Solutions to Exercise Sheet 6

1. We know that $\frac{d}{dx}\sinh x = \cosh x$. Therefore,

$$\int_{-1}^{1} \cosh x\, dx = [\sinh x]_{-1}^{1} = \sinh(1) - \sinh(-1) = 2\sinh 1.$$

2. We use the substitution $u = 1 + x^2$, so that $du = 2x\, dx$. We then obtain

$$\int \frac{x}{1+x^2}\, dx = \frac{1}{2}\int \frac{du}{u} = \frac{1}{2}\ln|u| + C = \frac{1}{2}\ln(1+x^2) + C.$$

Note that we could have written $\frac{1}{2}\ln|1+x^2| + C$, but this is not necessary in this case, because $1 + x^2$ is always positive anyway.

3. Note that

$$\int_0^{\frac{\pi}{4}} (\tan^2 x + \tan^4 x)\, dx = \int_0^{\frac{\pi}{4}} \tan^2 x(1 + \tan^2 x)\, dx.$$

We can then use the substitution $u = \tan x$, so that $du = (1 + \tan^2 x)\, dx$. The limit $x = 0$ then becomes $u = 0$, while $x = \frac{\pi}{4}$ becomes $u = 1$. Hence

$$\int_0^{\frac{\pi}{4}} (\tan^2 x + \tan^4 x)\, dx = \int_0^1 u^2\, du = \left[\frac{u^3}{3}\right]_0^1 = \frac{1^3}{3} - \frac{0^3}{3} = \frac{1}{3}.$$

4. We integrate by parts twice. First, let $u = 3x^2$ and $v = -e^{-x}$, which gives $du = 6xdx$ and $dv = e^{-x}\, dx$. Hence

$$\int 3x^2 e^{-x}\, dx = -3x^2 e^{-x} - \int 6x(-e^{-x})\, dx = -3x^2 e^{-x} + 6\int xe^{-x}\, dx.$$

Next, let $u = x$ and $v = -e^{-x}$, which gives $du = dx$ and $dv = e^{-x}\, dx$. Hence

$$\begin{aligned}\int 3x^2 e^{-x}\, dx &= -3x^2 e^{-x} + 6\left(-xe^{-x} + \int e^{-x}\, dx\right) \\ &= -3x^2 e^{-x} - 6xe^{-x} - 6e^{-x} + C \\ &= -3\left(x^2 + 2x + 2\right) e^{-x} + C.\end{aligned}$$

5. The integrand is a rational function and the degree of the numerator is larger than the degree of the denominator. So using polynomial long division, we rewrite the integrand as follows:

$$\frac{x^3+1}{x^2+1} = x - \frac{x-1}{x^2+1}.$$

It is convenient to split the last term again:

$$\frac{x^3+1}{x^2+1} = x - \frac{x}{x^2+1} + \frac{1}{x^2+1}.$$

Each term can then be integrated separately to give

$$\int \frac{x^3+1}{x^2+1}\,dx = \frac{1}{2}x^2 - \frac{1}{2}\ln(1+x^2) + \arctan x + C$$

(using the result from question 2 for the second term).

6. Factorise the denominator as follows:

$$x^2 + 5x + 6 = (x+2)(x+3).$$

Now find two numbers A and B that satisfy

$$\frac{2x+3}{x^2+5x+6} = \frac{A}{x+2} + \frac{B}{x+3} = \frac{A(x+3)+B(x+2)}{x^2+5x+6} = \frac{(A+B)x+3A+2B}{x^2+5x+6}.$$

We obtain the equations $A+B=2$ and $3A+2B=3$. Multiplying the first equation by 2 and subtracting the second equation, we obtain $-A=1$; so $A=-1$. It follows that $B=3$. Hence

$$\int \frac{2x+3}{x^2+5x+6}\,dx = \int \frac{3}{x+3}\,dx - \int \frac{1}{x+2}\,dx = 3\ln|x+3| - \ln|x+2| + C.$$

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