations, leading to a notion of homogeneity,
- contact manifold,
- sub-Riemannian manifold.

This will determine the kind of analysis is interesting / relevant on the Heisenberg group.

2.2. The natural dilations.

Since the Heisenberg Lie algebra is stratified via $\mathfrak{h}_n = V_1 \oplus V_2$, the natural dilations on the Lie algebra are given by

$$D_r(X_j) = rX_j \quad \text{and} \quad D_r(Y_j) = rY_j, \quad j = 1, \ldots, n, \quad \text{and} \quad D_r(T) = r^2T.$$

We check readily that the dilations D_r are morphism of the Lie algebra $\mathfrak{h}_n$:

$$\forall V_1, V_2 \in \mathfrak{h}_n, \ r > 0 \qquad [D_rV_1, D_rV_2] = D_r[V_1, V_2].$$

Indeed, by linearity and since $\mathbb{R}$ is the centre of the Lie algebra $\mathfrak{h}_n$, it suffices to check this for V_1, V_2 being a vector amongst $X_1, \ldots, X_n, Y_1, \ldots, Y_n$, and this is straightforward, for instance

$$[D_rX_j, D_rY_k] = [rX_j, rY_k] = r^2[X_j, Y_k] = r^2\delta j = kT = D_r[X_j, Y_k].$$

Note that if we have chosen the isotropic dilations $D_r(T) = rT$, they would not be morphism of the Lie algebra $\mathfrak{h}_n$.

We keep the same notation D_r for the corresponding dilations on the group $\mathbb{H}_n$, and we may also denote them just by $r\,\cdot$. They are characterised by

$$D_r \exp V = \exp D_rV, \quad V \in \mathfrak{h}_n,$$

so they are given by

$$D_r(x, y, t) = r(x, y, t) = (rx, ry, r^2t), \quad (x, y, t) \in \mathbb{H}_n, \ r > 0.$$

They are morphisms of the Lie group $\mathbb{H}_n$. As explained in Section 2.6, this equips the Heisenberg group with a homogeneous structure.

2.3. $\mathbb{H}_n$ **as a (the canonical) contact manifold.** Roughly speaking, an odd dimensional manifold M is contact when locally it looks like the Heisenberg group. The more geometric definition of a contact manifold M is that its tangent bundle TM admits a hyperplane distribution satisfying a condition of complete non-integrability that we now describe in two equivalent ways:

(1) Above each point $p \in M$, we assume that there is a distinguished hyperplane H_p in the tangent space T_pM; moreover, the dependence in p is smooth in the sense that we can find local smooth frames for the hyperplane distribution $H = \cup_{p\in M} H_pM$, that is, a family $V_1, \ldots, V_{2n}$ of vector fields giving a basis of H_pM above every $p \in U$ of any open subset U of M small enough; here $2n+1 = \dim M$. The condition of complete non-integrability is that $[V_i, V_j](p) \in H_pM$ unless $(i,j) = (2k-1, 2k)$, $k = 1, \ldots, n$ at every $p \in U$.

   Note that $H_pM \oplus (T_pM/H_pM)$ is then naturally equipped with the structure of Heisenberg Lie algebra.

(2) Equivalently, the condition is expressed by one-forms. This is the traditional geometric presentation.

   Recall that a one-form is an element of $\Omega^1(M) = \Gamma(T^*M)$, that is, roughly speaking, linear functionals depending smoothly on the points on the manifold.

   The distinguished hyperplane is described as the kernel of a particular one-form $\alpha \in \Omega^1(M)$: $H_p = \ker \alpha_p$, and the condition of complete non-integrability is the fact that the $(2n+1)$-form $\alpha \wedge (d\alpha)^{(\wedge n)} = \alpha \wedge d\alpha \wedge \ldots \wedge d\alpha$ vanishes nowhere.

   Note that then $\alpha \wedge (d\alpha)^{(\wedge n)} \in \Omega^{2n+1}(M)$ identifies with a volume form on M.

Before giving a sketch of the proof for the equivalence between the conditions in (1) and (2), let us show that the Heisenberg group $\mathbb{H}_n$ is a contact manifold. Indeed, the canonical distinguished hyperplane is given globally by

$$H = \mathfrak{g}_1 = \mathbb{R}X_1 \oplus \ldots \oplus \mathbb{R}X_n \oplus \mathbb{R}Y_1 \oplus \ldots \oplus \mathbb{R}Y_n,$$

while the canonical distinguished one-form is

$$\alpha = dt + \frac{1}{2}\sum_{j=1}^{n} x_j dy_j - y_j dx_j. \tag{2.1}$$

Indeed, we compute

$$\alpha(X_{j_0}) = \alpha(\partial_{x_{j_0}}) - \frac{y_{j_0}}{2}\alpha(\partial_t) = \frac{1}{2}y_{j_0} - \frac{y_{j_0}}{2} = 0$$

and similarly $\alpha(Y_{j_0}) = 0$, so $\mathfrak{g}_1 \subset \ker \alpha$ and even equality

$$\mathfrak{g}_1 = \ker \alpha,$$

since $\mathfrak{g}_1$ is a hyperplane distribution. Moreover,

$$d\alpha = \sum_{j=1}^{n} dx_j \wedge dy_j, \qquad \text{so} \qquad \alpha \wedge (d\alpha)^{(\wedge n)} = -c_n dt \wedge dx_1 \wedge dy_1 \wedge \ldots \wedge dx_n \wedge dy_n,$$

with a computable constant $c_n > 0$ (note $c_1 = 1$ for $n = 1$). Note that the volume form identified with $\alpha \wedge (d\alpha)^{(\wedge n)}$ is the Haar measure $\mu_{\mathbb{H}_n}$ of $\mathbb{H}_n$ given in (1.5).

Let us now sketch the proof for the equivalence between the conditions in (1) and (2) above; this will require some knowledge on contact manifolds. If (1) holds, the form dual to the V_j's will satisfy the complete non-integrability condition. If (2) holds, then the well known Darboux theorem for contact manifolds say that we can find local coordinates (x, y, t) on M such that the one-form α writes as in (2.1), and the canonical basis for $\ker \alpha$ for these coordinates provides the V_j's of (1).

Contact manifolds appear naturally in the geometry of classical mechanics, see Arnold's classic textbook [2] (perhaps for the motivated reader). They form an important class of manifolds which contain the real hypersurfaces of $\mathbb{C}^n$, arguably the most common class of Cauchy-Riemann manifolds. This is the important link between the Heisenberg group and analysis on complex manifolds mentioned in Section 1.8.2.

2.4. $\mathbb{H}_n$ **as a Sub-Riemannian manifold.**

Definition 2.1. (1) Consider a distribution H of the manifold M, that is, a subbundle $H = \cup_{p\in M} H_p$ of the tangent bundle TM: $H_p \subset T_pM$.

The distribution on H of M *satisfies the Hörmander condition* when for we have that any tangent vector at any point $p \in M$ can be presented as a linear combination of vectors of the vector fields $X_{j_1}, [X_{j_1}, X_{j_2}], [X_{j_1}, [X_{j_2}, X_{j_3}]], \ldots, [X_{j_1}, [X_{j_2}, [X_{j_3}, \ldots, X_{j_k}], \ldots$ at p where all vector fields X_{j_i} are in H.

(2) A sub-Riemannian manifold is a triple (M, H, g) where H is a distribution satisfying the Hörmander condition on the differentiable manifold M and g is a smooth section of positive-definite quadratic forms on H.

In this case, H is called the *horizontal distribution*, a vector field in H is called *horizontal*, a curve γ on M is called horizontal when $\dot\gamma(t) \in H_{\gamma(t)}$ for all time t etc.

A sub-Riemannian manifold carries the natural intrinsic metric, called the *Carnot-Carathéodory metric* or CC-metric for short, defined as

$$d(p,q) = \inf \int_0^1 |\dot\gamma(t)| dt$$

where the infimum is taken over all horizontal curves $\gamma : [0,1] \to M$ joining $p = \gamma(0)$ to $q = \gamma(1)$; here $|\dot\gamma(t)| := \sqrt{g_{\gamma(t)}(\dot\gamma(t), \dot\gamma(t)}$ denotes the g-length of the horizontal vector $\dot\gamma(t)$ at $\gamma(t)$.

When the sub-Riemannian manifold M is equipped with a (fixed positive Radon) measure, then we can define the associated sub-Laplacian as a div-grad [1, Chapter 21]. An intrinsic canonical measure exists in the case of a contact manifold and on the Heisenberg group and more generally, in the case of Carnot groups.

In the case of the Heisenberg group, the intrinsic sub-Laplacian on $\mathbb{H}_n$ is the operator

$$\begin{aligned}
\mathcal{L}_{\mathbb{H}_n} &:= \sum_{j=1}^n (X_j^2 + Y_j^2) \\
&= \sum_{j=1}^n \left(\partial_{x_j} - \frac{y_j}{2}\partial_t\right)^2 + \left(\partial_{y_j} + \frac{x_j}{2}\partial_t\right)^2 \\
&= \sum_{j=1}^n \partial_{x_j}^2 + \partial_{y_j}^2 + \frac{y_j^2 + x_j^2}{4}\partial_t^2 - y_j\partial_{x_j}\partial_t + x_j\partial_{y_j}\partial_t.
\end{aligned}$$

The operator $\mathcal{L}$ on $\mathbb{H}_n$ often plays the role of the Laplacian $\partial_1^2 + \ldots + \partial_N^2$ on $\mathbb{R}^N$. However, the Laplacian is elliptic (its principal symbol or characteristic polynomial $-(\zeta_1^2 + \ldots + \zeta_N^2)$ never vanishes), whereas $\mathcal{L}$ is hypoelliptic:

Definition 2.2. An operator D on an open set of $\mathbb{R}^N$ is hypoelliptic when the solutions of the equation $Df = g$ with $g \in C^\infty$, are also C^∞.

An elliptic operator is hypoelliptic, but the converse is not true. For instance, the sub-Laplacian $\mathcal{L}$ is known to be hypoelliptic (by a famous result due to Hörmander recalled in Theorem 2.3) but it is not elliptic: its principal symbol is $-\sum_{j=1}^n \left(\zeta_{x_j} - \frac{y_j}{2}\zeta_t\right)^2 + \left(\zeta_{y_j} + \frac{x_j}{2}\zeta_t\right)^2$, and its restriction to $x_j = 0 = y_j$, $j = 1, \ldots, n$, and $\zeta_{x_j} = 0 = \zeta_{y_j}$, $j = 1, \ldots, n$, vanishes identically.

Let us state the famous result by Hörmander on sum of squares [18]:

Theorem 2.3. *Let $V_0, V_1, \ldots, V_n$ be smooth real vector fields on an open subset Ω of $\mathbb{R}^N$. Assume that among the operators*

$$V_{j_1}, [V_{j_1}, V_{j_2}], [V_{j_1}, [V_{j_2}, V_{j_3}]], \ldots, [V_{j_1}, [V_{j_2}, [V_{j_3}, \ldots, V_{j_k}], \ldots$$

where $j_i = 0, 1, \ldots, n$, there exist n which are linearly independent at any given point in Ω. Then the second order differential operator $V_1^2 + \ldots + V_n^2 + V_0 + c$ (where c is a constant) is hypoelliptic.

The assumption means that the iterated commutator brackets of the V_j's span linearly the whole vector space at every point, i.e.

$$\forall p \in U \qquad \dim\left(\operatorname{span}_{\substack{1\leq i_1,\ldots,i_{2k}\leq n \\ k=1,2,\ldots}} [V_{i_1}, [V_{i_2}, [\ldots, [V_{i_k}, V_{i_{k+1}}, \ldots] V_{i_{2k}}](p)\right) = N.$$

This generalises readily to the case of a manifold M satisfying the Hörmander condition, that is, such that that the iterated commutator brackets of the vector fields $V_0, V_1, \ldots, V_n$ generate the tangent bundle TM. This explains the vocabulary in Definition 2.1 (1).

2.5. **So, which kind of analysis. . . ?** The analysis on sub-Riemannian manifolds or sub-elliptic Partial Differential Equations (PDE's) (e.g. PDE's involving sub-Laplacians) becomes rapidly intractable with Euclidean tools. The reason is that the integral kernels of operators in this context are often much more singular than their Euclidean counterparts, even using the CC-distance instead of a Riemannian/Euclidean one. Even the very powerful micro-local analysis does not yield interesting results. It can be argued that micro-local analysis is the culmination of the meeting of the phase-space analysis with pseudo-differential theory stemming from the study of Euclidean singular integral operators.

There have been many attempts to start developing tools to analyse sub-Riemannian or sub-elliptic settings. However, they are far from being remotely as effective as what is known in the Riemannian/Euclidean or elliptic settings. It is current research to determine what should be the 'right' tools comparable to e.g. micro-local analysis.

The starting point is usually agreed upon since the works of Elias Stein and his collaborators in the 70's (see e.g. [24, 11]): the Hörmander condition should be abstractly lifted to a nilpotent Lie group above each point. This tallies with the geometric viewpoint developped in the 80's: the metric tangent space in the sense of Gromov of a sub-Riemannian manifold is a (Carnot) nilpotent Lie group [3]. For instance, the metric tangent space of a contact manifold equipped with a compatible sub-Riemannian structure is the Heisenberg group above every point. More general Carnot groups may appear.

For these reasons, in these lecture notes, we will discuss the settings of homogeneous Lie groups (including Carnot groups) and the singular integral theory on spaces of homogeneous types.

2.6. **Definitions for the study of homogeneous Lie groups.** Here, we follow the vocabulary of G. Folland and E. Stein [12]. See also [10].

2.6.1. *Definition of a homogeneous Lie group.*

Definition 2.4. (i) A family of *dilations* of a Lie algebra $\mathfrak{g}$ is a family of linear mappings

$$\{D_r,\, r > 0\}$$

from $\mathfrak{g}$ to itself which satisfies:

– the mappings are of the form

$$D_r = \operatorname{Exp}(A \ln r) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} (\ln(r) A)^\ell,$$

where A is a diagonalisable linear operator on $\mathfrak{g}$ with positive eigenvalues, Exp denotes the matrix exponential or the exponential of linear maps on finite dimension vector space and $\ln(r)$ the natural logarithm of $r > 0$,

– each D_r is a morphism of the Lie algebra $\mathfrak{g}$, that is, a linear mapping from $\mathfrak{g}$ to itself which respects the Lie bracket:

$$\forall X, Y \in \mathfrak{g},\ r > 0 \qquad [D_r X, D_r Y] = D_r[X, Y].$$

(ii) A *homogeneous* Lie group is a connected simply connected Lie group whose Lie algebra is equipped with dilations.

(iii) We call the eigenvalues of A the *dilations' weights* or weights. The set of dilations' weights, or in other worlds, the set of eigenvalues of A is denoted by $\mathcal{W}_A$.

We can realise the mappings A and D_r in a basis of A-eigenvectors as the diagonal matrices

$$A \equiv \begin{pmatrix} \upsilon_1 & & & \\ & \upsilon_2 & & \\ & & \ddots & \\ & & & \upsilon_N \end{pmatrix} \quad \text{and} \quad D_r \equiv \begin{pmatrix} r^{\upsilon_1} & & & \\ & r^{\upsilon_2} & & \\ & & \ddots & \\ & & & r^{\upsilon_N} \end{pmatrix}.$$

We may assume that the dilations' weights are increasingly ordered:

$$0 < \upsilon_1 \leq \ldots \leq \upsilon_N .$$

For example, $\mathfrak{h}_n$ has weights $1 \leq \ldots \leq 1 \leq 2$ when equipped with the dilations discussed in Section 2.1, so in this case $\mathcal{W}_A = \{1, 2\}$.

Let $w_1, \ldots, w_s \in \mathbb{R}$ denote the weights listed without multiplicity:

$$\mathcal{W}_A = \{\upsilon_1, \ldots, \upsilon_N\} = \{w_1, \ldots, w_s\}, \quad \text{with} \quad 0 < w_1 < \ldots < w_s .$$

For each weight w_j, we denote by $\mathfrak{g}_{w_j}$ the w_j-eigenspace for A. The Lie algebra decomposes into a direct sum

$$\mathfrak{g} = \oplus_{j=1}^{s} \mathfrak{g}_{w_j} \qquad \text{satisfying} \qquad [\mathfrak{g}_{w_i}, \mathfrak{g}_{w_j}] \subseteq \mathfrak{g}_{w_i + w_j},\ 1 \leq i, j \leq s .$$

It follows that the $\mathfrak{g}$ is nilpotent, and so is G. Keeping the same notation for the dilations on the group, they are the maps D_r, with $r > 0$, on G characterised by $\delta_r \exp V = \exp \delta_r V$, $V \in \mathfrak{g}$.

Note that the dilations D_r on G are automorphisms of the group G; this explains why homogeneous Lie groups are often presented as Lie groups equipped with dilations.

We may write

$$rx := D_r(x) \quad \text{for } r > 0 \text{ and } x \in G.$$

The dilations on the group or on the Lie algebra satisfy

$$D_{rs} = D_r D_s, \quad r, s > 0.$$

Example 2.5. (1) The abelian group $(\mathbb{R}^n, +)$ is homogeneous when equipped with the usual dilations $D_r x = rx$, $r > 0$, $x \in \mathbb{R}^n$.

(2) The Heisenberg group $\mathbb{H}_n$ is homogeneous when equipped with the dilations defined in Section 2.1.

2.6.2. *Carnot, stratified and graded groups as homogeneous groups.* Let us recall some particular cases of homogeneous groups often considered in other publications, potentially with a different vocabulary from G. Folland and E. Stein [12].

When the weights of the dilations may be assumed to be integers [10, Section 3.1], the group is said to be *graded*. If in addition $\mathfrak{g}_{w_1}$ with $w_1 = 1$ generates the whole Lie algebra $\mathfrak{g}$, the Lie group G is called *stratified*; it is also known as *Carnot* when a scalar product has been fixed on $\mathfrak{g}_1$ (this can always be assumed). The main example of Carnot groups is the Heisenberg group.

Schematically, we have the following implications

$$\text{stratified} \implies \text{graded} \implies \text{homogeneous} \implies \text{nilpotent}.$$

However, none of them can be reversed [10, Section 3.1]: not all (connected simply connected) nilpotent Lie groups can be equipped with a homogeneous structure, there are homogeneous groups that do not admit any gradation, and finally it is not always possible to find a stratification on every graded group.

2.6.3. *Homogeneous functions.* A scalar function f defined on G or $G \setminus \{0\}$ is homogeneous of degree m (with respect to the dilations D_r, $r > 0$) when

$$\forall r > 0 \qquad f \circ D_r = r^m f, \quad \text{i.e.} \quad f(rx) = r^m f(x).$$

This notion extends to measurable functions on G and distributions $\mathcal{D}'(G)$ and $\mathcal{D}'(G \setminus \{0\})$.

Convention: In these lecture notes, we allow ourselves to write e.g. $\mathcal{D}'(G)$ for $\mathcal{D}(\mathbb{R}^N)$ where $\dim G = N$ and similarly for other functional spaces.

Example 2.6. We fix a canonical basis $X_1, \ldots, X_N$ for $\mathfrak{g}$ where the dilation matrix A is diagonal. This yields a natural choice for the Lebesgue measure on $\mathfrak{L}$, and therefore a natural choice for the Haar measure μ_G on G via the exponential mapping:

$$\int_G f d\mu_G = \int_{\mathbb{R}^N} f(\exp \sum_{j=1}^N x_j X_j) dx_1 \ldots dx_N.$$

This measure is homogeneous of degree 0 (with respect to the dilations D_r, $r > 0$).

We see that

$$\begin{aligned}
\int_G f \circ D_r \, d\mu_G &= \int_{\mathbb{R}^N} f(\exp D_r \sum_{j=1}^N x_j X_j) dx_1 \ldots dx_N \\
&= \int_{\mathbb{R}^N} f(\exp \sum_{j=1}^N x_j r^{\upsilon_j} X_j) dx_1 \ldots dx_N \\
&= r^{-Q} \int_G f(x) d\mu_G(G),
\end{aligned}$$

where Q is the *homogeneous dimension* of the homogeneous group G:

$$Q := \sum_{j=1}^N \upsilon_j = \sum_{i=1}^s w_i \dim \mathfrak{g}_i$$

This justifies that μ_G is 0-homogeneous. Indeed, the dilations of the distribution $\kappa \in \mathcal{D}'(G)$ are the distributions $\kappa \circ D_r$, $r > 0$ defined via

$$(\kappa \circ D_r, \varphi) = r^{-Q}(\kappa, \varphi \circ D_{r^{-1}}), \qquad \varphi \in C_c^\infty(G),$$

thereby coinciding with the case of e.g. a function $\kappa \in L^1(\mathbb{R}^n)$. The distribution $\kappa \in \mathcal{D}'(G)$ is homogeneous of degree m (with respect to the dilations D_r, $r > 0$) when $\kappa \circ D_r = r^m \kappa$, that is, when

$$\forall \varphi \in C_c^\infty(G), \ r > 0 \qquad (\kappa, \varphi \circ D_r) = r^{-m-Q}(\kappa, \varphi).$$

Example 2.7. The Dirac measure δ_0 at the neutral element 0 is homogeneous of degree $-Q$ since

$$\int_G \varphi \circ D_r \, d\delta_0 = \varphi(r0) = \varphi(0) = \int_G \varphi \, d\delta_0, \qquad \varphi \in C_c^\infty(G).$$

Note that the homogeneous dimension in the case of the Heisenberg group is $2n+2 > \dim \mathbb{H}_n = 2n+1$. More generally, for any (non-abelian) homogeneous Lie group, $Q > \dim G$.

An important type of homogeneous functions are the homogeneous quasi-norms:

Definition 2.8. A *homogeneous quasi-norm* is a continuous non-negative function

$$G \ni x \longmapsto |x| \in [0, \infty),$$

satisfying

(i) (symmetric) $|x^{-1}| = |x|$ for all $x \in G$,
(ii) (1-homogeneous) $|rx| = r|x|$ for all $x \in G$ and $r > 0$,
(iii) (definite) $|x| = 0$ if and only if $x = 0$.

The associated ball and sphere centred at $x \in G$ with radius $R > 0$ are defined by

$$B(x, R) := \{y \in G : \ |x^{-1}y| < R\}, \qquad S(x, R) := \{y \in G : \ |x^{-1}y| = R\}.$$

We could of course consider an Euclidean norm $|\cdot|_E$ on $\mathfrak{g}$ by declaring the X_j's to be orthonormal. We may also regard this norm as a function on G via the exponential mapping, that is,

$$|x|_E = |\exp_G^{-1} x|_E.$$

However, this norm is of limited use for our purposes, since it does not interact in a simple fashion with dilations.

It is easy to construct homogeneous quasi-norms:

$$|(x_1, \ldots, x_n)|_p = \left(\sum_{j=1}^{n} |x_j|^{\frac{p}{v_j}} \right)^{\frac{1}{p}}, \ 0 < p < \infty, \quad \text{and for } p = \infty, \ |(x_1, \ldots, x_n)|_\infty = \max_{1 \le j \le n} |x_j|^{\frac{1}{v_j}}.$$

In Definition 2.8 we do not require a homogeneous quasi-norm to be smooth away from the origin but some authors do. Quasi-norms with added regularity always exist as well but, in fact, a distinction between different quasi-norms is usually irrelevant for many questions of analysis because the following properties hold:

(i) Every homogeneous Lie group G admits a homogeneous quasi-norm that is smooth away from the unit element.
(ii) Any two homogeneous quasi-norms $|\cdot|$ and $|\cdot|'$ on G are mutually equivalent:

$$|\cdot| \asymp |\cdot|' \quad \text{in the sense that} \quad \exists a, b > 0 \quad \forall x \in G \qquad a|x|' \le |x| \le b|x|'.$$

Moreover, the usual Euclidean topology coincides with the topology associated with any homogeneous quasi-norm (i.e. the topology induced by the $|\cdot|$-balls). The terminology of 'quasi-norm' is justified by the fact every homogeneous quasi-norm satisfies the triangle inequality up to a constant:

$$\exists C > 0 \quad \forall x, y \in G \qquad |xy| \le C\,(|x| + |y|)\,.$$

The constant C above satisfies necessarily $C \ge 1$ since $|0| = 0$ implies $|x| \le C|x|$ for all $x \in G$. It is natural to ask whether a homogeneous Lie group G may admit a homogeneous quasi-norm $|\cdot|$ which is actually a norm or, equivalently, which satisfies the triangle inequality with constant $C = 1$. For instance, on the Heisenberg group $\mathbb{H}_n$, the homogeneous quasi-norm

$$|(x, y, t)| := \left((|x|^2 + |y|^2)^2 + 16t^2\right)^{1/4}.$$

It turns out to be a norm (cf. [7]). In the stratified case, the norm built from the CC distance $|x| = d(x, 0)$ is also 1-homogeneous. This can be generalised to all homogeneous Lie groups.

2.6.4. *Homogeneous operators.* The notion of homogeneity also extends to operators. For instance, an operator $T : \mathcal{D}(G) \to \mathcal{D}'(G)$ is homogeneous of degree m when

$$T(\varphi \circ D_r) = r^m (TF) D_r, \qquad \varphi \in \mathcal{D}(G).$$

We can also consider operators between other functional spaces e.g. $L^p(G)$, $\mathcal{S}(G)$, $\mathcal{S}'(G)$ etc.

Example 2.9. We check readily that the left-invariant vector fields $X_1, \ldots, X_N$ corresponding to the canonical basis of $\mathfrak{g}$ are 1-homogeneous, while the left-invariant vector field T is 2-homogeneous.

The following example is very relevant for the theme of this school:

Example 2.10. If $\kappa \in \mathcal{D}'(G)$ is m-homogeneous, then the associated convolution operator $T_\kappa : C_c^\infty(G) \to \mathcal{D}'(G)$ is $(-m-Q)$-homogeneous since

$$\begin{aligned} T_\kappa(\varphi \circ D_r)(g) &= \int_G \varphi(r(gh^{-1})\kappa(h)dh = \int_G \varphi(rg\ rh^{-1})\kappa(h)dh \\ &= r^{-Q} \int_G \varphi(rg\ h^{-1})\kappa(r^{-1}h)dh = r^{-Q-m} \int_G \varphi(rg\ h^{-1})\kappa(h)dh \\ &= r^{-Q-m}(T_\kappa\varphi)(rg). \end{aligned}$$

Recall that the convolution of two functions $f_1, f_2 \in C_c(G)$ is defined as

$$f_1 * f_2(g) := \int_G f_1(h) f_2(h^{-1}g)dh = \int_G f_1(gh^{-1}) f_2(h)dh.$$

This extends for instance to $f_1, f_2 \in L^2(G)$ and also to $f_1 \in \mathcal{S}'(G)$ while $f_2 \in \mathcal{S}(G)$ etc. The (right) convolution operator T_κ associated with $\kappa \in \mathcal{D}'(\mathbb{H}_)n)$ is the operator defined as

$$T_\kappa \varphi = \varphi * \kappa, \qquad \varphi \in C_c^\infty(G).$$

The distribution κ is then called the (right) convolution kernel of T_κ.

Example 2.11. The convolution operator with convolution kernel the Dirac measure δ_0 is the identity operator. As δ_0 is $(-Q)$-homogeneous, the identity operator is 0-homogeneous by Example 2.10; this is also trivially true.

One checks readily that a convolution operator T_κ is invariant under left translation.

Definition 2.12. An operator $T : C_c^\infty(G) \to \mathcal{D}'(G)$ is left-invariant when it commutes with the action of left-translations on functions:

$$T(\varphi(g_0\ \cdot)) = (T\varphi)(g_0\ \cdot), \qquad \varphi \in C_c^\infty(G),\ g_0 \in G.$$

The same vocabulary holds for operators between other functional spaces e.g. $L^p(G)$, $\mathcal{S}(G)$, $\mathcal{S}'(G)$ etc.

2.7. **Singular integrals on spaces of homogeneous types.** The operators appearing 'in practice' in the theory of partial differential equations on $\mathbb{R}^n$ often have kernels κ satisfying the following properties:

(1) the restriction of $\kappa(x, y)$ to $(\mathbb{R}^n_x \times \mathbb{R}^n_y) \setminus \{x = y\}$ coincides with a smooth function $\kappa_o = \kappa_o(x, y) \in C^\infty((\mathbb{R}^n_x \times \mathbb{R}^n_y) \setminus \{x = y\})$;
(2) away from the diagonal $x = y$, the function κ_o decays rapidly;
(3) at the diagonal, κ_o is singular but not completely wild: κ_o and some of its first derivatives admit a control of the form $|\kappa_o(x, y)| \leq C_x |x - y|^k$ for some power $k \in (-\infty, \infty)$ with C_x varying slowly in x.

In general, we want our operator T to map continuously some well-known functional space to another. For example, we are looking for conditions to ensure that our operator extends to a bounded operator from L^p to L^q. This is the subject of the theory of singular integrals on $\mathbb{R}^n$, especially when the power k above equals $-n$. In the classical Euclidean case, we refer to the monograph [25] by E. Stein for a detailed presentation of this theory.

Here, let us present the main lines of the generalisation of the theory of singular integrals to the setting of 'spaces of homogeneous type' due to Coifman and Wiess. We omit the proofs, referring to [5, Chapitre III] for details. Note that there is no (apparent) trace of a group structure.

We first need to relax the notion of distance to encompass the positive definite symmetric map that satisfy the triangle inequality up to a constant:

Definition 2.13. A *quasi-distance* on a set X is a function $d : X \times X \to [0, \infty)$ such that

(1) $d(x, y) > 0$ if and only if $x \neq y$;
(2) $d(x, y) = d(y, x)$;
(3) there exists a constant $K > 0$ such that

$$\forall x, y, z \in X \qquad d(x, z) \leq K (d(x, y) + d(y, z)) .$$

We call

$$B(x, r) := \{y \in G \ : \ d(x, y) < r\},$$

the *quasi-ball of radius r around x.*

Definition 2.14. A *space of homogeneous type* is a topological space X equipped with a quasi-distance d such that

(1) The quasi-balls $B(x, r)$ form a basis of open neighbourhood at x;
(2) homogeneity property

    there exists $N \in \mathbb{N}$ such that for every $x \in X$ and every $r > 0$ the ball $B(x, r)$ contains at most N points x_i such that $d(x_i, x_j) > r/2$.

The constants K in Definition 2.13 and N in Definition 2.14 are called the constants of the space of homogeneous type X.

Some authors (like in the original text of [5]) prefer using the vocabulary pseudo-norms, pseudo-distance, etc. instead of quasi-norms, quasi-distance, etc. In this lecture notes, following e.g. both Stein [25] and Wikipedia, we choose the perhaps more widely adapted convention of the term quasi-norm.

Examples of spaces of homogeneous type:

(1) A homogeneous Lie group equipped with the quasi-distance associated to any homogeneous quasi-norm.
(2) The unit sphere $\mathbb{S}^{n-1}$ in $\mathbb{R}^n$ with the quasi-distance

$$d(x, y) = |1 - x \cdot y|^{\alpha},$$

where $\alpha > 0$ and $x \cdot y = \sum_{j=1}^{n} x_j y_j$ is the real scalar product of $x, y \in \mathbb{R}^n$.
(3) The unit sphere $\mathbb{S}^{2n-1}$ embedded in $\mathbb{C}^n$ with the quasi-distance

$$d(z, w) = |1 - (z, w)|^{\alpha},$$

where $\alpha > 0$ and $(z, w) = \sum_{j=1}^{n} z_j \bar{w}_j$.
(4) Any compact Riemannian manifold.

The proof that these spaces are effectively of homogeneous type comes easily from the following lemma:

Lemma 2.15. *Let X be a topological set equipped with a quasi-distance d satisfying (1) of Definition 2.14.*

Assume that there exist a Borel measure μ on X satisfying

$$0 < \mu\left(B(x,r)\right) \le C\mu\left(B(x,\frac{r}{2})\right) < \infty. \tag{2.2}$$

Then X is a space of homogeneous type.

The condition (2.2) is called the *doubling condition*. For instance, the Riemannian measure of a Riemannian compact manifold or the Haar measure of a homogeneous Lie group satisfy the doubling condition; we omit the proof of these facts, as well as the proof of Lemma 2.15.

Let (X,d) be a space of homogeneous type. The hypotheses are 'just right' to obtain a covering lemma. We assume now that X is also equipped with a measure μ satisfying the doubling condition (2.2). A maximal function with respect to the quasi-balls may be defined. Then given a level, any function f can be decomposed 'in the usual way' into good and bad functions $f = g + \sum_j b_j$. The Euclidean proof of the Singular Integral Theorem can be adapted to obtain

Theorem 2.16 (Singular integrals)**.** *Let (X,d) be a space of homogeneous type equipped with a measure μ satisfying the doubling condition given in* (2.2)*.*

Let T be an operator which is bounded on $L^2(X)$:

$$\exists C_o \qquad \forall f \in L^2 \quad \|Tf\|_2 \le C_o\|f\|_2. \tag{2.3}$$

We assume that there exists a locally integrable function κ on $(X\times X)\backslash\{(x,y)\in X\times X : x=y\}$ such that for any compactly supported function $f \in L^2(X)$, we have

$$\forall x \notin \mathrm{supp} f \qquad Tf(x) = \int_X \kappa(x,y)f(y)d\mu(y).$$

We also assume that there exist $C_1, C_2 > 0$ such that

$$\forall y, y_o \in X \qquad \int_{d(x,y_o)>C_1 d(y,y_o)} |\kappa(x,y)-\kappa(x,y_o)|d\mu(x) \le C_2. \tag{2.4}$$

Then for all p, $1 < p \le 2$, T extends to a bounded operator on L^p because

$$\exists A_p \qquad \forall f \in L^2\cap L^p \quad \|Tf\|_p \le A_p\|f\|_p;$$

for $p = 1$, the operator T extends to a weak-type (1,1) operator since

$$\exists A_1 \qquad \forall f \in L^2\cap L^1 \quad \mu\{x \ : \ |Tf(x)| > \alpha\} \le A_1\frac{\|f\|_1}{\alpha};$$

the constants A_p, $1 \le p \le 2$, depend only on C_o, C_1 and C_2.

Remark 2.17. In the statement of the fundamental theorem of singular integrals on spaces of homogeneous types, cf. [5, Théorème 2.4 Chapitre III], the kernel κ is assumed to be square integrable in $L^2(X\times X)$. However, the proof requires only that the kernel κ is locally integrable away from the diagonal, beside the L^2-boundedness of the operator T. We have therefore chosen to state it in the form given above.

Let us discuss the two main hypotheses of Theorem 2.16.

About Condition (2.4) *in the Euclidean case.* As explained at the beginning of this section, we are interested in 'nice' kernels $\kappa_o(x,y)$ with a control of the form $|\kappa_o(x,y)| \le C_x|x-y|^k$ with a particular interest for $k=-n$, and similar estimates for their derivatives with power $-n-1$. Hence they should satisfy Condition (2.4). They are called Calderón-Zygmund kernels, which we now briefly recall:

Definition 2.18. A Calderón-Zygmund kernel on $\mathbb{R}^n$ is a measurable function κ_o defined on $(\mathbb{R}^n_x \times \mathbb{R}^n_y)\backslash\{x=y\}$ satisfying for some γ, $0<\gamma\leq 1$, the inequalities

$$\begin{aligned}
|\kappa_o(x,y)| &\leq A|x-y|^{-n}, \\
|\kappa_o(x,y)-\kappa_o(x',y)| &\leq A\frac{|x-x'|^\gamma}{|x-y|^{n+\gamma}} \qquad \text{if } |x-x'|\leq \frac{|x-y|}{2}, \\
|\kappa_o(x,y)-\kappa_o(x,y')| &\leq A\frac{|y-y'|^\gamma}{|x-y|^{n+\gamma}} \qquad \text{if } |y-y'|\leq \frac{|x-y|}{2}.
\end{aligned}$$

Sometimes the condition of Calderón-Zygmund kernels refers to a smooth function κ_o defined on $(\mathbb{R}^n_x \times \mathbb{R}^n_y)\backslash\{x=y\}$ satisfying

$$\forall \alpha,\beta \quad \exists C_{\alpha,\beta} \qquad \left|\partial_x^\alpha \partial_y^\beta \kappa_o(x,y)\right| \leq C_{\alpha,\beta}|x-y|^{-n-\alpha-\beta}.$$

For a detailed discussion, the reader is directed to [25, ch.VII].

Definition 2.19. A *Calderón-Zygmund operator* on $\mathbb{R}^n$ is an operator $T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ such that the restriction of its kernel κ to $(\mathbb{R}^n_x\times\mathbb{R}^n_y)\backslash\{x=y\}$ is a Calderón-Zygmund kernel κ_o. In other words, $T:\mathcal{S}(\mathbb{R}^n)\to\mathcal{S}'(\mathbb{R}^n)$ is a Calderón-Zygmund operator if there exists a Calderón-Zygmund kernel κ_o satisfying

$$Tf(x)=\int_{\mathbb{R}^n}\kappa_o(x,y)f(y)dy,$$

for $f\in\mathcal{S}(\mathbb{R}^n)$ with compact support and $x\in\mathbb{R}^n$ outside the support of f.

The Calderón-Zygmund conditions imply Condition (2.4) for the operator T and its formal adjoint T^* but they are not sufficient to imply the L^2-boundedness.

About Condition (2.3). The difficulty with applying the main theorem of singular integrals (i.e. Theorem 2.16) is often to know that the operator is L^2-bounded. In the particular case of a convolution operator T_κ on $\mathbb{R}^n$, the Plancherel formula for the Euclidean Fourier transform yields

$$\text{(2.5)} \quad \|T_\kappa f\|_{L^2(\mathbb{R}^n)}=\|f*\kappa\|_{L^2(\mathbb{R}^n)}=\|\widehat{f}\,\widehat{\kappa}\|_{L^2(\mathbb{R}^n)}, \quad \text{which implies} \quad \|T_\kappa\|_{\mathscr{L}(L^2(\mathbb{R}^n))}=\|\widehat{\kappa}\|_{L^\infty(\mathbb{R}^n)},$$

and therefore the hypothesis $\|\widehat{\kappa}\|_{L^\infty(\mathbb{R}^n)}<\infty$ provides the desired L^2-boundedness.

2.8. **Comments.**

2.8.1. *Application to multiplier problems.* The theory of singular integrals on $\mathbb{R}^n$, aka the Calderón-Zygmund theory, is applied for instance to Fourier multipliers. Indeed, the Mihlin multiplier theorem [20, 21] states that if a function σ defined on $\mathbb{R}^n\backslash\{0\}$ has at least $[d/2]+1$ continuous derivatives that satisfy

$$\text{(2.6)} \qquad \forall\alpha\in\mathbb{N}_0^d,\ |\alpha|\leq[d/2]+1, \qquad |\partial^\alpha\sigma(\xi)|\leq C_\alpha|\xi|^{-|\alpha|},$$

then the Fourier multiplier operator M_σ associated with σ, initially defined on Schwartz functions via

$$\text{(2.7)} \qquad M_\sigma\phi:=\mathcal{F}^{-1}\{\sigma\widehat{\phi}\},$$

admits a bounded extension on $L^p(\mathbb{R}^d)$ for all $1<p<\infty$. Above $[t]$ is the integer part of t and $\mathcal{F}\phi=\widehat{\phi}$ denotes the Euclidean Fourier transform of a function ϕ. Hörmander improved the Mihlin multiplier theorem by showing [17] that a sufficient condition for M_σ to be bounded on $L^p(\mathbb{R}^d)$ is the membership of σ locally uniformly to a Sobolev space $H^s(\mathbb{R}^d)$ for some $s>d/2$, that is,

$$\text{(2.8)} \qquad \exists\eta\in\mathcal{D}(0,\infty),\ \eta\not\equiv 0, \qquad \sup_{r>0}\|\sigma(r\,\cdot)\ \eta(|\cdot|^2)\|_{H^s}<\infty.$$

If a multiplier satisfies the Hörmander condition in (2.8) with s near enough $d/2$, then it satisfies the Mihlin condition in (2.6). Note that proceeding as in (2.5), $\|M_\sigma\|_{\mathscr{L}(L^2(\mathbb{R}^n))} = \|\sigma\|_{L^\infty(\mathbb{R}^n)}$, so the the L^2-boundedness necessary to apply a theorem like Theorem 2.16 for convolution operator on $\mathbb{R}^n$ is obtained easily from the use of the Euclidean Fourier transform.

Anisotropic analogues of the Hörmander condition in (2.8) have been studied by Rivière [23]. Note that the anisotropic setting is covered by the Singular Integral theory on spaces of homogeneous type by Coifman & Weiss presented above.

The theory of singular integral may also be applied to prove the continuity $L^p(G) \to L^p(G)$ of spectral multipliers $F(\mathcal{L}_G)$ of a sub-Laplacian $\mathcal{L}_G$ on a Carnot group G. Here, $F : \mathbb{R} \to \mathbb{R}$ is a measurable function, and the L^2-boundedness usually comes from functional analysis: $\|F(\mathcal{L}_G)\|_{\mathscr{L}(L^2(G))} \leq \|F\|_{L^\infty(\mathbb{R})}$.

2.8.2. *What is there beside $L^p \to L^p$ continuity?* Theorem 2.16 insists on the L^p-boundedness of certain operators. However, the theory of singular integrals is in fact more like a philosophy: operators can still be reasonably well-behaved when their integral kernels have singularities on the diagonal as long as the singularity corresponds to a certain homogeneity that is in a way integrable.

The theory of singular integrals was mainly due to Calderón and Zygmund in the 40's and 50's, with major improvement by Elias Stein in the 60's. This has led the way to the theory of pseudo-differential operators, and then micro-local analysis in the 70's.

3. The Group Fourier transform on $\mathbb{H}_n$

The Euclidean Fourier transform on $\mathbb{R}^n$ is well known and understood by analysts, in particular the Plancherel theorem

$$\|f\|_{L^2(\mathbb{R}^n)} = \|\widehat{f}\|_{L^2(\mathbb{R}^n)} \quad \text{and} \quad \|T_\kappa\|_{\mathscr{L}(L^2(\mathbb{R}^n))} = \|\widehat{\kappa}\|_{L^\infty(\mathbb{R}^n)}, \tag{3.1}$$

where $f \in L^2(\mathbb{R}^n)$, and $T_\kappa : \varphi \mapsto \varphi * \kappa$ denotes the convolution operator with convolution kernel $\kappa \in L^\infty(\mathbb{R}^n)$. From the viewpoint of representation theory, this is an easy case since the group $(\mathbb{R}^n, +)$ is commutative. Non-commutative cases are often much more complicated.

For certain questions, it is possible to use some weaker properties of commutativity. For instance, this is the case of spectral multipliers in one self-adjoint operator $\mathcal{L}$ on $L^2(M)$; although the space M may not have any particular structure and $\mathcal{L}$ may be unbounded, the spectral theorem for $\mathcal{L}$ yields a decomposition of $L^2(M)$ and an understanding of $\|\cdot\|_{\mathscr{L}(L^2(M))}$ for these spectral multipliers for continuous bounded functions. Note that the C^*-algebra $\{F(\mathcal{L}) \in \mathscr{L}(L^2(M)), F : \mathbb{R} \to \mathbb{R}$ continuous bounded$\}$ is commutative. This generalises to a strongly commuting family of self-adjoint operators on a space $L^2(M)$.

Here, we are going to give some elements of representations theory, especially for the Heisenberg group $\mathbb{H}_n$ and its group Fourier transform.

3.1. Elements of representation theory.

Recall that a *representation* $(\mathcal{H}_\pi, \pi)$ of G is a pair consisting of a Hilbert space $\mathcal{H}_\pi$ and a group morphism π from G to the set of unitary transforms on $\mathcal{H}_\pi$. Here, the representations will always be assumed (unitary) strongly continuous, and their Hilbert spaces separable. A representation is said to be *irreducible* if the only closed subspaces of $\mathcal{H}_\pi$ that are stable under π are $\{0\}$ and $\mathcal{H}_\pi$ itself. Two representations π_1 and π_2 are equivalent if there exists a unitary transform $\mathbb{U}$ called an *intertwining map* that sends $\mathcal{H}_{\pi_1}$ on $\mathcal{H}_{\pi_2}$ with $\pi_1 = \mathbb{U}^{-1} \circ \pi_2 \circ \mathbb{U}$. The *dual set* $\widehat{G}$ is obtained by taking the quotient of the set of irreducible representations by this equivalence relation. We may still denote by π the elements of $\widehat{G}$ and we keep in mind that different representations of the class are equivalent through intertwining operators.

Example 3.1. The group $\mathrm{GL}_n(\mathbb{C})$ acts on the Hilbert space $\mathbb{C}^n$ via $\pi(M)v = Mv$, $M \in \mathrm{GL}_n(\mathbb{C})$, $v \in \mathbb{C}^n$.

The irreducible representations of $(\mathbb{R}^n, +)$ are one dimensional and are given by the characters $x \mapsto e^{i2\pi x\cdot\xi}$, $\xi \in \mathbb{R}^n$. The Fourier transform is defined as

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)\overline{e^{i2\pi x\cdot\xi}}dx, \quad f \in L^1(\mathbb{R}^n).$$

In fact it may be defined on $\mathcal{S}'(\mathbb{R}^n)$. The Plancherel theorem (3.1) holds.

Pontryagin duality: The case of $\mathbb{R}^n$ generalises (except for the $\mathcal{S}'$ part) to any locally compact abelian group G. Indeed, any irreducible representation is one dimensional and therefore may be identified with a character. The dual $\widehat{G}$ of G identifies with the space of characters on G, and it is equipped naturally with a structure of locally compact abelian group. Moreover, $\widehat{\widehat{G}} \sim G$. We can define the Fourier transform

$$\widehat{f}(\chi) = \int_G f(x)\overline{\chi(x)}dx, \qquad f \in L^1(G)$$

where dx denotes a (fixed) Haar measure on G. The Plancherel theorem (3.1) holds on G.

The non-commutative case is more complicated but more interesting as well. In the case of a reasonable group G (technically: separable locally compact, unimodular, and of type I), $\widehat{G}$ is equipped with a metrisable topology, and moreover a measure $\mu_{\widehat{G}}$, called the Plancherel measure that is characterised as the unique measure on $\widehat{G}$ for which the Plancherel formula explained below holds. The notion of Fourier transform is the following. If π is a representation of G and $f \in L^1(G)$, we define

$$\pi(f) = \mathcal{F}_G f(\pi) = \widehat{f}(\pi) := \int_G f(x)\pi(x)^* dx \in \mathscr{L}(\mathcal{H}_\pi).$$

The Fourier transform of f is the measurable fields of operators $\widehat{f}(\pi) \in \mathscr{L}(\mathcal{H}_\pi)$ where $\pi \in \widehat{G}$, understood as a class of operator modulo intertwining operators. The Plancherel formula is

$$\|f\|^2_{L^2(G)} = \int_{\widehat{G}} \|\pi(f)\|^2_{HS(\mathcal{H}_\pi)} d\mu_{\widehat{G}}(\pi).$$

It holds for $f \in L^1(G) \cap L^2(G)$ and extends unitarily to $L^2(G)$. The second part of the Plancherel theorem is that if $T : L^2(G) \to L^2(G)$ is invariant under left-translation, then there exists a measurable field of operators $\widehat{T}(\pi) \in \mathscr{L}(\mathcal{H}_\pi)$ where $\pi \in \widehat{G}$ such that $\widehat{Tf} = \widehat{T}\widehat{f}$ for any $f \in L^2(G)$. These results were proved by Dixmier in the 60's [8].

Many classes of Lie groups are 'reasonable' in the sense above (technically: separable locally compact, unimodular, and of type I): semi-simple, compact, nilpotent etc. Moreover, in this case, any representation π of the group yields a representation of its Lie algebra via $\pi(X) = \partial_{t=0}\pi(\exp tX)$, and also of the universal enveloping algebra. Note that the action is on a subspace of $\mathcal{H}_\pi$, namely the space $\mathcal{H}_\pi^\infty$ of smooth vectors.

In the case of a compact Lie group G, the unitary dual $\widehat{G}$ is discrete and well understood via the highest weight theory. The Plancherel measure is completely explicit.

In the case of a nilpotent Lie group G, the unitary dual $\widehat{G}$ is also well understood via the orbit method due to Kirillov. The Plancherel measure is also completely explicit. In the next sections, we will give a description of the case of the Heisenberg group $\mathbb{H}_n$.

3.2. **Schrödinger representations π_λ.** The Schrödinger representations of the Heisenberg group $\mathbb{H}_n$ are the infinite dimensional unitary representations of $\mathbb{H}_n$, where, as usual, we allow ourselves to identify unitary representations with their unitary equivalence classes. They are parametrised by $\lambda \in \mathbb{R}\backslash\{0\}$. We denote these representations π_λ. Each π_λ acts on the Hilbert space $\mathcal{H}_{\pi_\lambda} = L^2(\mathbb{R}^n)$ in the way we now describe. An element of $L^2(\mathbb{R}^n)$ will very often be denoted as a function h of the variable $u = (u_1, \ldots, u_n) \in \mathbb{R}^n$.

First let us define π_1 corresponding to $\lambda = 1$. It is the representation of the group $\mathbb{H}_n$ acting on $L^2(\mathbb{R}^n)$ via

$$\pi_1(x, y, t)h(u) := e^{i(t+\frac{1}{2}xy)}e^{iyu}h(u + x),$$

for $h \in L^2(\mathbb{R}^n)$ and $(x, y, t) \in \mathbb{H}_n$. Here xy denotes the scalar product in $\mathbb{R}^n$ of x and y, and similarly for yu. Consequently its infinitesimal representation is given by

$$\left\{\begin{array}{rcl} \pi_1(X_j) & = & \partial_{u_j} \quad \text{(differentiate with respect to } u_j), \quad j = 1, \ldots, n, \\ \pi_1(Y_j) & = & iu_j, \quad \text{(multiplication by } iu_j), \quad j = 1, \ldots, n, \\ \pi_1(T) & = & i\mathrm{I}, \quad \text{(multiplication by } i). \end{array}\right.$$

The Schrödinger representations π_λ on the group are realised here using

$$\pi_\lambda(x, y, t) := \left\{\begin{array}{ll} \pi_1 \circ D_{\sqrt{\lambda}}(x, y, t) & \text{if } \lambda > 0, \\ \pi_{-\lambda}(x, -y, -t) & \text{if } \lambda < 0, \end{array}\right.$$

that is,

$$\pi_\lambda(x, y, t)h(u) = e^{i\lambda(t+\frac{1}{2}xy)}e^{i\sqrt{\lambda}yu}h(u + \sqrt{|\lambda|}x),$$

for $h \in L^2(\mathbb{R}^n)$ and $(x, y, t) \in \mathbb{H}_n$ where we use the following convention:

$$\sqrt{\lambda} := \mathrm{sgn}(\lambda)\sqrt{|\lambda|} = \begin{cases} \sqrt{\lambda} & \text{if } \lambda > 0, \\ -\sqrt{|\lambda|} & \text{if } \lambda < 0. \end{cases}$$

We keep the same notation, here π_λ for the corresponding infinitesimal representation. The infinitesimal representation of π_λ acts on the canonical basis of $\mathfrak{h}_n$ via

$$\pi_\lambda(X_j) = \sqrt{|\lambda|}\partial_{u_j}, \ \pi_\lambda(Y_j) = i\sqrt{\lambda}u_j, \ j = 1, \ldots, n, \quad \text{and} \quad \pi_\lambda(T) = i\lambda \mathrm{I}.$$

Consequently, the group Fourier transform of the sub-Laplacian

$$\mathcal{L} = \sum_{j=1}^{n} (X_j^2 + Y_j^2)$$

is

$$\pi_\lambda(\mathcal{L}) = |\lambda| \sum_{j=1}^{n} (\partial_{u_j}^2 - u_j^2).$$

A direct characterisation implies that the space of smooth vectors of π_λ is

$$\mathcal{H}_{\pi_\lambda}^\infty = \mathcal{S}(\mathbb{R}^n).$$

This is true more generally for any representation of a connected simply connected nilpotent Lie group realised on some $L^2(\mathbb{R}^m)$ via the orbit method, see [6, Corollary 4.1.2].

3.3. **The group Fourier transform on the Heisenberg group.** We could have realised the equivalence classes $[\pi_\lambda]$ of Schrödinger representations in various ways. For instance by composing with the unitary operator $U_\lambda : L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ given by $Uf(x) = |\lambda|^{\frac{n}{2}} f(\sqrt{\lambda}x)$, one would have obtained a slightly different, although equivalent, representation. Another realisation is with the Bargmann representations.

The group Fourier transform of a function $f \in L^1(\mathbb{H}_n)$ at π_1 is

$$\widehat{f}(\pi_1) = \mathcal{F}_{\mathbb{H}_n}(f)(\pi_1) = \pi_1(f) = \int_{\mathbb{H}_n} \kappa(x, y, t)\pi_1(x, y, t)^* dx dy dt,$$

that is, the operator on $L^2(\mathbb{R}^n)$ given by

$$\pi_1(f)h(u) = \int_{\mathbb{R}^{2n+1}} f(x, y, t)e^{i(-t+\frac{1}{2}xy)}e^{-iyu}h(u - x)dx dy dt$$

Proposition 3.2. *Let $f \in \mathcal{S}(\mathbb{H}_n)$. Then for each $\lambda \in \mathbb{R}\backslash\{0\}$ the operator $\widehat{f}(\pi_\lambda)$ acting on $L^2(\mathbb{R}^n)$ is the Hilbert-Schmidt operator with integral kernel*

$$K_{f,\lambda} : \mathbb{R}^n \times \mathbb{R}^n \longrightarrow \mathbb{C},$$

given by

$$K_{f,\lambda}(u, v) = (2\pi)^{n+\frac{1}{2}} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}\frac{u + v}{2}, \lambda)d\xi,$$

and Hilbert-Schmidt norm

$$\begin{aligned} \|\widehat{f}(\pi_\lambda)\|_{HS(L^2(\mathbb{R}^n))} &= (2\pi)^{\frac{3n+1}{2}} |\lambda|^{-\frac{n}{2}} \|\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\cdot, \cdot, \lambda)\|_{L^2(\mathbb{R}^{2n})} \\ &= (2\pi)^{\frac{3n+1}{2}} |\lambda|^{-\frac{n}{2}} \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\xi, w, \lambda)|^2 d\xi dw \right)^{\frac{1}{2}}. \end{aligned}$$

Here, we have used the following notation for the Euclidean Fourier transform

$$\mathcal{F}_{\mathbb{R}^N}\varphi(\xi) = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^N} \varphi(x)e^{-ix\xi}dx, \qquad \varphi \in L^1(\mathbb{R}^N), \ \xi \in \mathbb{R}^N.$$

Furthermore, we have

$$\int_{\mathbb{H}_n} |f(x,y,t)|^2 dxdydt = c_n \int_{\lambda\in\mathbb{R}\backslash\{0\}} \|\widehat{f}(\pi_\lambda)\|^2_{HS(L^2(\mathbb{R}^n))} |\lambda|^n d\lambda,$$

where $c_n = (2\pi)^{-(3n+1)}$.

In particular, Proposition 3.2 implies that the Plancherel measure $\mu_{\mathbb{H}_n}$ on the Heisenberg group is supported in $\{[\pi_\lambda],\ \lambda \in \mathbb{R}\backslash\{0\}\}$, see (3.2). Moreover, we have

$$d\mu_{\mathbb{H}_n}(\pi_\lambda) \equiv c_n |\lambda|^n d\lambda, \quad \lambda\in\mathbb{R}\backslash\{0\}.$$

The constant c_n depends on our choice of realisation of $\pi_\lambda \in [\pi_\lambda]$.

Proof of Proposition 3.2. We have for $h \in L^2(\mathbb{R}^n)$ and $u\in\mathbb{R}^n$,

$$\begin{aligned}\widehat{f}(\pi_\lambda)h(u) &= (2\pi)^{n+\frac{1}{2}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}\frac{u+v}{2}, \lambda) h(v) dv d\xi \\ &= \int_{\mathbb{R}^n} K_{f,\lambda}(u,v) h(v) dv,\end{aligned}$$

where $K_{f,\lambda}$ is the integral kernel of $\widehat{f}(\pi_\lambda)$ hence given by

$$K_{f,\lambda}(u,v) = (2\pi)^{n+\frac{1}{2}} \int_{\mathbb{R}^n} e^{i(u-v)\xi} \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi, \sqrt{\lambda}\frac{u+v}{2}, \lambda) d\xi.$$

Using the Euclidean Fourier transform, we may rewrite this as

$$K_{f,\lambda}(u,v) = (2\pi)^{\frac{3}{2}n+\frac{1}{2}} \mathcal{F}_{\mathbb{R}^n}\left\{ \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\frac{u+v}{2}, \lambda)\right\}(v-u).$$

The $L^2(\mathbb{R}^n\times\mathbb{R}^n)$-norm of the integral kernel is

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} |K_{f,\lambda}(u,v)|^2 dudv \\ &\quad = (2\pi)^{3n+1} \int_{\mathbb{R}^n\times\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^n}\left\{ \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\,\cdot, \sqrt{\lambda}\frac{u+v}{2}, \lambda)\right\}(v-u)|^2 dudv \\ &\quad = (2\pi)^{3n+1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^n}\left\{ \mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\,\cdot, w_2, \lambda)\right\}(w_1)|^2 |\lambda|^{-\frac{n}{2}} dw_1 dw_2,\end{aligned}$$

after the change of variable $(w_1, w_2) = (v-u, \sqrt{\lambda}\frac{u+v}{2})$. The (Euclidean) Plancherel formula on $\mathbb{R}^n$ in the variable w_1 (with dual variable ξ_1) then yields

$$\begin{aligned}&\int_{\mathbb{R}^n\times\mathbb{R}^n} |K_{f,\lambda}(u,v)|^2 dudv \\ &\quad = (2\pi)^{3n+1} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\sqrt{|\lambda|}\xi_1, w_2, \lambda)|^2 |\lambda|^{-\frac{n}{2}} d\xi_1 dw_2 \\ &\quad = (2\pi)^{3n+1} |\lambda|^{-n} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\xi, w_2, \lambda)|^2 d\xi dw_2,\end{aligned}$$

after the change of variable $\xi = \sqrt{|\lambda|}\xi_1$. Since $f\in\mathcal{S}(\mathbb{H}_n)$, this quantity is finite. Since the integral kernel of $\widehat{f}(\pi_\lambda)$ is square integrable, the operator $\widehat{f}(\pi_\lambda)$ is Hilbert-Schmidt and its Hilbert-Schmidt norm is the L^2-norm of its integral kernel. This shows the first part of the statement.

To finish the proof, we now integrate each side of the last equality against $|\lambda|^n d\lambda$ and then use again the (Euclidean) Plancherel formula on $\mathbb{R}^{2n+1}$ in the variable (ξ, w_2, λ). We obtain

$$\begin{aligned}\int_{\mathbb{R}\backslash\{0\}} \int_{\mathbb{R}^n\times\mathbb{R}^n} &|K_{f,\lambda}(u,v)|^2 du dv\ |\lambda|^n d\lambda \\ &= (2\pi)^{3n+1} \int_{\mathbb{R}\backslash\{0\}} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} |\mathcal{F}_{\mathbb{R}^{2n+1}}(f)(\xi, w_2, \lambda)|^2 d\xi dw_2 d\lambda \\ &= (2\pi)^{3n+1} \int_{\mathbb{R}^{2n+1}} |f(x,y,t)|^2 dx dy dt.\end{aligned}$$

This concludes the proof of Proposition 3.2. □

It follows from the Plancherel formula in Proposition 3.2 that the Schrödinger representations π_λ, $\lambda \in \mathbb{R}\backslash\{0\}$, are almost all the representations of $\mathbb{H}_n$ modulo unitary equivalence. 'Almost all' here refers to the Plancherel measure $\mu_{\mathbb{H}_n} = c_n|\lambda|^n d\lambda$ on $\widehat{\mathbb{H}}_n$. The other representations are finite dimensional and in fact 1-dimensional. They are given by the unitary characters of $\mathbb{H}_n$

$$\chi_w : (x,y,t) \mapsto e^{i(xw_1+yw_2)}, \quad w = (w_1, w_2) \in \mathbb{R}^n \times \mathbb{R}^n \sim \mathbb{R}^{2n}.$$

We can summarise this paragraph by writing

$$\widehat{\mathbb{H}}_n = \{[\pi_\lambda],\ \lambda \in \mathbb{R}\backslash\{0\}\} \bigcup \left\{[\chi_w],\ w \in \mathbb{R}^{2n}\right\} \overset{\mu_{\mathbb{H}_n}\text{-a.e.}}{=} \{[\pi_\lambda],\ \lambda \in \mathbb{R}\backslash\{0\}\}\,. \tag{3.2}$$

References

[1] A. Agrachev, D. Barilari and U. Boscain, *A comprehensive introduction to sub-Riemannian geometry*, Cambridge Studies in Advanced Mathematics, **181**, From the Hamiltonian viewpoint, With an appendix by Igor Zelenko, Cambridge University Press, Cambridge, 2020.

[2] V. Arnold, *Mathematical methods of classical mechanics*, Graduate Texts in Mathematics, **60**, Translated from the 1974 Russian original by K. Vogtmann and A. Weinstein, Corrected reprint of the second (1989) edition, Springer-Verlag, New York.

[3] A. Bellaïche, The tangent space in sub-Riemannian geometry, in *Sub-Riemannian geometry*, Progr. Math., **144**, pp 1–78, Birkhäuser, Basel, 1996.

[4] A. Bonfiglioli, E. Lanconelli and F. Uguzzoni, *Stratified Lie groups and potential theory for their sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[5] R. Coifman and G. Weiss, *Analyse harmonique non-commutative sur certains espaces homogènes*, Lecture Notes in Mathematics, Vol. 242, Springer-Verlag, Berlin-New York, 1971.

[6] L. Corwin and F. Greenleaf, *Representations of nilpotent Lie groups and their applications. Part I*, Cambridge Studies in Advanced Mathematics, **18**, Basic theory and examples, Cambridge University Press, Cambridge, 1990.

[7] J. Cygan, Subadditivity of homogeneous norms on certain nilpotent Lie groups, *Proc. Amer. Math. Soc.*, **83**, 1981, No 1, pp 69–70. ISSN = 0002-9939,

[8] J. Dixmier, *C^*-algebras*, North-Holland Mathematical Library, Vol. 15, Translated from the French by Francis Jellett, North-Holland Publishing Co., Amsterdam-New York-Oxford, 1977.

[9] A. Dooley, Anthony, and F. Ricci, The contraction of K to $\overline{N}M$, *J. Funct. Anal.*, **63**, 1985, No 3, pp 344–368.

[10] Fischer, Veronique and Ruzhansky, Michael, *Quantization on nilpotent Lie groups*, Progress in Mathematics, **314**, Birkhäuser/Springer, [Cham], 2016.

[11] G. Folland, Applications of analysis on nilpotent groups to partial differential equations, *Bull. Amer. Math. Soc.*, **83**, 1977, No 5, pp 912–930.

[12] G. Folland and E. Stein, *Hardy spaces on homogeneous groups*, Mathematical Notes, **28**, Princeton University Press, Princeton, N.J.; University of Tokyo Press, Tokyo, 1982.

[13] B. Hall, *Lie groups, Lie algebras, and representations*, Graduate Texts in Mathematics, **222**, Second edition, An elementary introduction, Springer, Cham, 2015.

[14] S. Helgason, *Geometric analysis on symmetric spaces*, Mathematical Surveys and Monographs, **39**, Second edition, American Mathematical Society, Providence, RI, 2008.

[15] S. Helgason, *Differential geometry, Lie groups, and symmetric spaces*, Graduate Studies in Mathematics, **34**, Corrected reprint of the 1978 original, American Mathematical Society, Providence, RI, 2001.

[16] S. Helgason, *Groups and geometric analysis*, Mathematical Surveys and Monographs, **83**, Integral geometry, invariant differential operators, and spherical functions, Corrected reprint of the 1984 original, American Mathematical Society, Providence, RI, 2000.

[17] Hörmander, L., Estimates for translation invariant operators in L^p spaces, *Acta Math.* **104** (1960), pp 93–140.

[18] L. Hörmander, Hypoelliptic second order differential equations, Acta Math., **119**, 1967, pp 147–171.

[19] B. Krötz, S. Thangavelu and Y. Xu Heat kernel transform for nilmanifolds associated to the Heisenberg group, Rev. Mat. Iberoam., **24**, 2008, No 1, pp 243–266.

[20] Mihlin, S. G., On the multipliers of Fourier integrals, *Dokl. Akad. Nauk SSSR (N.S.)* **109** (1956), pp 701–703.

[21] Mihlin, S. G. Fourier integrals and multiple singular integrals, *Vestnik Leningrad. Univ. Ser. Mat. Meh. Astr.* **12** (1957), no. 7, pp 143–155.

[22] A. Nagel, E. Stein and S. Wainger, Balls and metrics defined by vector fields. I. Basic properties, Acta Math., **155**, 1985, No 1-2, pp 103–147.

[23] Rivière, N. M., Singular integrals and multiplier operators, *Ark. Mat.* **9** (1971), pp 243–278.

[24] L. Rothschild and E. Stein, Hypoelliptic differential operators and nilpotent groups, *Acta Math.*, **137**, 1976, No 3-4, 247–320.

[25] E. Stein, *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, *Princeton Mathematical Series*, **43**, Princeton University Press, Princeton, NJ, 1993.

[26] S. Thangavelu, Harmonic analysis on Heisenberg nilmanifolds, *Rev. Un. Mat. Argentina*, **50**, 2009, No 2, pp 75–93.

[27] L. Tu, *An introduction to manifolds*, Universitext, Second edition, Springer, New York, 2011.

[28] V. Varadarajan, *Lie groups, Lie algebras, and their representations*, Graduate Texts in Mathematics, **102**, Reprint of the 1974 edition, Springer-Verlag, New York, 1984.