Solutions to Exercise Sheet 7

1. We use the substitution $u = \sin x$. Then $du = \cos x\, dx$ and $x = 0$ becomes $u = 0$, while $x = \frac{\pi}{3}$ becomes $u = \frac{\sqrt{3}}{2}$. Hence

$$\int_0^{\frac{\pi}{3}} \cos x \sin^5 x\, dx = \int_0^{\frac{\sqrt{3}}{2}} u^5\, du = \left[\frac{1}{6}u^6\right]_0^{\frac{\sqrt{3}}{2}} = \frac{1}{6}\left(\frac{3^3}{2^6} - 0\right) = \frac{9}{128}.$$

2. Factorise the denominator and use the method of partial fractions. We want to find A, B, and C such that

$$\begin{aligned}\frac{x^2+3x+5}{x^3-x^2+2x-2} = \frac{x^2+3x+5}{(x-1)(x^2+2)} &= \frac{A}{x-1} + \frac{Bx+C}{x^2+2} \\ &= \frac{A(x^2+2)+(Bx+C)(x-1)}{x^3-x^2+2x-2} \\ &= \frac{(A+B)x^2+(C-B)x+2A-C}{x^3-x^2+2x-2}.\end{aligned}$$

This gives the system of equations $A+B=1$, $C-B=3$, and $2A-C=5$. Adding all of them, we obtain $3A=9$; so $A=3$. It then follows that $B=1-3=-2$ and $C=2\cdot 3-5=1$. Thus

$$\begin{aligned}\int \frac{x^2+3x+5}{(x-1)(x^2+2)}\, dx &= \int \frac{3}{x-1}\, dx - \int \frac{2x}{x^2+2}\, dx + \frac{1}{x^2+2}\, dx \\ &= 3\ln|x-1| - \ln|x^2+2| + \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.\end{aligned}$$

(The last of these terms is derived with the substitution $u = x/\sqrt{2}$, which gives

$$\int \frac{dx}{x^2+2} = \frac{1}{\sqrt{2}}\int \frac{du}{u^2+1} = \frac{1}{\sqrt{2}}\arctan\left(\frac{x}{\sqrt{2}}\right) + C.)$$

3. First note that the degree of the numerator is the same as the degree of the denominator, so we must first perform polynomial long division to write

$$x^2+8x+12 = 1\cdot(x^2+6x+9) + (2x+3)$$

and so, combining this with factorisation of the denominator, we obtain

$$\frac{x^2+8x+12}{x^2+6x+9} = 1 + \frac{2x+3}{(x+3)^2}.$$

We can now look for the partial fraction form

$$\frac{2x+3}{(x+3)^2} = \frac{A}{(x+3)} + \frac{B}{(x+3)^2} = \frac{Ax+3A+B}{(x+3)^2},$$

for suitable A and B. Comparing coefficients, we obtain the solution $A = 2$ and $B = 3 - 3A = -3$. Hence

$$\begin{aligned}\int \frac{x^2+8x+12}{x^2+6x+9}\,\mathrm{d}x &= \int \left(1 + \frac{2}{(x+3)} - \frac{3}{(x+3)^2}\right) dx \\ &= x + 2\ln|x+3| + \frac{3}{x+3} + C.\end{aligned}$$

4. As arccos is an inverse function, we integrate by parts with $u = \arccos x$ and $v = x$, which gives $du = \frac{-1}{\sqrt{1-x^2}}\,dx$ and $dv = dx$. Hence

$$\int_{-1}^{0} \arccos x\,dx = [x \arccos x]_{-1}^{0} + \int_{-1}^{0} \frac{x}{\sqrt{1-x^2}}\,dx.$$

Note that $[x\arccos x]_{-1}^{0} = 0 - (-1)\pi = \pi$. For the integral on the right-hand side, we use the substitution $w = 1 - x^2$, so $dw = -2x\,dx$, while the limit $x = -1$ becomes $w = 0$ and $x = 0$ becomes $w = 1$. Hence

$$\int_{-1}^{0} \frac{x}{\sqrt{1-x^2}}\,dx = -\int_{0}^{1} \frac{1}{2\sqrt{w}}\,dx = -\left[\sqrt{w}\right]_0^1 = -1 + 0 = -1.$$

So $\int_{-1}^{0} \arccos x\,dx = \pi - 1$.

5. Let $\sqrt{2}x = \sin u$, so that $\sqrt{2}\,dx = \cos u\,du$ and $1 - 2x^2 = 1 - \sin^2 u = \cos^2 u$. Thus

$$\int \frac{dx}{\sqrt{1-2x^2}} = \frac{1}{\sqrt{2}} \int \frac{\cos u\,du}{\sqrt{\cos^2 u}} = \frac{1}{\sqrt{2}} \int du = \frac{u}{\sqrt{2}} + C = \frac{1}{\sqrt{2}} \arcsin(\sqrt{2}x) + C.$$

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