Solutions to Exercise Sheet 8

1. (a) We compute

$$\int_1^\infty e^{-x/5}\,dx = \left[-5e^{-x/5}\right]_1^\infty = \lim_{x\to\infty}(-5e^{-x/5}) + 5e^{-1/5} = 5e^{-1/5},$$

as the limit is zero.

(b) The function $f(x) = (x-1)^{-4}$ satisfies $\lim_{x\to 1^-} f(x)\,dx = \infty$, thus we need to take the limit $x \to 1^-$. Note that

$$\int (x-1)^{-4}\,dx = -\frac{1}{3}(x-1)^{-3} + C$$

and $\lim_{x\to 1^-}(x-1)^{-3} = -\infty$, so

$$\int_0^1 \frac{dx}{(x-1)^4} = -\frac{1}{3}\lim_{x\to 1^-}(x-1)^{-3} + \frac{1}{3}(-1)^{-3} = \infty.$$

(c) We can use the same antiderivative as above. Since $\lim_{x\to 1^+}(x-1)^{-3} = \infty$, it follows that

$$\int_1^2 \frac{dx}{(x-1)^4} = -\frac{1}{3} + \frac{1}{3}\lim_{x\to 1^+}(x-1)^{-3} = \infty.$$

(d) With two integrations by parts, we compute

$$\int x^2 e^x\,dx = x^2e^x - 2\int xe^x\,dx = (x^2-2x)e^x + 2\int e^x\,dx = (x^2-2x+2)e^x + C.$$

Hence

$$\int_{-\infty}^3 x^2e^x\,dx = 5e^3 - \lim_{x\to-\infty}(x^2-2x+2)e^x.$$

L'Hopital's rule gives

$$\lim_{x\to-\infty}(x^2-2x+2)e^x = \lim_{x\to-\infty}\frac{x^2-2x+2}{e^{-x}} = \lim_{x\to-\infty}\frac{2x-2}{-e^{-x}} = \lim_{x\to-\infty}\frac{2}{e^{-x}} = 0.$$

Therefore,

$$\int_{-\infty}^3 x^2e^x\,dx = 5e^3.$$

2. The trapezium rule approximation for $\int_a^b f(x)\,dx$, using N intervals, is

$$\frac{1}{2}h(y_0 + 2(y_1 + \cdots + y_{N-1}) + y_N),$$

where $h = (b-a)/N$ and $y_k = f(a+kh)$.

The following table gives the relevant values for $f(x) = \ln(x)$ on the interval $[1,2]$ with 8 intervals, i.e., $x_k = 1 + kh$ with $h = 1/8$, for $k = 0,\ldots,8$ and $y_k = \ln(x_k)$. For 4 intervals we just take every other value.

x_k	1	1.125	1.25	1.375	1.5	1.625	1.75	1.875	2
y_k	0	0.11778	0.22314	0.31846	0.40547	0.48551	0.55962	0.62861	0.69315

Thus the approximation to $\int_1^2 \ln(x)\,dx$ with 4 intervals is

$$\frac{1}{2} \cdot \frac{1}{4} \cdot (0 + 2 \cdot (0.22314 + 0.40547 + 0.55962) + 0.69315) = 0.3837 \quad \text{(4 d.p.)}$$

while the approximation with 8 intervals is

$$\frac{1}{2} \cdot \frac{1}{8} \cdot (0 + 2 \cdot (0.11787 + \cdots + 0.62861) + 0.6931) = 0.3856 \quad \text{(4 d.p.)}.$$

3. We can use the same data as for the previous question, but with the approximation of Simpson's rule:

$$\frac{1}{3}h((y_0 + 4y_1 + y_2) + (y_2 + 4y_3 + y_4) + \cdots + (y_{N-2} + 4y_{N-1} + y_N))$$
$$= \frac{1}{3}h(y_0 + 4y_1 + 2y_2 + 4y_3 + \cdots + 2y_{N-2} + 4y_{N-1} + y_N).$$

Thus the approximation to $\int_1^2 \ln(x)\,dx$ with 4 intervals now becomes

$$\frac{1}{3} \cdot \frac{1}{4} \cdot (0 + 2 \cdot 0.40547 + 4 \cdot (0.22314 + 0.55962) + 0.69315) = 0.3863 \quad \text{(4 d.p.)}$$

while the approximation with 8 intervals is

$$\frac{1}{3} \cdot \frac{1}{8} \cdot (0 + 2 \cdot (0.22314 + 0.40547 + 0.55962) +$$
$$4 \cdot (0.11778 + 0.31846 + 0.48551 + 0.62861) + 0.69315) = 0.3863 \quad \text{(4 d.p.)}.$$

Since the two calculations agree to 4 decimal places, we can have some confidence that this is the right answer; in fact it is, as the exact answer is $2\ln 2 - 1$.

RM, 07/11/2017