Solutions to Exercise Sheet 4

1. We compute

$$\frac{d}{dx}\sinh x = \frac{1}{2}\frac{d}{dx}(e^x - e^{-x}) = \frac{1}{2}(e^x + e^{-x}) = \cosh x$$

and

$$\frac{d}{dx}\cosh x = \frac{1}{2}\frac{d}{dx}(e^x + e^{-x}) = \frac{1}{2}(e^x - e^{-x}) = \sinh x.$$

2. (a) We look at the derivatives of $\sin^2(3x)$ and $(x-2)^2\ln(2x)$ separately.
Write $y = \sin^2(3x)$ and use the substitutions $v = 3x$ and $u = \sin v$, so that $y = u^2$. Then using the chain rule, we find that

$$\frac{d}{dx}\sin^2(3x) = \frac{dy}{du}\frac{du}{dv}\frac{dv}{dx} = 2u\cos v \cdot 3 = 6\sin(3x)\cos(3x).$$

The chain rule also gives $\frac{d}{dx}\ln(2x) = \frac{2}{2x} = \frac{1}{x}$. Letting $f(x) = (x-2)^2$ and $g(x) = \ln(2x)$, we use the product rule to get

$$\frac{d}{dx}(x-2)^2\ln(2x) = f'(x)g(x) + f(x)g'(x) = 2(x-2)\ln(2x) + \frac{(x-2)^2}{x}.$$

Putting everything together:

$$\frac{d}{dx}\left(\sin^2(3x) + (x-2)^2\ln(2x)\right) = 6\sin(3x)\cos(3x) + 2(x-2)\ln(2x) + \frac{(x-2)^2}{x}.$$

(b) This becomes easier if we first reformulate the expression. By the rules for exponential functions and logarithms,

$$5^{10x} = e^{10x\ln 5} \quad \text{and} \quad \log_{10} x^5 = 5\log_{10} x = \frac{5\ln x}{\ln 10}.$$

Hence

$$\frac{d}{dx}\left(5^{10x} - \log_{10} x^5\right) = 10\ln 5 \cdot e^{10x\ln 5} - \frac{5}{x\ln 10} = 10\ln 5 \cdot 5^{10x} - \frac{5}{x\ln 10}.$$

3. (a) Differentiating with respect to the parameter t gives $x'(t) = 3t^2 + 4$, $y'(t) = 4t + 7$. The point $(4,5)$ occurs when $t = 1$ and so at this point $\frac{dy}{dx} = \frac{y'(1)}{x'(1)} = \frac{11}{7}$.

(b) Differentiating with respect to x, while treating y as an implicit function of x, we find that

$$2x(y+2x) + x^2\left(\frac{dy}{dx} + 2\right) + 6y\frac{dy}{dx} = 0.$$

Re-arranging for $\frac{dy}{dx}$, we get $(x^2 + 6y)\frac{dy}{dx} = -2x(y+3x)$, which gives $\frac{dy}{dx} = \frac{-2x(y+3x)}{x^2+6y}$. Substituting the point $(1, -\frac{1}{3})$, we find $\frac{dy}{dx} = \frac{16}{3}$.

4. First compute $f'(x) = 3x^2 - 8x + 4 = (3x - 2)(x - 2)$. This is defined everywhere, so the only candidates for the local minima and maxima are the solutions of $f'(x) = 0$, i.e., $x = \frac{2}{3}$ and $x = 2$.

   Next we compute $f''(x) = 6x - 8$. This satisfies $f''(\frac{2}{3}) = -4 < 0$ and $f''(2) = 4 > 0$. So we have a local maximum at $\frac{2}{3}$ and a local minimum at 2.

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