Solutions to Exercise Sheet 9

1. This is just another way of saying: find the general antiderivative of $9x$. The solution is
$$\int 9x\,dx = \frac{9}{2}x^2 + C.$$

2. We use separation of variables. Thus
$$x\frac{dy}{dx} = yx^2$$
becomes
$$\int \frac{dy}{y} = \int x\,dx.$$
This implies that
$$\ln|y| = \frac{x^2}{2} + C$$
Solving the equation for y, we find that
$$y = \pm e^{x^2/2+C} = Ae^{x^2/2},$$
where $A = \pm e^C$. Now we look for the value of the constant A that corresponds to the initial condition $y = 3$ at $x = 0$. This gives the equation $3 = Ae^0$, and so $A = 3$. Therefore the solution of the initial value problem is $y = 3e^{x^2/2}$.

3. (a) The easiest way to solve this problem is by separation of variables. We first note that we can rewrite it as
$$\frac{dy}{dx} = \frac{1}{3}y(x^2 - 1).$$
Thus
$$\int \frac{dy}{y} = \frac{1}{3}\int (x^2 - 1)\,dx,$$
and therefore,
$$\ln|y| = \frac{1}{3}\left(\frac{x^3}{3} - x\right) + C.$$
It follows that
$$y = Ae^{x^3/9 - x/3},$$
where $A = \pm e^C$.

(b) Divide through by x to rewrite the problem as

$$\frac{dy}{dx} + 2xy = 6x.$$

We use an integrating factor m, which is given as the solution of

$$\frac{dm}{dx} = 2xm,$$

namely $m = e^{x^2}$. Now the function $z = my$ satisfies

$$\frac{dz}{dx} = \frac{d}{dx}(my) = m\frac{dy}{dx} + \frac{dm}{dx}y = m\frac{dy}{dx} + 2xmy = 6mx.$$

Hence

$$z = \int 6xe^{x^2}\,dx = 3e^{x^2} + C,$$

and therefore

$$y = \frac{z}{m} = 3 + Ce^{-x^2}.$$

(c) First we try to find a particular solution with the method of undetermined coefficients. That is, we try to find a solution of the equation of the form $y = (Ax + B)e^{-5x}$. Then

$$\frac{dy}{dx} + 4y = Ae^{-5x} - 5(Ax + B)e^{-5x} + 4(Ax + B)e^{-5x} = (-Ax + A - B)e^{-5x}.$$

The right hand side should coincide with xe^{-5x}, so we choose $A = -1$ and $B = -1$. That is, the function $y = -(x + 1)e^{-5x}$ solves the equation.
Next we consider the homogeneous equation

$$\frac{dy}{dx} + 4y = 0. \tag{1}$$

Using separation of variables, we see that $y = Ce^{-4x}$ is the general solution of (1). Now the general solution of the equation from the question is the sum of the two:

$$y = Ce^{-4x} - (x + 1)e^{-5x}.$$

(d) Again we try to find a particular solution with the method of undetermined coefficients. We guess that a particular solution of the form $y = A\sin x + B\cos x$ may exist. If so, then

$$\begin{aligned}\frac{dy}{dx} + 5y &= (A\cos x - B\sin x) + 5(A\sin x + B\cos x)\\ &= (A + 5B)\cos x + (5A - B)\sin x.\end{aligned}$$

So we need to solve the equations $A + 5B = 0$ and $5A - B = 1$. Multiplying by 5 in the second equation and then adding both, we get $26A = 5$; so $A = \frac{5}{26}$, and then $B = -\frac{1}{26}$. Thus we have the solution

$$y = \frac{5}{26}\sin x - \frac{1}{26}\cos x.$$

The general solution of the corresponding homogeneous equation is

$$y = Ce^{-5x}.$$

So the general solution of the equation from the question is

$$y = Ce^{-5x} + \frac{5}{26}\sin x - \frac{1}{26}\cos x.$$

RM, 14/11/2017