Solutions to Exercise Sheet 5

1. We first check that l'Hopital's rule applies: we have the limits $\lim_{h\to 0}\sin h = 0$, $\lim_{h\to 0}(\cos h - 1) = 0$, and of course $\lim_{h\to 0} h = 0$, so this is the case. Next we calculate $\frac{d}{dh}\sin h = \cos h$, $\frac{d}{dh}(\cos h - 1) = -\sin h$, and $\frac{d}{dh}h = 1$. Therefore,

$$\lim_{h\to 0}\frac{\sin h}{h} = \lim_{h\to 0}\frac{\cos h}{1} = 1$$

and

$$\lim_{h\to 0}\frac{\cos h - 1}{h} = \lim_{h\to 0}\frac{-\sin h}{1} = 0.$$

2. We compute $\ell'(x) = 1$, $\ell''(x) = 0$, and all higher derivatives vanish as well. Hence the Maclaurin polynomial of order 1 is

$$\ell(0) + \ell'(0)x = 1 + x = x + 1.$$

If we take the Maclaurin polynomial of any higher order, we just add 0 a few times, so the result will be the same.

3. Let $f(x) = \cos x - 1$ and $g(x) = 2x^2$. Since $\lim_{x\to 0} f(x) = \lim_{x\to 0} g(x) = 0$, we can (and should) use L'Hopital's rule:

$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{-\sin x}{4x}.$$

Now we still have $\lim_{x\to 0} f'(x) = \lim_{x\to 0} g'(x) = 0$, so we use L'Hopital's rule again:

$$\lim_{x\to 0}\frac{f'(x)}{g'(x)} = \lim_{x\to 0}\frac{f''(x)}{g''(x)} = \lim_{x\to 0}\frac{-\cos x}{4} = -\frac{1}{4}.$$

4. We are trying to solve $f(x) = x^2 - 3 = 0$ by Newton's method, so we first compute $f'(x) = 2x$. Thus the iterative formula is

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - 3}{2x_n} = \frac{x_n}{2} + \frac{3}{2x_n}$$

Applying this, starting with $x_0 = 2$, we get

$$\begin{aligned} x_1 &= \frac{2}{2} + \frac{3}{4} = 1.75, \\ x_2 &= \frac{1.75}{2} + \frac{3}{3.5} = 1.732143, \\ x_3 &= \frac{1.732143}{2} + \frac{3}{3.464286} = 1.732051, \\ x_4 &= \frac{1.732051}{2} + \frac{3}{3.464102} = 1.732051 \end{aligned}$$

We can stop here and conclude that $\sqrt{3} = 1.73205$ to 5 decimal places.

5. We differentiate $f(x)$ three times and evaluate each derivative at $x = 1$, obtaining

$$f(x) = x^3 - x, \quad f'(x) = 3x^2 - 1, \quad f''(x) = 6x, \quad f'''(x) = 6,$$
$$f(1) = 0, \quad f'(1) = 2, \quad f''(1) = 6, \quad f'''(1) = 6.$$

Substituting into the formula for the Taylor polynomial at $x = 1$ gives

$$T_{3,1}(x) = \sum_{k=0}^{3} \frac{f^{(k)}(1)}{k!}(x-1)^k = 2(x-1) + 3(x-1)^2 + (x-1)^3.$$

(Check: if you expand, you get exactly $f(x)$, which should be expected as $f(x)$ is actually a polynomial of degree 3 and so is its own best approximation.)

6. We have the standard series

$$\cos(x) = 1 - \frac{x^2}{2} + \frac{x^4}{4!} + \cdots,$$

and

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots.$$

so we can do this using the algebra of series, without differentiation. Thus

$$\begin{aligned} \cos(2x) &= 1 - \frac{(2x)^2}{2} + \frac{(2x)^4}{4!} + \cdots \\ &= 1 - 2x^2 + \frac{2}{3}x^4 + \cdots \end{aligned}$$

and

$$\begin{aligned} \frac{1}{1+x^2} &= \frac{1}{1-(-x^2)} \\ &= 1 + (-x^2) + (-x^2)^2 + (-x^2)^3 + \cdots \\ &= 1 - x^2 + x^4 + \cdots. \end{aligned}$$

So

$$\begin{aligned} \frac{2\cos(2x)}{1+x^2} &= 2\left(1 - 2x^2 + \frac{2}{3}x^4 + \cdots\right)\left(1 - x^2 + x^4 + \cdots\right) \\ &= 2\left(1 - x^2 + x^4 - 2x^2 + 2x^4 + \frac{2}{3}x^4 + \cdots\right) \\ &= 2 - 6x^2 + \frac{22}{3}x^4 + \cdots. \end{aligned}$$

RM, 23/10/2017